# A TREATISE

OF

# MECHANICS,

BY


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TRANSLATED FROM THE FRENCH, AND ELUCIDATED WITH EXPLANATORY NOTES,

BY


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IN TWO VOLUMES.

VOL I


LONDON LONGMAN AND CO.

DUBLIN A. MILLIKEN, AND HODGES AND SMITH.

MDCCCXLII

DUBLIN:
PRINTED AT THE UNIVERSITY PRESS,
BY GRAISBERRY AND GILL

# PREFACE.

THOUGH there are several works in our language that treat of detached parts of mechanics in a clear and comprehensive manner, there is no single treatise in which the entire subject of rational mechanics, in all its bearings and generality, is brought before the reader The consequence of this has been, that the *Mechanique* of Poisson is the work which is now in the hands of all who propose to acquire such a knowledge of the principles of mechanics as may qualify and enable them to extend their reading to the more abstruse works of Laplace, La Grange, and also of Poisson himself The object of the present translation is to render this treatise of Poisson more accessible to the English reader

As several analytic operations and integrations are taken for granted by the author, it has been suggested that the work would be still more easily understood, if the most difficult of these operations were given in detail in the form of notes appended to the end of each volume It is likely the experienced reader may consider many of them to be superfluous, but as the object

of the translator was to render this work accessible to the English student, whose acquaintance with the higher branches of the calculus was not so extensive as the author takes for granted, he trusts he will not be deemed unnecessarily diffuse if he has insisted in the notes more at length on several points, than the accomplished reader may consider to be necessary

The translator did not venture to make any change in the text, though in some few cases the author appears to have fallen into mistake, see Nos 390 and 607 He also retained the numerical coefficient which is given by the author for the dilatation of gas, which it appears from the experiments of Rudberg, ought to be 00,364, instead of 00,375, see Scientific Memoirs, No 7 The reader, however, will find all these points adverted to in the corresponding notes

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## ERRATA

Page 52, line 9, *for* (e) *read* (b), and *for* (c) *read* (d)
—— 101, —— 2, *for* AME *read* ANF
—— 113, —— 11, for $\frac{x}{2}\left(\frac{a}{x}\right.$ read $\frac{x}{2}\left(\frac{a}{2}\right.$, &c , and for $\frac{1}{4}(ax-x)^2$ read $\frac{1}{4}(ax-x^2)$, and last line *for* $a^3$ read $a^2$
—— 155, last line, for $\epsilon^3$ read $\epsilon^2$
—— 180, line 1, for $i'nh$ read $inh$
—— 186, —— 21, *for* after displaced *read* in a sensible manner
—— 203, last line, for $2gh$ read $\sqrt{2gh}$
—— 244, last line, for $nx$ read $nz$
—— 286, line 14, *for* $\frac{d\theta^2}{dt^2}+$ read $\frac{d\theta^2}{dt^2}=$
—— 360, —— 5, for $de$ read $d\epsilon$, and line 11, *after* equations *read* ($h$)
—— 370, —— 10 *for* C *read* D
—— 373, —— 3 from bottom, for $3nt$ read $3n\psi$
—— 383, —— 9 *for* $\psi$ *read* $\phi$
—— 401, —— 13, for $2\pi\rho f'h$ read $2\pi\rho' fh$
—— 402, —— 5, for $r^3$ read $r^2$
—— 466, —— 2 from bottom, for $c\sqrt{(x'+c)^2}$ read $c+\sqrt{(x'+c)^2}$
—— 475, —— 1, put the bracket [ before $(dxd^2y$, and $]^{\frac{1}{2}}$ after $dxd^2y)^2$
—— 475, —— 11, for $(dxd^2y$ read $[(dxd^2y$, and for $dxd^2y)^2$ read $dxd^2y)^2]^{\frac{1}{2}}$
—— 496, —— 18, for $b=$ read $l=$
—— 514 last line, for $g=g\gamma\omega a$ read $q=g\gamma\omega a$
—— 522, line 3 from bottom, *for* $1-h^2+4h\sin^2\frac{1}{2}(\theta-a)$ *read* $(1-h)^2+4h\sin^2\frac{1}{2}(\theta-a)$,
—— 525, —— 10, *for* $\cos\frac{n\pi(x'-x)}{a}=\phi x$ read $\cos\frac{n\pi(x'-a)}{a}dx=\phi x$
—— 529, —— 3 from bottom, *for* $\frac{nx}{2a}=n\epsilon$, read $\frac{n\pi}{2a}=n\epsilon$
—— 600, —— 12, after $\int^{n-2}x\,dx$ read $\sqrt{\beta_1-x^2}$

# INTRODUCTION.

THAT which can affect our senses in any manner whatever, is termed *matter*.

*Bodies* are such portions of matter as are bounded in every direction; they must consequently have a determinate *form* and *volume*. By the *mass* of a body, is understood the quantity of matter of which it is composed.

A *material point* is a body infinitely small in all its dimensions, so that the length of every line comprised in its interior, is infinitely small, that is to say, less than any assignable length. A body of finite dimensions may be considered as an aggregate of an infinite number of material points, and its mass as the sum of all their infinitely small masses.

2. A body is in *motion*, when this body, or its parts, occupy successively different places in space. But *space* being indefinite and every where the same, we can only judge of the state of rest or motion of a body, by comparing it, either with other bodies, or with ourselves; and on this account, all the motions which we observe are necessarily relative motions.

All bodies are *moveable*; but matter never moves spontaneously, for there is no reason why a material point should move in one direction rather than in another, and in fact, if we consider a body at the instant it passes from a state of rest to a state of motion, we may always observe, that this change

is owing to the action of an extraneous cause, i. e. of one without which we may conceive that this body may exist.

Any cause which excites motion in a body, or which only tends to excite it, when its effect is interrupted or prevented by any other cause, is called *force*

3 When several forces are applied at once to the same body, they modify each other's effects, in virtue of the connexion that exists between its parts, and which hinders them from assuming the motion, that the force to which it is subjected, tends to impress on each of them. It may happen, that these forces completely destroy each other's effects, so that the body does not move at all, this particular state of the moveable body is termed *equilibrium*, in which the body remains at rest, though solicited by several forces, or in other words, the forces are said to constitute an *equilibrium*.

*Mechanics* is the science which treats of the equilibrium and motion of bodies The part of which the object is, in general, to discover the conditions of equilibrium, is called *statics*. The part of which the object is to determine the motion which a body assumes, when the forces which are applied to it do not constitute an equilibrium, is called *dynamics*.

As geometers have succeeded, as will be hereafter shewn, in reducing all questions relating to motion to mere problems + of equilibrium, the natural mode of proceeding would be to treat first of statics, and then to enter on the consideration of dynamics; but in order to facilitate the understanding of the subject, it seems preferable, in a treatise designed for instruction, to direct our attention to the elementary and simple parts of dynamics, before we enter on the consideration of the general questions of equilibrium. This is the order which will be pursued in the following treatise.

4. In the case of a force acting on a material point, there are always three things to be considered, namely, the position of this point, the intensity of the force, and its direction, that

is, the rectilineal space which it tends to cause to be described at its point of application Nevertheless, we must not suppose that a material point is the same with what is called a *point* in geometry, where this term denotes the extremity of a line, or the intersection of two lines which cut each other, neither is the space which a material point describes a line having only one dimension; but as this body is infinitely small in every direction, and as the breadth and thickness of the space, which the force tends to make it describe, are also infinitely small, the position and direction of this force can be determined in the same manner, as the position and direction of a right line are determined in geometry.

Thus, the position of the point of application of a force in space, will be determined, in general, by means of its three coordinates parallel to the intersections of three rectangular planes; and this will not leave any ambiguity with respect to the direction, when we take into account both the sign and magnitude of each coordinate. Sometimes also polar coordinates are employed, these are the radius vector of the given point, or its distance from the origin, the angle this radius makes with a fixed line drawn through the origin, and the angle comprised between a plane passing through these lines, and a fixed plane passing through the second.

5. In order to measure forces, it is necessary to take some known force as unit, and then to express in numbers the relations of the other forces to this unit, this requires that we should precisely define what is meant when one force is said to be equal to another, double, triple, or quadruple .... of another, independently altogether of the particular nature of these different causes of motion.

Two forces are said to be *equal*, when being applied in opposite directions to the same material point, or to two points connected by a right line, which is of an invariable length, they constitute an equilibrium.

If, after having recognized that two forces are equal, we

apply them in the same direction to the same point, we shall have a *double* force; if we combine in this manner three equal forces, we shall have a *triple* force; if we combine four, we shall have a *quadruple* force, and so on.

When, therefore, we say that a force applied to a material point is a certain multiple of another force, we understand that the first may be considered as made up of a certain number of forces respectively equal to the second, and acting in the same direction. It is by considering them in this manner, that forces can be regarded, whatever their particular nature may be, as quantities susceptible of being measured, which may, therefore, be expressed in numbers, like every other description of quantity, by referring them to the unit of their species. We may likewise represent their intensities by lines proportional to these numbers, which lines we lay off in their several directions, commencing at their respective points of application; one advantage effected by this is, that mechanical theorems may be stated with great simplicity.

6. The points of application, and the intensities of forces, being thus determined, it only remains to shew how their directions are ascertained.

Let M (fig. 1) be the point of application of a force, and let its direction be represented by the line MD, in which case this force tends to cause the point M to move from M to D; through the point M, conceive three rectangular axes MA, MB, MC, to be drawn, these will be, in general, parallel to the axes of the coordinates, and in the direction of the positive coordinates; let $\alpha$, $\beta$, $\gamma$, be the obtuse or acute angles which the direction MD makes with these axes, so that

$$\text{AMD} = \alpha, \quad \text{BMD} = \beta, \quad \text{CMD} = \gamma;$$

this direction will be completely determined, when these three angles are given.

In fact, if we only consider the two angles $\alpha$ and $\beta$, the line MD must exist at the same time on two right cones, of

which the common summit is M, and of which the respective axes are MA and MB. Therefore it is necessary that $\alpha$ and $\beta$ should be such that these two cones may intersect; this will have place in the direction of two lines, drawn from M, situated in the same plane perpendicular to the plane AMB, and which will make with the axis MC, two angles that are supplements the one of the other. Therefore the line MD may have also two different positions, but as the angle $\gamma$ is also given, we know whether it is acute or obtuse, and we can select between these two positions that which answers to the direction of the force. This construction also shews, that the three angles $\alpha$, $\beta$, $\gamma$, cannot be arbitrarily taken. In fact, there exists between the cosines of the angles which the same right line MD makes with three rectangular axes, the equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \tag{1}$$

which can be demonstrated by taking on MD, reckoning from the point M, a line equal to unity, and then forming a rectangular parallelopiped, of which this line is the diagonal, and of which the three adjacent sides lie on the three axes MA, MB, MC. These three sides will be the cosines of the angles $\alpha, \beta, \gamma$; and as by a known theorem, the sum of their squares must be equal to the square of the diagonal, there results the equation which has been stated above.

7. In this treatise, the division of the circumference into 360°, of the degree into 60 minutes, and of the minute into 60 seconds, will be adopted. The letter $\pi$ will be constantly employed to represent the semicircumference of which the radius is equal to unity, so that we shall have

$$\pi = 3{,}1415926 \ldots .$$

The fourth part of the circumference corresponds to a right angle, or to the angle $324000''(a)$, hence it follows, that the length of an arc corresponding to an angle, containing any number such as $n$ of seconds, will be the fourth term of a

proportion, of which the three first terms are $\frac{1}{2}\pi$ $n$ and 32400″, and denoting this length by $\omega$ there will result(*b*)

$$\omega = \frac{n}{206264,8\ldots}.$$

The logarithm, in the common system, of this constant divisor is

$$5,3144251.$$

In numerical calculations, the arcs computed in this manner ought to be employed in place of angles, which will not be comprised under the trigonometric signs *sin*, *cos*, *tang*.

In order that we may be able, by means of the angles $\alpha$, $\beta$, $\gamma$, to represent the direction of a force in all possible positions about its point of application, it is only necessary that they be reckoned from zero up to 180° inclusively. If, for example, the axis MC lies above the plane of the two other axes MA and MB, the angle $\gamma$ will be less or greater than 90°, according as the right line MD exists above or below this plane: it will be zero when the direction MD coincides with MC, and equal to 180°, when MD coincides with MC′ the production of MC. The cosines of $\alpha$, $\beta$, and $\gamma$, may therefore be positive or negative; but their sines will be always positive, because these angles never exceed 180°.

In general, if we consider MD′ the production of any line MD, in the opposite direction, it is evident that the angles which it makes with the three axes are the supplements of $\alpha$, $\beta$, $\gamma$. Therefore if we make

$$\text{AMD}' = \alpha', \quad \text{BMD}' = \beta', \quad \text{CMD}' = \gamma',$$

we shall have

$$\cos\alpha' = -\cos\alpha, \quad \cos\beta' = -\cos\beta, \quad \cos\gamma' = -\cos\gamma;$$

hence it follows that the directions of two forces which act in opposite directions on the same point M, the one along MD and the other along MD′, are distinguished from each other, by the

signs of the cosines of the angles which respectively correspond to them.

8. In place of three angles $\alpha$, $\beta$, $\gamma$, connected together by equation (1), we may only employ two angles, perfectly independent of each other, to determine the direction of a force

In fact, let ME be the projection of MD on the plane AMB, and let $\delta$ be the angle which this projection makes with the axis MA, so that we have AME $= \delta$. When the angle $\delta$ is given, it will make known the position of the plane CME, and then the angle $\gamma$ will completely determine the position of the line MD comprised in this plane. It is necessary that the angle $\delta$ should be reckoned from MA in a definite direction, which may extend from zero to $360°$; the angle $\gamma$, as was before stated, is comprised between the limits zero and $180°$. The projection of the diagonal of the parallelopiped, adverted to in No. 6, on the plane AMB, will be the cosine of the angle DME, or equal to $\sin \gamma$ If we project a second time this projection on the axis MA, this second projection is equal to first multiplied by $\cos \delta$, moreover it coincides with the projection of the diagonal of the parallelopiped on this same axis MA, and consequently will be equal to $\cos \alpha$; therefore we shall have

$$\cos \alpha = \sin \gamma . \cos \delta .$$

In like manner we shall have

$$\cos \beta = \sin \gamma . \sin \delta .$$

These two formulæ will enable us to transform the equations in which the angles $\alpha$, $\beta$, $\gamma$, are made use of, into others in which we only employ $\gamma$ and $\delta$. It is easy to perceive that they satisfy equation (1).

9. There exists another equation that comprises equation (1) as a particular case, and which will be frequently useful.

In order to obtain it, let $x$, $y$, $z$, be the three coordinates of any point M (fig 2) referred to the three rectangular axes

$ox$, $oy$, $oz$. Let $r$ denote its radius vector OM, and $\alpha, \beta, \gamma$, the acute or obtuse angles which this radius makes with the three axes, so that, for example, we may have

$$zOM = \gamma.$$

If from the point M a perpendicular MN be let fall on the axis $Oz$, the right line ON will be the ordinate $z$, and in the right angled triangle MON we shall have

$$z = r.\cos\gamma,$$

in like manner there results

$$y = r.\cos\beta, \quad x = r.\cos\alpha.$$

Let M$'$ be another point and let $x'$, $y'$, $z'$, $r'$, $\alpha'$, $\beta'$, $\gamma'$, be respectively its coordinates, its radius vector, and the angles, relatively to this line; we shall have

$$x' = r'\cos\alpha', \quad y' = r'.\cos\beta', \quad z' = r'.\cos\gamma'.$$

Let $u$ denote the distance MM$'$, then we have the known relation

$$u^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$

and if $\varepsilon$ denote the angle MOM$'$, we shall have at the same time

$$u^2 = r^2 + r'^2 - 2rr'.\cos\varepsilon,$$

in the triangle of which $r$, $r'$, and $u$ are the three sides.

Because

$$x^2 + y^2 + z^2 = r^2, \quad x'^2 + y'^2 + z'^2 = r'^2,$$

the first value of $u^2$ is the same thing as

$$u^2 = r^2 + r'^2 - 2(xx' + yy' + zz');$$

by comparing it with the second, we obtain

$$rr'.\cos\varepsilon = xx' + yy' + zz',$$

and if in this equation, we substitute the preceding values of $x, y, z, x' y', z'$, there will result

$$\cos\varepsilon = \cos\alpha.\cos\alpha' + \cos\beta.\cos\beta' + \cos\gamma.\cos\gamma'; \qquad (2)$$

which it was proposed to find out.

When the two lines OM and OM′ coincide, the angles $\alpha', \beta', \gamma'$, are the same as $\alpha, \beta, \gamma$, and this formula is reduced to equation (1). When these two right lines are at right angles to each other, we have $\varepsilon = 90^\circ$, and consequently

$$\cos\alpha \cos\alpha' + \cos\beta . \cos\beta' + \cos\gamma . \cos\gamma' = 0.$$

By substituting in the values of $x, y, z$, those of $\cos\alpha$, $\cos\beta$, which have been found in the preceding number, we shall have

$$x = r . \sin\gamma . \cos\delta, \quad y = r . \sin\gamma \sin\delta, \quad z = r \cos\gamma;$$

in which formulæ, the three variables $r, \gamma, \delta$, are the three polar coordinates of the point M, such as they have been defined in No. 4, and which will consequently enable us to transform rectangular into polar coordinates

10 The consideration of projections which were made use of in No. 8, will be frequently adverted to in this treatise, it will therefore be useful here to explain their first principles.

The *projection* of a right line on another right line, is the part of the latter which is comprised between two perpendiculars let fall from the two extremities of the projected line on the other Thus, the differences $x' - x$, $y' - y$, $z' - z$, of the extreme coordinates are the projections of the same right line MM′ on the axes of $x, y, z$, and, from the first expression which is given above for $u^2$, it follows, that the sum of the squares of the projections of the same right line, on three rectangular axes, is equal to the square of this right line If the projected line, and the line on which it is projected, exist in the same plane, the projection is equal and parallel to the base of a right angled triangle, of which the projected line is the hypothenuse, so that if $l$ represents the length of this line, $\lambda$ that of its projection, and $\imath$ the angle contained by these two right lines, we shall have

$$\lambda = l . \cos\imath.$$

The projection of a plane surface on another plane, is the

part of this plane which is terminated by the projection of the outline of the projected surface, that is to say, by the curve formed by perpendiculars let fall from all the points of this outline. Now the preceding equation likewise obtains, if in place of $l$, the area of the projected surface be substituted, and in place of $\lambda$, the area of its projection, $\imath$ being then the inclination of one plane upon the other, which is evidently equal to the angle contained by the two perpendiculars to these planes.

In fact, if the area of the projected surface be divided into a number of indefinitely small elements perpendicular to the intersection of its plane with that on which the projection is made; the projection of each element will be equal to this element multiplied by the cosine of their mutual inclination, consequently as this inclination is the same and equal to $\imath$ for all the elements, the sum of their projections or $\lambda$, will be equal to their sum, or to the entire area $l$, multiplied by $\cos \imath$. Hence it follows, that the square of the area of a plane surface, is equal to the sum of the squares of its projections on three rectangular planes, this follows from equation (1), and from considering that the inclination on each plane is the angle which a normal to the given surface makes with the perpendiculars to this plane(*c*).

11. When in a question, we consider a system of parallel forces, we may suppose that one of the three rectangular axes MA, MB, MC, (fig. 1) is also parallel to them. Then two of the three angles $a, \beta, \gamma$, the two last, for instance, will be right angles for all these forces, and the equation (1) will be reduced to

$$\cos^2 a = 1;$$

hence it follows that $a = 0$ or $180°$.

In this manner, the direction of each force will be determined by stating, that it makes with the axis MA an angle, which either vanishes or is equal to $180°$, but in this particular case, it will be simpler to determine this direction by the

sign of the force, those forces which act in one direction being considered as positive, while those which act in an opposite direction are regarded as negative.

In fine, the case of parallel forces will be the only one in which *forces* are considered as *positive* and *negative*, in all other cases, the quantities which represent the magnitudes of forces are considered in computations as positive; and the variation of the sign will fall on the cosines of the angles which their directions make with fixed axes.

12. What precedes respects the preliminary definitions and details that are required for determining the magnitudes and directions of forces, but as the method of *infinitely small quantities* will be exclusively adopted in the following treatise, it is necessary to advert to in this introduction, the principles of the infinitesimal analysis, and among the formulæ which are immediately deducible from them, to enumerate those which will be useful in the sequel.

An *infinitely small quantity* is a magnitude less than any other given magnitude of the same nature.

When we consider the successive variations of a magnitude subject to the law of continuity, the notion of infinitely small quantities is necessarily suggested Thus, time increases by degrees less than any assignable interval, however small. The spaces traversed by the different points of a body, likewise increase by infinitely small quantities; for each point cannot go from one position to another, without traversing all the intermediate positions, and we cannot assign a distance, as small as we please, between two consecutive positions. Therefore, infinitely small quantities have a real existence, and are not to be considered as merely a means of investigation devised by geometers.

An infinitely small quantity may be double, triple, quadruple.... of another, and between such quantities any relations may exist, the determination of which is an essential object of the infinitesimal analysis.

If $a$ and $b$ be infinitely small quantities, and if the ratio of $a$ to $b$ be also infinitely small, $b$ is what is termed an infinitely small quantity of the second order. For example, the chord of an arc of a circle being supposed infinitely small, the versed sine of the same arc is an infinitely small quantity of the second order, because the ratio of the versed sine to the chord is always the same as that of the chord to the diameter, and consequently becomes infinitely small at the same time as the second ratio. In like manner, if $b$ be an infinitely small quantity of the second order, and if we suppose the ratio of $c$ to $b$ to be an infinitely small quantity of the first order, $c$ will be an infinitely small quantity of the third order, and so on. Hence it follows, that a product composed of a number $n$ of infinitely small factors of the first order, must be ranged in the class of infinitely small quantities of the order $n$. The area of a surface infinitely small in all its dimensions, is at least an infinitely small quantity of the second order; for it is less than the square of the longest line which can be drawn from one point to another of its outline, which is by hypothesis an infinitely small quantity. Likewise it is evident, that a volume of which all the dimensions are infinitely small, is at least an infinitely small quantity of the third order, for it is less than the cube of the longest line drawn from one point to another of its superficies.

This being premised, the fundamental principle of the infinitesimal calculus consists in this, that two finite quantities, which only differ from each other by an infinitely small quantity, must be considered as equal, since we cannot assign any inequality between them, however small. This is also true with respect to two infinitely small quantities of the first order, of which the difference is an infinitely small quantity of the second order, and in general two infinitely small quantities of any order whatever, which only differ from each other by an infinitely small quantity of a higher order, may be considered as quantities rigorously equal, and their ratio as equal to unity.

These principles may be expressed in a different manner, by stating that in a calculation we are permitted to neglect, without any apprehension of altering the results, either infinitely small quantities added to finite quantities, or infinitely small quantities of any order, which are added to those of an inferior order.

13. $dx$, the *differential* of an independent variable $x$, is an infinitely small increment ascribed to this variable; $dy$ the differential of $y$ a function of $x$, is the corresponding increment of this function, reduced to the same order of magnitude as that of the independent variable, by the suppression of infinitely small quantities of a higher order, hence it follows, that this differential $dy$ is always of the form $\mathrm{X}dx$, $\mathrm{X}$ being another function of $x$. It may happen, that for some particular values of $x$, the differential coefficient $\mathrm{X}$ becomes infinite, which will render the differential $\mathrm{X}dx$ indeterminate; but this case will not occur in mechanics. Let $fx$ be a given function of $x$, $c$ a constant arbitrary, and $\mathrm{F}x + c$ the complete or *indefinite* integral of $fxdx$. Let also $a$ and $b$ be two given constants. If the constant $c$ be determined in such a manner, that this integral may be nothing, or commence when $x = a$, and if then $x$ be assumed equal to $b$, the result $\mathrm{F}b - \mathrm{F}a$ will be what is termed the *definite* integral, taken between the limits $x = a$ and $x = b$. We shall denote it by $\int_a^b fxdx$, agreeably to the very commodious notation proposed by Fourier; consequently, we shall have

$$\mathrm{F}b - \mathrm{F}a = \int_a^b fxdx.$$

If we assign to $x$ an infinite number of values, increasing successively from $a$ to $b$ by infinitely small differences, and if these differences (whether equal or unequal) be assumed as the values of $dx$, it is easy to shew that the sum of all the values of the differential $fxdx$ will be equal to the definite integral

$Fb - Fa$. In fact, if infinitely small quantities of an order higher than the first, be neglected, we have by the definition of the differential

$$F(x + dx) - Fx = fxdx.$$

Therefore if $\delta_1, \delta_2, \delta_3, \ldots \delta_n$, denote an infinite number of infinitely small quantities, such that

$$\delta_1 + \delta_2 + \delta_3 + \ldots\ldots + \delta_n = b - a,$$

and if we take successively for $x$ and $dx$, the pairs of values $a$ and $\delta_1$, $a + \delta_1$ and $\delta_2$, $a + \delta_1 + \delta_2$ and $\delta_3$, ..... $b - \delta_{n-1}$ and $\delta_n$, there will result

$$F(a + \delta_1) - Fa = fa\delta_1,$$
$$F(a + \delta_1 + \delta_2) - F(a + \delta_1) = f(a + \delta_1)\,\delta_2,$$
$$F(a + \delta_1 + \delta_2 + \delta_3) - F(a + \delta_1 + \delta_2) = f(a + \delta_1 + \delta_2)\,\delta_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$Fb - F(b - \delta_n) = f(b - \delta_n)\,\delta_n,$$

the sum of these equations is

$$Fb - Fa = fa\delta_1 + f(a + \delta_1)\,\delta_2 + f(a + \delta_1 + \delta_2).\delta_3$$
$$\cdot \quad \cdot\cdot\cdot\cdot + f(b - \delta_n)\,\delta_n,$$

which expresses the theorem, that was proposed to be proved.

When the function $fx$ becomes infinite, between the two limits $a$ and $b$, this demonstration does not obtain, and the theorem fails. In this case of exception, which will not occur in this treatise, the definite integral has no connexion with the sum of the values of the differential, and it may be negative when all these values are positive, or positive when they are all negative. In order then to make this theorem obtain, we should take care, that $fx$ does not become infinite between $x = a$ and $x = b$, by causing the variable $x$ to pass from the first to the second of these limits, through a series of imaginary values(*d*).

The preceding theorem may be extended without difficulty to multiple integrals. Thus for example, if $f(x, y)$ be a given

function of two independent variables $x$ and $y$, and if we assign to these variables successively, series of values increasing by infinitely small differences, and if at the same time we assume the differences between the consecutive values of $x$ to be equal to $dx$, and those of the consecutive values of $y$ to be equal to $dy$, the sum of all the values of $f(x, y)\, dxdy$, will be the integral $\iint f(x, y)\, dxdy$, taken between suitable limits.

(14.) When the function $fx$ contains a quantity $a$ that has been considered as constant during the course of the integration, the value of the integral $\int_a^b fxdx$ will be itself a function of $a$ Questions occur, in which this integral not being known in a finite form, it will be nevertheless necessary to determine its differential with respect to $a$. Now, in this operation, two different cases present themselves, according as the limits $a$ and $b$ are independent of $a$, or depend on it in some way In the first case, it will be sufficient to difference $fx$ with respect to $a$, under the sign $\int$, so that we shall have

$$\frac{d \int_a^b fxdx}{da} = \int_a^b \frac{dfr}{da}\, dx.$$

In fact, it follows from the theorem of the preceding number, that the first member of this equation is the differential coefficient with respect to $a$ of the sum of the values of $fxdx$ comprised between $x = a$ and $x = b$, while the second member is the sum of the values between the same limits, of the differential coefficient of $fxdx$ taken relatively to $a$, and it is evident, that these two sums are identically the same. In the second case, when $a$ becomes $a + da$, the limit $b$ becomes $b + \frac{db}{da} da$, and on this account, the sum of the values of $fxdx$, or the integral $\int_a^b fxdv$ is increased by the value of $fxdx$,

which answers to $x = b$, and $dx = \frac{db}{da} . da$, that is to say, of $fb \frac{db}{da} da$; at the same time, the limit $a$ is changed into $a + \frac{da}{da} da$, this diminishes this integral by the value of $fxdx$, corresponding to $x = a$ and $dx = \frac{da}{da} da$, or of $fa \frac{da}{da} da$, therefore, on account of the simultaneous variation of the two limits $a$ and $b$, produced by that of $\alpha$, the integral will be increased by the differential

$$\left(\frac{db}{d\alpha} fb - \frac{da}{d\alpha} fa\right) d\alpha,$$

and its differential coefficient with respect to $\alpha$, by this coefficient of $d\alpha$. Consequently, by adding it to the second member of the preceding equation, we shall have

$$\frac{d . \int_a^b fx\, dx}{d\alpha} = \int_a^b \frac{dfx}{d\alpha} dx + \frac{db}{d\alpha} fb - \frac{da}{d\alpha} fa,$$

for the complete value of the differential coefficient of $\int_a^b fxdx$.

When $\alpha$ does not occur in $fx$, if this quantity be one of the two limits $b$ or $a$, and if these two limits do not depend the one on the other, this expression will be reduced to

$$\frac{d . \int_a^b fxdx}{db} = fb \text{ or } \frac{d \int_a^b fxdx}{da} = -fa;$$

which is, otherwise, evident of itself.

Similar remarks are applicable to multiple integrals, of which the differential coefficients with respect to a quantity which in the first instance is considered as constant, may likewise be obtained, by differentiating under the signs of integration, and by adding to the result the terms which depend on the variations of the limits, when they are functions of this quantity considered as variable

15. The integral calculus furnishes rules which enable us to determine either exactly, or by approximation, the numerical values of definite integrals, whether simple or multiple; so that a problem is considered to be resolved, when we are enabled to express the unknown quantities by integrals of this nature. The problem is then said to be reduced to *quadratures*, because on the one hand, a multiple integral is nothing else than a simple integral several times repeated, and also because on the other, an integral such as $\int_a^b fx dx$ may always be represented by a square equal to the area of the plane curve, in which $x$ and $fx$ are the coordinates of any point whatever, and $a$ and $b$ the abscissæ of the extreme points.

Among the different formulæ which are in use for determining the approximate values of this integral $\int_a^b fx dx$, we will cite the following, in which it is assumed that the functions $fx$ and $\frac{dfx}{dx}$ do not become infinite between the limits $a$ and $b$.

Retaining the preceding notation, and moreover making

$$\frac{dfx}{dx} = fx', \quad \frac{d^2fx}{dx^2} = fx'', \text{ \&c}$$

If the differences $\delta_1$, $\delta_2$, $\delta_3$, &c. are not infinitely small, but only very small, when they are all equal and represented respectively by $\delta$, we shall have by Taylor's theorem

$$\text{F}(a+\delta) = \text{F}a + \delta fa + \tfrac{1}{2}\delta^2 f'a + \text{\&c.}$$
$$\text{F}(a+2\delta) = \text{F}(a+\delta) + \delta f(a+\delta) + \tfrac{1}{2}\delta^2 f'(a+\delta) + \text{\&c.}$$
$$\text{F}(a+3\delta) = \text{F}(a+2\delta) + \delta f(a+2\delta) + \tfrac{1}{2}\delta^2 f'(a+2\delta) + \text{\&c.}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\text{F}(a+n\delta) = \text{F}(a+(n\delta-\delta)) + \delta f(a+n\delta-\delta) + \tfrac{1}{2}\delta^2 f'(a+n\delta-\delta) + \text{\&c.}$$

Therefore, if we suppose $n\delta = b - a$, we shall have, by taking the sum of these equations,

$$\mathrm{F}b - \mathrm{F}a = \delta . \Sigma f(a + \imath\delta) + \tfrac{1}{2}\,\delta^2 . \Sigma f'(a + \imath\delta)$$
$$+ \tfrac{1}{6}\,\delta^3\,\Sigma f'' . (a + \imath\delta) + \&c.$$

$\imath$ being an integer number or cypher, and the characteristics $\Sigma$ denoting the sums which extend to the $n$ values of $\imath$, taken between $\imath = 0$ and $\imath = n - 1$. If we assume $fx$ and $fx'$, $fx'$ and $fx''$, &c. successively, in place of $\mathrm{F}x$ and $fx$, we shall likewise have

$$fb - fa = \delta\Sigma f'(a + \imath\delta) + \tfrac{1}{2}\,\delta^2\,\Sigma f''(a + \imath\delta) + \&c.,$$
$$f'b - f'a = \delta\Sigma f''(a + \imath\delta) + \&c.$$

This being established, if we wish to neglect powers of $\delta$ higher than the square, in the value of $\mathrm{F}b - \mathrm{F}a$, we can, by means of the preceding equations, assume

$$\tfrac{1}{2}\,\delta^2\Sigma f'(a + \imath\delta) = \tfrac{1}{2}\,\delta(fb - fa) - \tfrac{1}{4}\,\delta^2(f'b - f'a),$$
$$\tfrac{1}{6}\,\delta^3\,\Sigma f''(a + \imath\delta) = \tfrac{1}{6}\,\delta^2\,(f'b - f'a),$$

for the values of its two last terms; its entire value will consequently become

$$\mathrm{F}b - \mathrm{F}a = \delta\Sigma f(a + \imath\delta) + \tfrac{1}{2}\,\delta\,(fb - fa) - \tfrac{1}{12}\,\delta^2\,(f'b - f'a),$$

or which is the same thing($e$),

$$\int_a^b fx dx = \delta\,[\tfrac{1}{2} fa + f(a + \delta) + f(a + 2\delta) \ldots\ldots$$
$$\ldots\ldots + f(a + n\delta - \delta) + \tfrac{1}{2} fb] - \tfrac{1}{12}\delta^2\,(f'b - f'a).$$

This formula will be more exact according as the difference $\delta$, or $\frac{1}{n}(b - a)$, is less, and as the values of $fx$ vary less rapidly between the limits $a$ and $b$. For the most part we can neglect the term depending on $\delta^2$; the formula will then only contain the values of $fx$, which may be given in numbers, although the form of this function is not known

16 In the theory of infinitely small quantities, curves are treated as polygons, which consist of an infinite number of infinitely small sides. This implies that the chord of an infinitely small arc is equal to this arc, or that we can assume

the ratio of their respective lengths to be equal to unity; which can be plainly demonstrated in the following manner. Let $\text{M}mm'\text{M}'$ (fig. 3) be an infinitely small arc of a curve, let the chords $\text{M}m$, $mm'$, $m'\text{M}'$, be drawn, and let the third be produced to meet $\text{MT}$ the production of the first, in a point $\text{K}$. The arc $mm'$ is greater than the chord $mm'$, and less than the line $m\text{K} + \text{K}m'$, therefore, if we prove that this line and this chord, when infinitely small, differ only by an infinitely small quantity of a higher order, and that consequently we can assume their ratio to be equal to unity, it will follow *à fortiori*, that the ratio of the arc $mm'$ to its chord, is that of equality. Now if in the arc $\text{M}mm'\text{M}'$, there does not exist any *singular* point in which the direction of the curve changes abruptly, chords drawn from one of its points to two other will contain an angle differing from two right angles by an infinitely small quantity. Consequently the angle $\text{TKM}'$ the supplement of $\text{MKM}'$ will be an infinitely small quantity, denoting it by $\delta$, and moreover, making,

$$m\text{K} = a, \quad m'\text{K} = b, \quad mm' = c,$$

we shall have in the triangle $m\text{K}m'$, the equation

$$c^2 = a^2 + b^2 + 2ab \cos \delta,$$

which can be converted into

$$c^2 = (a+b)^2 - 4ab . \sin^2 \tfrac{1}{2} \delta,$$

(because $\qquad \cos \delta = 1 - 2 \sin^2 \tfrac{1}{2} \delta$).

Consequently we shall have

$$\frac{c^2}{(a+b)^2} = 1 - \frac{4ab}{(a+b)^2} . \sin^2 \tfrac{1}{2} \delta,$$

which expresses the square of the ratio of the chord $mm'$ to the line $m\text{K} + \text{K}m'$. Moreover as

$$\frac{4ab}{(a+b)^2} = 1 - \left(\frac{a-b}{a+b}\right)^2,$$

it follows that the coefficient of $\sin^2 \tfrac{1}{2}\delta$, can never become in-

finite, because it is always less than unity. Therefore, if we neglect infinitely small quantities of the second order, the ratio of $c$ to $a+b$ will be always expressed by unity.

17. Considering a curve as an infinitesimal polygon, the tangents will be the productions of the infinitely small sides, at the point M, where the side is M$m$, the tangent will be the indefinite line T′M$m$T. If $x$, $y$, $z$, denote the three rectangular coordinates of the point M, those of the point $m$ will be $x+dx$, $y+dy$, $z+dz$. If $ds$ denotes the element of the curve, i. e. its side M$m$, the differentials $dx$, $dy$, $dz$, will be its projections on the axes of $x$, $y$, $z$; consequently, if $\alpha$, $\beta$, $\gamma$, be the three angles which the direction of the line MT makes with lines parallel to these axes respectively, drawn through the point M, we have

$$\cos\alpha = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds}, \qquad (1)$$

and likewise at the same time,

$$dx^2 + dy^2 + dz^2 = ds^2.$$

Assuming on the curve CM$mm'$M′ a fixed point C, and supposing that $s$ denotes the arc CM reckoned from this origin, this arc can be considered as the independent variable, consequently $x$, $y$, $z$, will be given functions of $s$, depending on the equations of the curve. In this case, $ds$ will be positive, but $dx$, $dy$, $dz$, and consequently $\cos\alpha$, $\cos\beta$, $\cos\gamma$, may be either positive or negative. The angles $\alpha$, $\beta$, $\gamma$, always refer to $m$T the production of the side M$m$, or to the part M$m$T of the tangent, the angles relatively to the other part MT′ will be the supplements of $\alpha$, $\beta$, $\gamma$, (No. 7.) As the direction of the tangent at the point M is determined by equations (1), we can deduce from them also the equation of the *normal plane* at this same point; but this equation may be directly obtained in the following manner. Let $k$ denote the radius of a sphere, the centre of which is at the point M, its equation will be

$$(x'-x)^2 + (y'-y)^2 + (z'-z)^2 = k^2,$$

$x'$, $y'$, $z'$, denoting the coordinates at the other extremity of the radius $k$. The equation of the sphere having the same radius, and of which the centre is at the point $m$, may be deduced from this, by substituting $x + dx$, $y + dy$, $z + dz$, in place of $x$, $y$, $z$, and if we subtract one of these equations from the other, we shall have, by neglecting infinitely small quantities of the second order($f$),

$$(x' - x)\,dx + (y' - y)\,dy + (z' - z)\,dz = 0;$$

which is the equation for the intersection of two spherical surfaces. As it is the equation of a plane of which $x'$, $y'$, $z'$, are the coordinates, it will be that of the plane of this curve, and consequently the required equation of the normal plane, because the intersection of these two spheres is a circle perpendicular to the line TT′ which passes through their centres M and $m$.

If we divide this equation by $ds$, and then substitute for $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, their values given by equation (1), it will become

$$(x' - x)\cos\alpha + (y' - y)\cos\beta + (z' - z)\cos\gamma = 0.$$

Therefore if

$$a\,(x' - x) + b\,(y' - y) + c\,(z' - z) = 0,$$

represents the equation of a plane drawn through a point of which the coordinates are $x$, $y$, $z$, and perpendicular to the line of which the direction is determined by the angles $\alpha$, $\beta$, $\gamma$, it must coincide with the preceding; this requires that we should have

$$a = h.\cos\alpha, \quad b = h\cos\beta, \quad c = h\cos\gamma,$$

$h$ being an indeterminate factor. Moreover from equation (1) of No. 6, we can infer

$$a^2 + b^2 + c^2 = h^2;$$

by means of which we are enabled to determine the value of

$h$ in every particular, with the exception of the sign. Hence then there results

$$\cos\alpha = \frac{a}{h}, \quad \cos\beta = \frac{b}{h}, \quad \cos\gamma = \frac{c}{h}; \tag{2}$$

which coincides with those known formulæ by means of which the direction of a perpendicular to a given plane can be determined. The reason why the sign of $h$ is undetermined, is because the plane has two sides, and the angles $\alpha, \beta, \gamma$, may refer to this line considered indifferently as existing on one side or the other.

18. The *angle of contact* is the indefinitely small angle contained between two consecutive tangents. Thus if M$m$ and $mm'$ (fig. 4) be consecutive sides of the curve, this angle at the point $m$ is the supplement of M$mm'$, or the angle T$mt$, made by the tangent T$mt$ and the consecutive one $mm't$. If we denote it by $\delta$, and suppose that the angles $\alpha, \beta, \gamma$, refer always to the direction of MT, and if $\alpha', \beta', \gamma'$, denote what they become with respect to the direction of $mt$, we shall have in virtue of equation (2) of No. 9,

$$\sin^2\delta = 1 - (\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')^2.$$

Likewise by Taylor's theorem we have

$$\cos\alpha' = \cos\alpha + d.\cos\alpha + \tfrac{1}{2}.d^2\cos\alpha + \&c.,$$

$$\cos\beta' = \cos\beta + d.\cos\beta + \tfrac{1}{2}.d^2\cos\beta + \&c.,$$

$$\cos\gamma' = \cos\gamma + d.\cos\gamma + \tfrac{1}{2}.d^2\cos\gamma + \&c.$$

Now, if we substitute these values in those of $\sin^2\delta$, and if we take into account the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

and its differential

$$\cos\alpha.d.\cos\alpha + \cos\beta.d.\cos\beta + \cos\gamma.d.\cos\gamma. = 0,$$

it is apparent that the finite quantities, and also the infinitely small quantities of the first order mutually destroy each other,

hence, therefore, if we neglect infinitely small quantities of a higher order than the second, we shall have

$$\sin^2\delta = -(\cos\alpha\, d^2.\cos\alpha + \cos\beta d^2.\cos\beta + \cos\gamma d^2.\cos\gamma);$$
(g)

Moreover, if the preceding equation be differentiated, we shall obtain

$$\cos\alpha d^2\cos\alpha + \cos\beta d^2.\cos\beta + \cos\gamma d^2.\cos\gamma$$
$$+ (d.\cos\alpha)^2 + (d.\cos\beta)^2 + (d.\cos\gamma)^2 = 0,$$

hence the preceding value of $\sin^2\delta$ will become

$$\sin^2\delta = (d.\cos\alpha)^2 + (d.\cos\beta)^2 + (d.\cos\gamma)^2,$$

and this will be also the value of $\delta^2$, because the infinitely small arc is equal to its sine.

The differentials of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, may be deduced from formulæ (1) of the preceding number. For if the independent variable be not specified, we shall have

$$d\cos\alpha = \frac{ds^2\, d^2x - dx.ds\, d^2s}{ds^3},$$

and as we have

$$ds^2 = dx^2 + dy^2 + dz^2,$$
$$ds\, d^2s = dx\, d^2x + dy\, d^2y + dz\, d^2z,$$

consequently there results (h)

$$d.\cos\alpha = \frac{dy}{ds^3}.(dyd^2x - dxd^2y) + \frac{dz}{ds^3}(dzd^2x - dxd^2z);$$

and in like manner

$$d.\cos\beta = \frac{dx}{ds^3}(dxd^2y - dyd^2x) + \frac{dz}{ds^3}(dzd^2y - dyd^2z),$$

$$d.\cos\gamma = \frac{dx}{ds^3}(dxd^2z - dzd^2x) + \frac{dy}{ds^3}(dyd^2z - dzd^2y),$$

now, if we take the sums of the squares of these three variables, after some reductions, we shall find that the expression for $\sin^2\delta$ or $\delta^2$ may be exhibited under the form (i)

$$\delta^2 = \frac{1}{ds^4}\left[(dxd^2y - dyd^2x)^2 + (dzd^2x - dxd^2z)^2 + (dyd^2z - dzd^2y)^2\right].$$

The *osculating circle* is that which has two consecutive sides common with the curve, consequently at the point M, this circle is that which passes through the three points M, $m$, $m'$, therefore the centre is at O the intersection of two perpendiculars to M$m$ and $mm'$ in the plane of these two consecutive elements raised at their points of bisection, and its radius is the right line MO. If these two elements be supposed to be equal, this line will bisect the angle M$mm'$; and we may assume this to be the case, without any apprehension that the value of MO will be altered, for it is easy to be assured that the numerical ratio of the two infinitely small sides M$m$ and $mm'$ can only affect the magnitude of this radius by an infinitely small quantity, and it is consequently the same, whether the two consecutive sides be assumed to be equal or unequal. Denoting the length of the sides M$m$ by $ds$, and that of the radius $m$O by $\rho$, the projection of $\rho$ on M$m$ will be $\frac{1}{2}ds$; so that we shall have

$$\tfrac{1}{2}ds = \rho \cdot \cos \mathrm{M}m\mathrm{O};$$

and because this angle M$m$O is half the supplement of $\delta$, or equal to $\frac{1}{2}\pi - \frac{1}{2}\delta$, there will result

$$\tfrac{1}{2}ds = \rho \sin \tfrac{1}{2}\delta = \tfrac{1}{2}\rho\delta,$$

the arc $\frac{1}{2}\delta$ being substituted in place of its sine.

This being established, if the value of the radius of curvature was known in any other manner, we would have

$$\delta = \frac{ds}{\rho},$$

for the value of the angle of contact; and conversely, from the preceding value of $\delta^2$, which has been given above, there would result that of $\rho$

$$\rho = \frac{ds^3}{\left[(dxd^2y - dyd^2x)^2 + (dzd^2x - dxd^2z)^2 + (dyd^2z - dzd^2y)^2\right]^{\frac{1}{2}}}$$

19. In order to have a complete knowledge of the nature of the curve at any point M, its *osculating plane* should be also determined, that is, the plane of the two consecutive sides M$m$ and $mm'$. If this plane passes through the point M, its equation may be represented by

$$A(x'-x)+B(y'-y)+C(z'-z)=0;$$

$x'$, $y'$, $z'$ being any coordinates whatever, and because it must pass through the points $m$ and $m'$, the first and second differentials of this equation, namely,

$$A dx' + B dy' + C dz' = 0,$$

$$A d^2x' + B d^2y' + C d^2z' = 0,$$

must be satisfied like the equation itself, by making $x'=x$, $y'=y$, $z'=z$, so that we shall have

$$A dx + B dy + C dz = 0,$$

$$A d^2x + B d^2y + C d^2z = 0.$$

The values of A, B, C, which satisfy these two conditions, are, as may be easily verified($k$),

$$C = D(dx d^2y - dy d^2x),$$

$$B = D(dz d^2x - dx d^2z),$$

$$A = D(dy d^2z - dz d^2y),$$

D being an indeterminate factor. By substituting them in the equation of the osculating plane, and then suppressing the factor which is common to all its terms, it becomes

$$(z'-z)(dx d^2y - dy d^2x) + (y'-y)(dz d^2x - dx d^2z) + (x'-x)(dy d^2z - dz d^2y) = 0.$$

Naming $\lambda, \mu, \nu$, the angles that the normal to the osculating plane makes with the parallels to the axes of $x, y, z$, drawn through the point M, we shall have by means of equations (2) of No. 17,

$$\left.\begin{aligned}\cos\lambda &= \frac{1}{h}(dyd^2z - dzd^2y),\\ \cos\mu &= \frac{1}{h}(dzd^2x - dxd^2z),\\ \cos\nu &= \frac{1}{h}(dxd^2y - dyd^2x),\end{aligned}\right\}\quad(3)$$

$h^2$ denoting the sum of the squares of the three numerators

The infinitely small angle contained between two consecutive normals, which is the angle between two consecutive osculating planes, can be determined in the same manner as the angle between two tangents has been already determined. If we denote it by $\varepsilon$, by a method similar to that employed in the preceding number, we shall have

$$\varepsilon^2 = (d\,.\cos\lambda)^2 + (d\,\cos\mu)^2 + (d\,.\cos\nu)^2.$$

20. As the centre of curvature exists at the same time on the osculating plane, and on the intersection of two consecutive normal planes, we are enabled to determine its coordinates by means of the equations of these three planes, which now we can consider as known.

The equation of the normal plane at M being (No. 17)

$$(x'-x)\,dx + (y'-y)\,dy + (z'-z)\,dz = 0,$$

that of the consecutive plane may be deduced from it, by substituting $x+dx$, $y+dy$, $z+dz$, in place of $x, y, z$, consequently, the differential of the equation of the first of these two planes taken with respect to $x, y, z$, namely,

$$(x'-x)\,d^2x + (y'-y)\,d^2y + (z'-z)\,d^2z = ds^2,$$

will be that of their intersection.

From these two equations, we infer($l$)

$$(x'-x)(dxd^2y - dyd^2x) = (z'-z)(dyd^2z - dzd^2y) - ds^2dy,$$
$$(y'-y)(dyd^2x - dxd^2y) = (z'-z)(dxd^2z - dzd^2x) - ds^2dx;$$

and by means of the equation of the osculating plane, there results

$$z'-z=\frac{\rho^2}{ds^4}\left[dy\,(dyd^2z-dzd^2y)-dx\,(dzd^2x-dxd^2z)\right],$$

$\rho$ denoting, in order to abridge, the same expression as in No. 18 We can obtain in the same manner

$$y'-y=\frac{\rho^2}{ds^4}\left[dx\,(dxd^2y-dyd^2x)-dz\,(dyd^2z-dzd^2y)\right],$$

$$x'-x'=\frac{\rho^2}{ds^4}\left[dz\,(dzd^2x-dxd^2z)-dy\,(dxd^2y-dyd^2x)\right];$$

hence the three coordinates $x', y', z'$, of O the centre of the curve may be known, and, consequently, the direction of the curvature of which the radius of the osculating circle only determines the magnitude. If the squares of these values of $x'-x$, $y'-y$, $z'-z$, be added together, we shall obtain, after all reductions($m$),

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=\rho^2,$$

hence it follows that the quantity $\rho$ is the distance of the point O from the point M, or the radius of curvature MO, as we know from other considerations.

21 The formulæ given in the five preceding numbers, contain every thing which respects the direction and curvature of any *line*, either of single or double curvature. With respect to any *surface*, we have also to consider its curvature, and the direction of the tangential plane. As to the curvature, the reader will find it fully discussed in the twenty-first volume of the Journal of the Polytechnic School, so that here we shall only concern ourselves with what respects the tangential plane and normal. At the point M, of which the coordinates are $x, y, z$, the equation of the tangential plane may be represented by

$$\text{A}\,(x'-x)+\text{B}\,(y'-y)+\text{C}\,(z'-z)=0,$$

where $x', y', z'$, are indeterminate coordinates. This plane must pass through M′ another point of the surface infinitely

near to M, consequently this equation must be satisfied by making $x' = x + dx$, $y' = y + dy$, $z' = z + dz$, or, its differential taken relatively to $x', y', z'$, must be true, when $x, y, z$, are substituted in place of these variables. Therefore we shall have

$$\mathrm{A}dx + \mathrm{B}dy + \mathrm{C}dz = 0.$$

The equation of the surface being

$$dz = pdx + qdy,$$

in which $p$ and $q$ are known functions of $x, y, z$, the preceding becomes

$$(\mathrm{A} + p\mathrm{C})\,dx + (\mathrm{B} + q\mathrm{C})\,dy = 0;$$

and as it must obtain for every direction which the line MM can assume, and consequently for every relation which can subsist between $dx$ and $dy$, the coefficients of each of these differentials must be separately equal to cypher, hence results

$$\mathrm{A} + p\mathrm{C} = 0, \quad \mathrm{B} + q\mathrm{C} = 0.$$

If the values of A and B be deduced from these equations, and if they be then substituted in the equation of the tangential plane, we shall have

$$z' - z - p\,(x' - x) - q\,(y' - y) = 0.$$

If $a, b, c$, denote the angles, which the normal at the point M, makes with the productions of the coordinates $x, y, z$, we shall have by means of the equations (2) of No. 17,

$$\left.\begin{aligned} \cos a &= -\frac{p}{\sqrt{1 + p^2 + q^2}}, \\ \cos b &= -\frac{q}{\sqrt{1 + p^2 + q^2}}, \\ \cos c &= \pm\frac{1}{\sqrt{1 + p^2 + q^2}}. \end{aligned}\right\} \qquad (4)$$

The radical will be positive or negative in these three formulæ, according as the part of the normal in question makes

an acute or obtuse angle $c$ with the line drawn through the the point M, in the positive direction of $z$. If $\omega$ represents the element of the surface, of which the projection on the plane of $x$ and $y$ is $dxdy$, we shall have

$$dxdy = \pm \omega \cos c,$$

according as $c$ is acute or obtuse, for this element, which is infinitely small in every direction, lies in the tangential plane, of which the angle $c$, or its supplement, is the inclination to the plane passing through $x$ and $y$; and the theorem of No. 10 is true also in the case of the projection of an infinitely small plane surface. Consequently from what has been just established, we have

$$\omega = dxdy\sqrt{1+p^2+q^2},$$

the sign of the radical being always considered as positive. If L be a given function of $x$, $y$, $z$, and if

$$\text{L} = 0,$$

represents the equation of the surface, which we have considered, we shall obtain, by differentiating it successively with respect to $x$ and $y$

$$\frac{d\text{L}}{dx} + p\frac{d\text{L}}{dz} = 0, \quad \frac{d\text{L}}{dy} + q\frac{d\text{L}}{dz} = 0.$$

If the values of $p$ and $q$ be determined by means of these equations, and then substituted in the equation of the tangential plane, it will assume the form

$$(x'-x)\frac{d\text{L}}{dx} + (y'-y)\frac{d\text{L}}{dy} + (z'-z)\frac{d\text{L}}{dz} = 0.$$

At the same time, formulæ (4) become

$$\cos a = \text{V}\frac{d\text{L}}{dx}, \quad \cos b = \text{V}\frac{d\text{L}}{dy}, \quad \cos c = \text{V}\frac{d\text{L}}{dz}, \tag{5}$$

in which equations, for the sake of abridging, we put

$$\left[\left(\frac{d\mathrm{L}}{dx}\right)^2+\left(\frac{d\mathrm{L}}{dy}\right)^2+\left(\frac{d\mathrm{L}}{dz}\right)^2\right]^{-\frac{1}{2}}=\mathrm{V}$$

22 We shall here make a remark that will be useful in verifying and in deducing from one another, formulæ which refer to different axes.

Suppose that in any question all things were alike with respect to the three axes of the coordinates $x, y, z$. If an equation $\mathrm{X}=0$, obtains with respect to the axis of $x$, a similar one $\mathrm{Y}=0$ will have place for the axis of $y$, and likewise a third $\mathrm{Z}=0$, for the axis of $z$, and these two last equations $\mathrm{Y}=0$, $\mathrm{Z}=0$, may be inferred from $\mathrm{X}=0$, by simple changes of the letters $x, y, z$. The following is the manner in which these permutations are effected.

In X let all quantities relative to the axis of $x$, be substituted in place of the corresponding quantities which respect the axis of $y$, then substitute these quantities instead of those which respect the axis of $z$, and finally, these last in place of the first, which respect the axis of $x$. By this mutual permutation Z is deduced from X, by a second permutation of the same nature effected on Z, we shall obtain Y, and by a third permutation effected on Y, we shall light on the first equation X For example, if these operations were instituted on equations (3) of No 19, of which the first respects the axis of $x$, the second the axis of $y$, and the third the axis of $z$, let the coordinates $x, y, z$, and the angles $\lambda, \mu, \nu$, which correspond to them respectively, be written down in the same line, but distributed into two classes, then in a second line let these six quantities be also arranged into two classes, but in a different order, so that we may have

$$\begin{array}{ll} x, y, z, & \lambda, \mu, \nu; \\ z, x, y, & \nu, \lambda, \mu. \end{array}$$

This being done, in the first equation (3), let each of the quantities of the superior line be replaced by a corresponding quantity in the inferior line, it is evident that by this permu-

tation no change is produced in $h$, and the third equation (3) will be obtained. If now in this last, the quantities of the inferior line be substituted in place of those which correspond to them in the superior line, the second equation (3) will be given; and by operating in the same manner on this equation, we shall arrive at the first equation (3), from which we set out.

Each of these operations implies a change in the axes of the coordinates, in which we first make the axes of $x$ and of $y$ revolve in their plane, in such a manner that the axis of the positive $xs$ may fall on the axis of the positive $ys$, then this last on the axis of the negative $xs$, and afterwards this axis of the positive $ys$ thus displaced, and the axis of the positive $zs$ are made to revolve in such a manner, that the first may fall on the axis of the positive $zs$, and this last on the primitive axis of the positive $xs$; so that, finally, each axis of the positive coordinates takes the place of another axis of the positive coordinates. It is on this account, that the equations relative to the three axes of the coordinates may be inferred the one from the other, by simple permutations of letters, and without any change of sign, which would not be the case unless the three coordinates, and the quantities which relate to them, were changed simultaneously, in the manner which has been pointed out.

23. As the following general observation is of frequent occurrence, it will be useful to advert to it here, at the conclusion of this introduction. The equations which will come under our observation will frequently contain abstract numbers, such as the number $\pi$, logarithms, trigonometrical lines, &c; they will also contain other quantities of different natures, which will likewise be represented by numbers expressing their ratio to different units arbitrarily selected, however each unit must be the same for all quantities of the same species. But, if the magnitude of one or more of these units be changed, the numbers which express corresponding quantities, will be changed in the inverse ratio of this magnitude; notwithstand-

ing, however, this change, which is entirely arbitrary, the equations in which they occur must still subsist For this purpose, it is necessary that their form should satisfy certain conditions, which may be easily verified in each particular case, and which, in the most extended acceptation, may be termed the conditions of the *homogeneity of quantities* Every equation that does not satisfy these, will, on this sole account, be inaccurate, and ought to be rejected.

Thus if F denotes a given function, and if

$$F\,(f, f' \ldots\ldots l, l', \ldots\ldots m, m, \ldots \quad t, t' \ldots\ldots) = 0, \quad \text{(a)}$$

in which equation $f, f'$, .... denote forces, $l, l'$, .... lines, $m, m'$, .... masses, $t, t'$, ... times, then if $n, n', n'', n'''$, .... represent abstract numbers, and if, at the same time, we diminish the unit of force in the ratio of one to $n$, the linear unit in the ratio of one to $n'$, the unit of mass in the ratio of one to $n''$, the unit of time in the ratio of one to $n'''$, the numbers $f, f'$, .... $l, l'$, .... $m, m'$, ... $t, t'$, .... will become $nf, nf'$, .... $n'l$, $n'l'$, .... $n''m, n''m$ .... $n'''t, n'''t$ ...., and the equation (a) must still obtain, that is to say, we must still have

$$F\,(nf, n'f', \ldots\ldots n'l, n'l', \ldots\ldots n''m, n''m', \ldots\ldots n'''t, n'''t \ldots\ldots) = 0,$$

whatever may the magnitudes of $n, n', n'', n'''$. If the equation (a) contains the surfaces $s, s'$, .... and the volumes $v, v'$, .... their dimensions must be referred to the same unit as the lines $l, l'$, ...., and these quantities will consequently become $n'^2 s$, $n'^2 s'$, ... $n'^3 v, n'^3 v'$, .... by the change of this unit.

The equation of No. 18, which gives the value of $\rho$, evidently satisfies this condition, for it only contains finite lines, or infinitely small quantities, $\rho, ds, dx, dy, dz, d^2x, d^2y, d^2z$; and when the linear unit is changed, and each of the lines is multiplied, in the manner pointed out, by the same number $n'$, this number disappears, and so the equation will not be changed. The equation in the same number which expresses the value of $\delta^2$, also satisfies the condition of homogeneity, since $\delta^2$ is

an abstract number which like this value does not change with the magnitude of the linear unit. It will be impossible for equation (a) to contain only one quantity of the same species, when it contains but two, for example two forces $f$ and $f'$, if it be resolved with respect to one of them, we we shall have

$$f' = \mathrm{F}(f, l, l, \ldots \; m, m', \ldots \; t, t', \ldots),$$

it is necessary in order to secure the homogeneity of these quantities, that $f$ should be a common factor to all the terms of the new function F, or in other words, we must have

$$f' = \mathrm{N}f,$$

N being a factor which does not contain any quantity of the same nature as $f$ and $f'$, nor vary with the unit of force. Sometimes the principle of the homogeneity of quantities will appear not to have place, when there is taken for the unit of force, one of the forces which are considered in the question, or for the linear unit, the distance between two of the material points to which the question refers, or for the unit of mass that of one of the bodies of the problem, &c But, then, if these units be arbitrarily changed, and if the force, the line, the mass, and time, which have been in the first place taken for units, be now expressed by $\phi$, $\lambda$, $\mu$, $\theta$, the other quantities of these different species which occur in the equation (a) will become $\frac{f}{\phi}, \frac{f'}{\phi}, \ldots \frac{l}{\lambda}, \frac{l'}{\lambda}, \ldots \frac{m}{\mu}, \frac{m'}{\mu}, \ldots \frac{t}{\theta}, \frac{t'}{\theta}, \ldots$, we must therefore have

$$\mathrm{F}\left(\frac{f}{\phi}, \frac{f}{\phi}, \ldots \frac{l}{\lambda}, \frac{l'}{\lambda}, \ldots \frac{m}{\mu}, \frac{m'}{\mu}, \ldots \frac{t}{\theta}, \frac{t'}{\theta}, \ldots\right) = 0;$$

which equation may be written

$$\mathrm{F}_1(\phi, f, f', \ldots \lambda, l, l', \ldots \mu, m, m', \ldots \theta, t, t', \ldots) = 0,$$

and this should now satisfy the condition of homogeneity · $\mathrm{F}_1$ indicates here a function which, in each case, may be deduced from the given function F

# BOOK THE FIRST.

## STATICS

## FIRST PART.

### CHAPTER I

#### OF THE COMPOSITION AND EQUILIBRIUM OF FORCES APPLIED TO THE SAME POINT.

24. When a material point is subjected to the simultaneous action of several forces which do not constitute an equilibrium, it moves in some determinate direction, and this motion with which the point is actuated, may be ascribed to an unique force acting in this direction. This force is termed the *resultant* of the forces which produce motion in the moveable, and these last are called the *components* of the first. If the resultant be applied to the point in a direction directly opposite to that in which it acts, it will constitute an equilibrium with the components, because it tends to impress on the moveable a motion equal and opposite to that which it would receive from the simultaneous action of the components, and there is no reason consequently why it should move in one direction(*a*) rather than in the other. If all the components act in the same direction and along the same right line, it follows from what has been stated in No. 5, about the measure of forces,

that the resultant is equal to their sum. If the point is solicited by two forces directly opposed to each other, and if the greater be resolved into two others, of which one is equal to the less, this last will be destroyed, and what remains, namely, the excess of the greater above the less, will be the resultant From these two propositions it follows, that if there be any number of components, of which some act in the direction of the same right line, and others in the opposite direction, the resultant will be equal to the sum of those which act in one direction, minus the sum of those which act in the opposite direction, and it will act in the direction of the greater sum. When the two sums are equal, the resultant is cypher, and the given forces constitute an equilibrium.

24. There is also another case in which the magnitude and direction of the resultant may be very easily determined.

Let MA, MB, MC, (fig. 5), be the directions of three equal forces applied to the point M, if these forces exist in the same plane, and if the three angles AMB, BMC, CMA, are respectively equal to each other, or each to 120°, the point M will remain in equilibrio, for there is no reason why it should deviate from the plane in which the three forces act, nor why it should move in one rather than in another of these three angles. Consequently, each of the three forces will be equal and opposite to the resultant of the other two. Now, if on the directions MA and MB of two of the forces, the equal lines MG and MH be taken to represent their intensities, and if the rhombus GMHK be completed, the diagonal MK will fall on MD the production of MC, the angle MGK will be 60°, as will also be each of the other angles of the same triangle, which will be equilateral; hence we shall have MK = MG, consequently MK, the diagonal of the rhombus constructed on the two forces MG and MH, represents the resultant of these two forces in magnitude and direction.

This proposition is contained in another, which we now proceed to demonstrate in the case of two equal forces, the

directions of which make any *angle whatever* with each other, and which we shall afterwards extend to the case of unequal forces.

(26) The resultant of two equal forces always divides the angle comprised between their directions into two equal parts; for there is no reason why it should be nearer to one of these two forces than to the other, nor why its direction should deviate from their plane on one side, rather than on the other; therefore, as its direction is known, it is only necessary for us to determine its magnitude.

For this purpose, let MA and MB (fig 6) be the directions of the components, and let P denote their common value. Likewise let the angle AMB be represented by $2x$, and let MD be the direction of the resultant, so that we may have AMD $=$ BMD $= x$. Its intensity depends solely on the quantities P and $x$; therefore, if it be denoted by R, we shall have

$$R = f(P, x).$$

In this equation, R and P are the only quantities of which the numerical expression varies with the unit of force; by the principle of the homogeneity of quantities (No. 23), the function $f(P, x)$ must therefore be of the form $P\phi x$. Hence we have

$$R = P\phi x;$$

and the question is reduced to the determination of the form of the function $\phi x$.

For this purpose, let the four lines MA′, MA″, MB′, MB″, be drawn arbitrarily through the point M, so that the four angles A′MA, A″MA, B′MB, B″MB, may be equal to each other, and respectively represented by $z$. Let the force P acting in the direction MA, be decomposed into two equal forces, acting in the directions MA′, MA″, that is to say, let the force P be regarded as the resultant of two equal forces of which the value is unknown, and which act in the directions MA′, MA″; if this value be denoted by Q we shall have

$$P = Q\phi z$$

for the same relation must exist between the quantities P, Q, $z$, as between the quantities R, P, $x$. Likewise the force P acting in the direction MB, may be decomposed into two forces respectively equal to Q, and acting in the directions MB′ and MB″, in this manner the two forces P will be replaced by the four forces Q; consequently, the resultant of these last ought to coincide in magnitude and direction with the force R, which is the resultant of the two forces P. Now, if Q′ denotes the resultant of the two forces Q, which act in the respective directions MA′ and MB′, this force will act in the direction MD, and since A′MD = B′MD = $x - z$, we shall have

$$Q' = Q\phi(x - z).$$

In like manner, the resultant of the two other forces Q will act in the direction MD, since this line also divides the angle A″MB″ into two equal parts, and because A″MD = B″MD = $x + z$, we shall have

$$Q'' = Q\phi(x + z);$$

Q″ denoting this second resultant. As the two forces Q′ and Q″ act in the direction of the same right line MD, their resultant, which is likewise that of the four forces Q, must be equal to their sum, consequently we must have

$$R = Q' + Q''.$$

But we have already

$$R = P\phi x = Q\phi z\phi x;$$

and by substituting this value of R and those of Q′ and Q″ in the preceding equation, there results, by suppressing the factor Q, which is common to all the terms,

$$\phi x\phi z = \phi(x + z) + \phi(x - z). \qquad (1)$$

This is the equation which must be resolved in order to obtain the expression of $\phi x$

27. It is evident that it may be satisfied by assuming

$$\phi x = 2\cos ax;$$

$a$ being a constant arbitrary, so that we may have at the same time,

$$\phi z = 2 \cos az,$$
$$\phi(x + z) = 2 \cos a(x + z),$$
$$\phi(x - z) = 2 \cos a(x - z);$$

and, in fact, if these values be substituted in equation (1), there results the known equation,

$$2 \cos ax \cos az = \cos a(x + z) + \cos a(x - z).$$

Now it is to be remarked, that this expression of the function $\phi x$ is the only one which satisfies equation (1), and moreover, that in the present question, the constant quantity $a$ is unity; so that we have †

$$\phi x = 2 \cos x. \tag{2}$$

This is evident when $x = 0$, for then the directions of the two forces coincide, and the resultant R is equal to 2 P, which implies that $\phi x = 2$. 1°

If we admit that there is another value such as $a$ of $x$, for which we have also $\phi a = 2 \cos a$, then equation (2) will likewise subsist for all values $2a, 3a, 4a, \ldots \frac{1}{2}a, \frac{1}{4}a, \frac{1}{8}a, \ldots$ of $x$, and generally for 2°

$$x = \frac{ma}{2^n}, \tag{3}$$

$m$ and $n$ being any whole numbers whatever. In fact, if equation (2) is true for the three angles $x, z, x - z$, so that we have

$$\phi x = 2 \cos x, \quad \phi z = 2 \cos z, \quad \phi(x - z) = 2 \cos(x - z),$$

it will be also true for a fourth angle $x + z$, for, in virtue of equation (1), we shall then have

$$\phi(x + z) = 4 \cos x \cos z - 2 \cos(x - z),$$

which equation may be reduced to

$$\phi(x + z) = 2 \cos(x + z).$$

Thus, as equation (2) obtains for $x = 0$ and $x = a$, it follows that it subsists for $x = 2a$, and as it obtains for $x = a$ and $x = 2a$, it will also subsist for $x = 3a$, by continuing in this manner, it appears that it will obtain generally for $x = ma$.

Now if we make $ma = \beta$, we shall have

$$\phi\beta = 2\cos\beta,$$

from which it follows that equation (2) will likewise obtain for $x = \frac{1}{2}\beta$, for assuming $x = z = \frac{1}{2}\beta$, equation (1) will become

$$(\phi\tfrac{1}{2}\beta)^2 = 2\cos\beta + 2,$$

consequently we shall have ($b$)

$$\phi\tfrac{1}{2}\beta = 2\cos\tfrac{1}{2}\beta.$$

If in the next place, we make $x = z = \frac{1}{4}\beta$, we shall have by equation (1) and this last,

$$(\phi\tfrac{1}{4}\beta)^2 = 2\cos\tfrac{1}{2}\beta + 2, \quad \phi\tfrac{1}{4}\beta = 2\cos\tfrac{1}{4}\beta,$$

and by continuing in this manner, equation (2) will be demonstrated for $x = \dfrac{\beta}{2^n}$, that is to say, for all values of $x$ comprised in formula (3).

Now, as the numbers $m$ and $n$ may be rendered as great as we please, and may even become infinite, these values of $x$ may be made to increase by infinitely small quantities. Therefore formula (3) embraces all possible values of the angle $x$, and equation (2) is completely demonstrated, provided it is true for a particular value $x = a$, different from zero. Now by the theorem of No. 25, the resultant R is equal to P, in the case of $x = 60^\circ$, therefore in this case we have

$$\phi x = 1 = 2\cos 60^\circ,$$

hence equation (2) obtains for $x = 60^\circ$, and consequently for all values of $x$.

28. By means of this equation we shall have

$$R = 2P\cos x.$$

Hence if the resultant R and the two components P are represented, as in No. 25, by lines taken in their respective directions, reckoning from their point of application, the force R will be equal to twice the projection of P on its direction, or equal to the diagonal of the rhombus constructed on the two forces P.

Now let two unequal forces P and Q be applied to the point M (fig. 7) in the directions MA and MB, also let their intensities be represented by the lines MG and MH, taken in their respective directions, and let the parallellogram MGHK be completed, there are two cases to be considered, first, when the angle AMB is a right angle, and, secondly, when it is acute or obtuse. In the first case, let the two diagonals MK and GH be drawn intersecting in the point L, and through the points G and H, let GN and HO be drawn parallel to ML, meeting in N and O, the line drawn through the point M parallel to GH. MK and GH bisect each other at the point L, and since in a rectangle the two diagonals are equal, we have

$$GL = LH = LM.$$

Hence each of the parallellograms GLMN, HLMO is a rhombus, consequently it follows from the preceding proposition that the force MG may be considered as the resultant of the two forces MN and ML, and the force MH, as the resultant of MO and ML. Therefore, if instead of the two given forces there be substituted their components, we shall have for MH and MG the two forces MN and MO (which since they are equal and opposite, they mutually destroy each other) and the two forces ML, which added together give a resultant represented in magnitude and direction by the diagonal MK. In the second case, let GE and HF be drawn through the points G and H, perpendicular to the diagonal MK (fig. 8,) and the lines GN and HO parallel to this diagonal, likewise let NMO be drawn through the point M perpendicular to the same line. The two parallellograms GEMN and HFMO will be rectangles, and their

sides MN and MO will be equal, as being the altitudes of the two equal triangles GMK and HMK By the first case, we can replace the forces MG and MH by their rectangular components ME and MN, MF and MO, therefore, in place of the two given forces we shall have the two forces MN and MO, which destroy each other, as being equal and opposite, and the two forces ME and MF acting in the same direction, which being added together will give (because ME = FK) a resultant expressed in magnitude and direction by the diagonal MK Hence we may conclude generally, that the resultant of any two forces whatever, applied to the same point and represented by lines taken on their directions, reckoning from this point, is represented in magnitude and direction by the diagonal of the parallellogram constructed on the two given forces.

29. The following consequences may be immediately deduced from this theorem.

In the first place, it appears that all questions which can be proposed relatively to the composition of two forces into one or relatively to the decomposition of one force into two others, are reduced to the resolution of a triangle. In fact, the magnitudes of the resultant and of the two components are represented by the three sides MK, MG, GK, of the triangle MGK, and the three angles of this triangle are those which the resultant makes with each of the components and the supplement of the angle comprised between the components. It follows therefore, that any three of these six quantities, namely, the three forces and the three angles comprised between their directions, being given, the three remaining quantities may be found by the resolution of the triangle MGK; this, however, supposes that in the number of data, there is one force at least. For example, let P and Q be the values of the two components, and $m$ the angle contained between their directions, it is required to determine their resultant R and $x$ the angle which it makes with the force P. In the first place, we have the equation

All properties true of the sides of Δ are true of three forces in equilibrium acting on a [illegible]

$$R^2 = P^2 + Q^2 + 2PQ \cos m,$$

by means of which the value of R can be determined, and that of $x$ can be deduced from the proportion

$$\sin x : \sin m :: Q : R.$$

If the three forces P, Q, S, applied to the same point M, (fig. 9,) in the directions MA, MB, MC, are in equilibrio, each of these forces must be equal and directly opposite to the resultant of the two others, and as this resultant exists in the plane of these two forces, it follows that these three forces must also exist in the same plane Let MD be the production of MC, the resultant of P and Q will act in the direction of MD, and if it be represented by R, we shall have R = S. Moreover, if the force R be compared with each of its components, we have, agreeably to what is stated above,

$$R : Q :: \sin AMB : \sin AMD,$$

$$R : P :: \sin AMB : \sin BMD;$$

and since

$$\sin AMD = \sin AMC, \sin BMD = \sin BMC,$$

there results

$$S : Q : P :: \sin AMB : \sin AMC : \sin BMC;$$

this shows that when three forces are in equilibrio about the same point, the magnitude of each of them may be represented by the sine of the angle comprised between the directions of the other two.

If from O, a point assumed on the direction of the resultant R, or on its production, the perpendiculars OE and OF be let fall on the directions of the components P and Q; we shall have

$$OE = MO \sin AMD, \ OF = MO \sin BMD.$$

If, therefore, the two last terms of the proportion

$$P : Q :: \sin BMD : \sin AMD,$$

be multiplied by MO, there will result

$$P : Q :: OF : OE;$$

from which it appears that the components are in the inverse ratio of perpendiculars let fall on their directions, from any point in the direction of the resultant. Conversely, if the components P and Q are in the inverse ratio of OE and OF, perpendiculars let fall on their directions, from any point O taken in their plane, this point will exist on the direction of the resultant, for if the two last terms of the last proportion be divided by MO, the preceding will be obtained, which determines this direction.

30. The resultant of two forces being known, it is easy to deduce that of any number of forces applied to the same point and situated, or not, in the same plane. First, the resultant of two of these forces is taken, then this resultant is compounded with a third force, this will give a second resultant, which is compounded in the same manner with a fourth force, and so on, until all the given forces are exhausted. In this construction it is easy to perceive, that if the magnitudes of all the forces are represented by the sides of a portion of a polygon parallel to their respective directions, and traced in the direction of their actions, the resultant will be represented in magnitude and direction by the line which connects its two extreme points, and consequently closes the polygon. It is indifferent in what order the sides parallel to the forces succeed each other. When the polygon is closed of itself, the resultant vanishes, and the given forces constitute an equilibrium.

Hence it follows, that when there are but three forces which do not exist in the same plane, their resultant is represented in magnitude and direction by the diagonal of a parallelopiped, of which the three forces constitute the adjacent sides.

31. This reduction of any number of forces to one, may be effected in a simpler manner, by considering, first, the par-

ticular case of three rectangular forces, to which the general case may then be reduced.

Let X, Y, Z be the three components, R their resultant, $a$, $b$, $c$ the angles which it makes with X, Y, Z. From what has been already observed, R is evidently the diagonal of a parallelopiped of which X, Y, Z are the three adjacent sides, now as this parallelopiped is rectangular, it follows that

$$R^2 = X^2 + Y^2 + Z^2. \tag{a}$$

It likewise follows that if the extremity of the diagonal R be joined to those of the three sides X, Y, Z, three right angled triangles will be formed, of which R will be the common hypothenuse, hence we have

$$X = R \cos a, \quad Y = R \cos b, \quad Z = R \cos c; \tag{b}$$

which equations agree with the preceding, for the three angles $a$, $b$, $c$ are connected by the equation (No. 6.)

$$\cos^2 a + \cos^2 b + \cos^2 c = 1.$$

When the components X, Y, Z are given, equation (a) determines the value of the resultant, and equations (b) determine the direction, by means of the three angles $a$, $b$, $c$; if, on the other hand, the force R is given, and it is required to decompose it, into three rectangular forces X, Y, Z, which make with it the given angles $a$, $b$, $c$, the values of the required forces will be immediately determined by means of equation (b). If one of the components, the force Z for example, vanishes, R is then only the resultant of two forces X and Y; it exists in their plane, and its direction depends solely on the two angles $a$ and $b$. These angles and the value of R are then determined by the equations

$$R^2 = X^2 + Y^2, \quad X = R \cos a, \quad Y = R \cos b.$$

32. Let us now suppose that M (fig. 1) is the point of application of any number of given forces. Let these forces be represented by P, P′, P″, &c., and for greater clearness, let the line MD be the direction of the force P. It is unnecessary

to indicate the directions of the other forces in the figure. Let $\alpha$, $\beta$, $\gamma$ be the angles which the direction MD makes with the three rectangular axes MA, MB, MC, drawn arbitrarily through the point M. Likewise let $\alpha'$, $\beta'$, $\gamma'$ be the angles which the force P′ makes with the same axes, $\alpha''$, $\beta''$, $\gamma''$, those which correspond to the force P″, &c. All these angles are given, and are supposed to include every angle from zero to 180° (No. 7), in order that the forces P, P′, P″, &c., may have all possible positions about the point M. If each of these forces be resolved into three others in the direction of the axes MA, MB, MC, the components of the force P will be $P \cos \alpha$, $P \cos \beta$, $P \cos \gamma$; those of the force P′ will be $P' \cos \alpha'$, $P' \cos \beta'$, $P' \cos \gamma'$, &c., and these components will act in the direction of the axes or of their productions, according as they are positive or negative. For example, as the direction MD falls like the axis MC above AMB, the plane of the two other axes, the component $P \cos \gamma$ of the force P tends to elevate the point M, that is to say, it acts in the direction MC, and in this case $P \cos \gamma$ is a positive quantity, because $\gamma$ is less than 90°. On the contrary, if this direction MD falls below the plane AMB, we would have $\gamma > 90°$; and the component $P \cos \gamma$ will be negative, and, at the same time, it will tend to depress the point M, that is to say, it will act along the production of MC. Therefore, taking into account the signs of the components, it appears from what has been stated in No. 24, that all forces acting in the direction of the same axis and its production, are reduced to one sole force, equal to their difference.

In this manner the given forces P, P′, P″, &c, may be replaced by three rectangular forces, and if these last be denoted by X, Y, Z we shall have

$$\left.\begin{aligned} X &= P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' +, \&c. \\ Y &= P\cos\beta + P'\cos\beta' + P''\cos\beta'' +, \&c. \\ Z &= P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' +, \&c. \end{aligned}\right\} \quad (c)$$

The values of X, Y, Z may be either positive or negative,

and their signs make known the direction of their action. If the force X is positive, it follows that it acts in the direction of the axis MA, or in the direction of the components $P \cos a$, $P' \cos a'$, &c., which are positive, if it is negative, we must infer that it acts along the production of MA or in the direction of the negative components, and the same is true for the forces Y and Z.

This being agreed on, let R be the resultant of the given forces P, P', P'', &c., or of the three forces X, Y, Z, let also $a, b, c$ be the angles which its unknown direction makes with the axes MA, MB, MC. The values of R, $a, b, c$ will be given by equations (a) and (b), in which the formulæ (c) are substituted in place of X, Y, Z. The angles $a, b, c$ may be acute or obtuse, but because the force R must be always a positive quantity, the signs of their cosines must be always the same as those of the quantities X, Y, Z, in virtue of equations (b). In this manner, the force R will be completely determined in magnitude and direction.

33. The magnitude of the resultant R does not depend on the arbitrary direction of the axes MA, MB, MC, it depends solely on the magnitude of the given forces and on the angles comprised between their directions, and in fact, we may find an expression for it, which contains these quantities only.

For this purpose, let PMP', PMP'', P'MP''', &c, denote the angles contained between the directions of the forces P and P', P and P'', P' and P''', &c. By equation (2) of No 9, we shall have

$$\cos PMP' = \cos a \cos a' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma',$$
$$\cos PMP'' = \cos a \cos a'' + \cos \beta \cos \beta'' + \cos \gamma \cos \gamma'',$$
$$\cos P'MP'' = \cos a' \cos a'' + \cos \beta' \cos \beta'' + \cos \gamma' \cos \gamma'',$$
&c.

We shall likewise have

$$\cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1,$$
$$\cos^2 a' + \cos^2 \beta' + \cos^2 \gamma' = 1,$$
$$\cos^2 a'' + \cos^2 \beta'' + \cos^2 \gamma'' = 1,$$
&c.

and, this being the case, if the squares of formulæ (c) be added, we shall obtain, by taking into account equation (a),

$$\begin{aligned} R^2 = {} & P^2 + P'^2 + P''^2 +, \&c \\ & + 2PP' \cos PMP' + 2PP'' \cos PMP'' \\ & + 2P'P'' \cos P'MP'' +, \&c. \end{aligned}$$

for the value of the square of R which was required to be determined

34. We may also deduce from a consideration of equations (b) and (c), a property of the resultant which will be useful in one of the following numbers.

Let there be drawn through the point M, in any direction whatever, a right line, of which let the point O be the other extremity, and let $g, h, k$ denote the angles AMO, BMO, CMO, which this line makes with the three axes MA, MB, MC, if the angles comprised between this same right line MO and the directions of the forces R, P, P, &c , be denoted by RMO, PMO, P′MO, &c , we shall have, as has been just stated, by equation (2) of No. 9,

$$\cos RMO = \cos g \cos a + \cos h \cos b + \cos k \cos c,$$

$$\cos PMO = \cos g \cos \alpha + \cos h \cos \beta + \cos k \cos \gamma,$$

$$\cos P'MO = \cos g \cos \alpha' + \cos h \cos \beta' + \cos k \cos \gamma', \ \&c.$$

From the first of these formulæ and equations (b), we obtain

$$R \cos RMO = X \cos g + Y \cos h + Z \cos k,$$

and in virtue of the subsequent formulæ, if, after having multiplied the first of equations (c) by $\cos g$, the second by $\cos h$, the third by $\cos k$, we add them together, there will result (e)

$$R . \cos RMO = P . \cos PMO + P' \cos P'MO + \&c.;$$

which shows that the resultant R, resolved in any direction MO, is equal to the sum of the components P, P′, P″, &c., resolved in the same direction.

This being established, let the line MO be projected on the directions of the forces R, P, P′, P″, &c , and let these projections be denoted respectively by $r, p, p', p''$, &c., so that

$$r = \text{MO} \cos \text{RMO},$$

$$p = \text{MO} \cos \text{PMO}, \quad p' = \text{MO} \cos \text{P}'\text{MO}, \&\text{c.},$$

each of the quantities $r, p, p', p''$, &c , being considered as positive or negative, according as the projection which it represents, falls on the direction of the force, or on its production. Then, if the preceding equation be multiplied by MO, we shall have

$$\text{R}r = \text{P}p + \text{P}p + \text{P}''p'' + \&\text{c.} \qquad \text{(d)}$$

which expresses the property of the resultant that was proposed to be demonstrated.

35 In order that the forces P, P′, P″, &c., may be in equilibrio, it is *sufficient* that the resultant R should vanish, and this condition is *necessary*, if their point of application M is entirely free , but the equation $\text{R} = 0$, or

$$\text{X}^2 + \text{Y}^2 + \text{Z}^2 = 0,$$

cannot have place, unless we have separately

$$\text{X} = 0, \quad \text{Y} = 0, \quad \text{Z} = 0,$$

that is to say, in virtue of equation (c),

$$\left.\begin{array}{l} \text{P} \cos \alpha + \text{P}'.\cos \alpha' + \text{P}'' \cos \alpha'' + \&\text{c.} = 0, \\ \text{P} \cos \beta + \text{P}' \cos \beta' + \text{P}'' \cos \beta'' + \&\text{c.} = 0, \\ \text{P} \cos \gamma + \text{P}' \cos \gamma' + \text{P}'' \cos \gamma'' + \&\text{c.} = 0. \end{array}\right\} \qquad \text{(e)}$$

Therefore these are the *equations of equilibrium* of a material point, which is supposed to be entirely free. In this state, each of the forces that solicit it, must be equal and directly contrary to the resultant of all the others, which may be easily verified in the following manner.

Let R′ be the resultant of the forces P′, P″, &c. Denoting

the angles which it makes with the axes MA, MB, MC, by $a'$, $b'$, $c'$, &c., and in order to abridge, making

$$X' = P' \cos \alpha' + P'' \cos \alpha'' + \&c.,$$
$$Y' = P' \cos \beta' + P'' \cos \beta'' + \&c,$$
$$Z' = P' \cos \gamma' + P'' \cos \gamma'' + \&c.,$$

we shall have, by No. 32,

$$X' = R' \cos a', \quad Y' = R' \cos b', \quad Z' = R' \cos c',$$

and consequently, in virtue of the equations of equilibrium,

$$P \cos \alpha = - R'. \cos a',$$
$$P \cos \beta = - R'. \cos b',$$
$$P \cos \gamma = - R'. \cos c'.$$

If the squares of each of the members of these equations be taken, and if they be then added together, we shall have

$$P^2 = R^2,$$

because by (No. 6)

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$
$$\cos^2 a' + \cos^2 b' + \cos^2 c'^2 = 1,$$

therefore we shall have $P = \pm R'$, but as these forces must be respectively positive, we must take $P = R'$. The preceding equations will then become

$$\cos \alpha = - \cos a', \quad \cos \beta = - \cos b', \quad \cos \gamma = - \cos c',$$

consequently, the angles $\alpha, \beta, \gamma$, are the supplements of $a', b', c'$, and belong to a force, of which the direction is the production of the force $R'$ (No. 7). It follows, therefore, that the force P is equal and directly opposed to $R'$ the resultant of all the other forces $P'$, $P''$, &c.; which it was proposed to establish.

36. If the point M, to which the forces $P, P', P''$, &c, are applied, be subjected to exist on a given surface, it is no longer necessary, in order to the equilibrium, that their re-

sultant should vanish, it is sufficient if it be normal to the surface, because then the point M cannot slide in any direction on this surface, and moreover, this condition will be necessary; for if it was not satisfied, the resultant might be decomposed into two forces, the one normal to the surface, which will be destroyed, the other a tangent to the surface which there is nothing to prevent from causing the moveable to slide. It is only therefore necessary to determine in each case, the direction of the resultant of the forces P, P′, P″, &c., and then examine if it is perpendicular to the given surface, in order to know whether the equilibrium exists; but it is more convenient in practice to express, as has been done in the case of a free point, the conditions of the equilibrium by means of the equations which exist between the data of the question. Now the normal component of each of the forces which acts on the point M, is destroyed by the resistance of the surface; consequently, this resistance is equivalent to a force equal and contrary to all the forces destroyed. Hence it appears, that we may abstract from the consideration of the given surface, and regard the material point as entirely free, provided that to the given forces P, P′, P″, &c., a new force of an unknown magnitude and perpendicular to this surface be added.

Therefore, if N represents this force, and if $\lambda, \mu, \nu$, denote the angles which its direction makes with the axes MA, MB, MC, each of the equations of equilibrium already given, will be increased by a new term, so that instead of equations (e), we shall have

$$\left.\begin{aligned} N.\cos\lambda + P.\cos\alpha + P'.\cos\alpha' + P''.\cos\alpha'' + \&c. = 0, \\ N.\cos\mu + P.\cos\beta + P'.\cos\beta' + P''.\cos\beta'' + \&c. = 0, \\ N.\cos\nu + P.\cos\gamma + P'.\cos\gamma' + P''.\cos\gamma'' + \&c. = 0, \end{aligned}\right\} \quad (f)$$

Let $x, y, z$, denote the three coordinates of M referred to the axes parallel to MA, MB, MC, and let $L = o$, denote the equation of the given surface; the direction of the force being,

by hypothesis, that of the normal at the point M, we shall have, by means of equations (5) of No. 21,

$$\cos\lambda = \text{V}.\frac{d\text{L}}{dx}, \quad \cos\mu = \text{V}.\frac{d\text{L}}{dy}, \quad \cos\nu = \text{V}.\frac{d\text{L}}{dz},$$

in which, in order to abridge, we suppose

$$\text{V} = \pm\left[\left(\frac{d\text{L}}{dx}\right)^2 + \left(\frac{d\text{L}}{dy}\right)^2 + \left(\frac{d\text{L}}{dz}\right)^2\right]^{-\frac{1}{2}}.$$

The sign is undetermined, because we do not know beforehand in what direction of the normal, the force V acts; but V disappears in the elimination of N between equations (f), and by means of the formulæ (c), we obtain(*c*)

$$\text{Y}.\frac{d\text{L}}{dx} - \text{X}\frac{d\text{L}}{dy} = 0, \quad \text{Z}.\frac{d\text{L}}{dx} - \text{X}\frac{d\text{L}}{dz} = 0, \qquad \text{(g)}$$

for the two equations which are necessary and sufficient for the equilibrium of a material point subjected to exist on a given surface.

37. If the position of this point on this surface, is not known, equations (g) combined with the given equation $\text{L} = o$, will enable us to determine the coordinates of the different points of this surface, in which the material point may be in equilibrio. When this position is given, we shall only have to verify whether $x$, $y$, $z$, the coordinates of the points of application of the given forces, satisfy equations (g). But, in this case, these equations may be simplified by making one of the axes MA, MB, MC, as for instance the first, to coincide with one of the two parts of the normal, hence there will result

$$\cos\lambda = \pm 1, \quad \cos\mu = 0, \quad \cos\gamma = 0;$$

and this changes equations (f) into the following

$$\pm \text{N} + \text{P}.\cos\alpha + \text{P}'\cos\alpha' + \text{P}''.\cos\alpha'' + \&c. = 0,$$
$$\text{P}.\cos\beta + \text{P}'\cos\beta' + \text{P}''\cos\beta'' + \&c. = 0,$$
$$\text{P}.\cos\gamma + \text{P}.\cos\gamma' + \text{P}''\cos\gamma'' + \&c. = 0.$$

From the two last of these equations it appears, as is indeed evident of itself, that in the plane which touches the given surface at the point where the normal meets it, the components of the forces applied to the material point, constitute an equilibrium, in the same manner as if this surface did not exist at all.

The resistance N which the surface opposes to the forces P, P′, P″, &c., is equal and contrary to the *pressure* that it experiences from them. From equations (f) it appears, that in the case of equilibrium, this pressure is the resultant itself of these forces In practice, the magnitude of this force ought to be calculated by means of equation (a), in order to know whether the surface is capable of supporting it. If the moveable is merely laid on this surface, which is that of a solid body, it is moreover necessary that the direction of this pressure should be such as to press the moveable against this surface; which condition, as it cannot be expressed by any equation, must be verified in each case by determining the direction of this force, by means of equations (b). This verification can be effected in a simpler manner, by means of the first of the three preceding equations.

In fact, let us suppose for greater clearness, that the part of the normal with which the axis MA coincides, is situated in the concave part of the surface. As it is known whether the given angles $a, a', a''$, &c., are acute or obtuse; the sign of X, the sum of the components acting in the direction of this line will be known, and as the quantity N should be positive, it is necessary that in the equation in question, that is to say,

$$\pm N + X = 0,$$

we should take the sign $-$ or the sign $+$ before N, according as the sum X is positive or negative. In the first case, we shall have $\cos\lambda = -1$, and the pressure contrary to N will act in the direction MA; in the second case, we shall have

$\cos\lambda = 1$, and the pressure will act along the production of this determinate part of the normal.

38. When the material point M, on which the forces P, P′, P″, &c., act, is constrained to exist on two given surfaces, or on their curve of intersection, it is sufficient, in order to insure the equilibrium, that the resultant of all these forces should be decomposable into two forces perpendicular to the given surfaces, and which will be destroyed by their resistances. Consequently, if there be added to the forces P, P′, P″, &c., two forces perpendicular to these surfaces, but of an unknown magnitude, we may then altogether abstract from the consideration of these surfaces, and consider the material point as entirely free.

N, N′, being therefore these new forces; $\lambda$, $\mu$, $\nu$, the angles which determine the direction of N with respect to the axes MA, MB, MC, and $\lambda'$, $\mu'$, $\nu'$, those which determine the direction of N′ with respect to the same axes, equations (c) will become

$$\left.\begin{array}{l} N.\cos\lambda + N'.\cos\lambda' + P.\cos\alpha + P'.\cos\alpha' + P''.\cos\alpha'' + \&c. = 0, \\ N.\cos\mu + N'.\cos\mu' + P\cos\beta + P'.\cos\beta' + P''.\cos\beta'' + \&c. = 0, \\ N.\cos\nu + N'.\cos\nu' + P.\cos\gamma + P'.\cos\gamma' + P''.\cos\gamma'' + \&c. = 0, \end{array}\right\} \text{(h)}$$

Moreover, if $x$, $y$, $z$, denote the coordinates of the point M referred to axes parallel to MA, MB, MC, and if $L = o$, $L' = o$, represent the equations of the two given surfaces, the values of $\cos\lambda$, $\cos\mu$, $\cos\nu$, will be the same as in the last case, and those of $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$, will be obtained from them by changing L into L′. If these values be substituted in the three equations (h), and if N and N′ be then eliminated between them, the equation of equilibrium which the given forces P, P′, P″, &c., ought to satisfy, will be obtained; also, if the position of the moveable on the intersection of the two surfaces is not given, this equation of equilibrium combined with equations $L = o$, $L' = o$, will determine its three coordinates $x$, $y$, $z$.

When the position of the moveable is given on the curve on which it should remain, the equation of equilibrium of the forces $P, P', P''$, &c., may be obtained *at once*, by taking the axes MB and MC, to which the angles $\mu, \beta, \beta'$, &c $\nu, \gamma, \gamma'$, &c, refer, in the plane of the normals to the two given surfaces. The third axis MA will then fall on the tangent to their curve of intersection, it is, therefore, perpendicular to the normal forces N and N', so that we have $\lambda = 90^\circ$, $\lambda' = 90^\circ$, and likewise, in virtue of the first equation (h),

$$P \cos\alpha + P'.\cos\alpha' + P''.\cos\alpha'' + \&c. = 0,$$

for the required equation

This equation indicates, that the sum of the components of $P, P', P''$, &c., acting tangentially to the intersection of the given surfaces, is equal to cypher, this is in fact the condition which must be satisfied, in order that the point M shall not slide on this curve. After having assured ourselves that it is satisfied, the values of the forces N and N', and the direction in which they act, can be determined by means of the two last equations (h). If then two forces equal and contrary to N and N' be taken, and if these are reduced to one by the rule for the composition of forces, this will be the resultant of the forces $P, P', P''$, &c, and will make known the pressure exerted on the given curve, to which it will be perpendicular.

(39) It appears from what precedes, that when the moveable is constrained to exist on a given *curve*, there is only one equation of equilibrium, that there are two when it is restricted to move on a given *surface*, and three when the point is altogether free, so that the number of these equations increases, as they evidently ought, according as the possible movements of the moveable are less restricted. These different equations may be all embraced in one formula, which will consequently be the general equation of equilibrium applicable to any system whatever of material points. In order to obtain this formula, let us suppose that the moveable is transferred

from the point M, which it occupied in its position of equilibrium, to O another point infinitely near to M, and such that this displacement may be compatible with the condition to which the moveable is subjected, if it is not altogether free. Let $r, p, p', p''$, &c., denote the projections of the infinitely small line MO on the directions of the forces R, P, P′, P″, &c., in the first position of the material point, and let each of these projections be considered as positive or negative, according as it falls on the direction itself of the force to which it refers, or on its production. If the force R be supposed to be the resultant of P, P′, P″, &c., the product R$r$ will *always* vanish in the case of equilibrium; it will vanish for a material point entirely free, because then the resultant R should be equal to cypher, it will also vanish, for a point subjected to exist on a given surface or curve, because the force R must act in the direction of the normal, hence as the infinitely small line MO exists in the tangent plane, or on the tangent, $r$ its projection on the direction of R will be equal to cypher. Therefore by equation (d), which has been already demonstrated, and which obtains equally when the line MO is infinitely small, we have

$$Pp + P'p' + P''p'' + \&c. = 0, \qquad (i)$$

as often as the forces P, P′, P″, &c., constitute an equilibrium. Conversely, the equilibrium will subsist, when this equation obtains for all possible displacements of a material point entirely free, or constrained to exist on a given surface or curve. Every infinitely small line, such as MO, which the material point can be made to describe, consistently with the conditions to which it is subjected, is called its *virtual velocity*, and the principle of equilibrium contained in the equation which has been just expressed, is termed the *principle of virtual velocities*. Now, if it be successively applied to the case of a material point entirely free, to one subjected to exist on a surface, or constrained to remain on a given curve, we can, without any difficulty, arrive at the equations of equilibrium

which have been already obtained. Each of the equations (e) may be deduced from formula (i), by taking for MO, the displacement of M on one of the axes MA, MB, MC, the equations of equilibrium, which have place in the case of a point subjected to exist on a given surface, will be obtained by considering its displacements in the direction of two axes traced in the tangential plane, and formula (i) furnishes immediately the equation of equilibrium of a point constrained to exist on a given curve, by assuming for MO the element of this curve, and for $p, p', p''$, &c., the projections of this element on the directions of the forces P, P′, P″, &c. The angles which these directions make with the tangent to the curve being $a$, $a'$, $a''$, &c., we shall then have

$$p = \text{MO} \cos a, \quad p' = \text{MO} \cos a', \quad p'' = \text{MO} \cos a'', \text{ \&c.},$$

and if MO, which is a common factor to all the terms of equation (i), be suppressed, there will result

$$\text{P} \cos a + \text{P}' \cos a' + \text{P}'' \cos a'' + \text{\&c.} = 0,$$

as before

## CHAPTER II

### OF THE EQUILIBRIUM OF THE LEVER.

40. IN this chapter the *lever* is supposed to be an inextensible right line or curve, of an invariable form, which can only turn in a plane about C, (fig 10,) one of its points, that is supposed to be fixed, and which is termed the *prop* of the lever. Most commonly there are only two forces applied to this *machine*, of which one is employed in keeping the other in equilibrio; the first is termed the *power*, and the second the *resistance*. But, in order that our conclusions may be as general as possible, we shall suppose that any number of forces whatever directed in the plane of the lever, act on different points of this line; and the object of this chapter is to find the conditions of their equilibrium

It is not proposed, in the present treatise, to apply the laws of equilibrium, which will be explained here, to the different mechanic powers or machines. The student is referred to the various elementary treatises on statics for information on this subject, but as the law of the equilibrium of the lever is a fundamental principle of mechanics, it will be necessary for us to dwell on it here; and we now proceed to show how this principle is connected with that of the composition of forces which act on a detached point

41. When several forces are applied to a body which is supposed to be of an invariable form, the point of application of each of these forces may be transferred to another point of the body, assumed either on its direction or on its production. For example, if a given force P acts at E, the extremity of the lever, in the direction of the line AE, and if M be another point

belonging to this direction, which is supposed to be connected with the lever in an invariable manner, then we may replace the force P by another force of the same intensity, acting at the point M in the direction of the line MA. In fact, we can, in the first place, apply to the point M two equal forces, and acting in opposite directions, the one along MA, the other along its production MA′; moreover, if each of these forces be supposed equal to P, that which acts in the direction MA′ will destroy the force P applied to the point E in the direction EA, because these two equal forces act in opposite directions at the extremities of the line ME, which by hypothesis is of an invariable length, therefore, the only force which remains is P acting at the point M in the direction MA, by which, therefore, the given force P, which acts at the point E, is replaced.

Forces act frequently on those bodies, in which they excite, or have a tendency to excite, motion, either by pulling them by means of a string which is attached to them, or by pushing them by means of a bar that presses against their surface. As this string or this bar may be more or less extended or contracted, it is only when they cease to extend or contract that they are considered as invariable lines, which represent the direction of each force, the action of which is then the same as if it was exerted immediately at the points of the surface of the body at which these lines terminate Strictly speaking, a lever is not, as has been supposed above, a line of an invariable form, it is a bar which is susceptible of flexure, however small it may be, and which also contracts or extends by a small quantity in consequence of the forces which are applied to it. It would be extremely difficult to determine before hand, the form which it should assume, but it is only when it has attained this form, that it is considered as invariable, and it is to this figure, which differs very little from its natural form, that the conditions of equilibrium which we propose to investigate, are referred.

42 Let a second force Q act at F, the other extremity of

the lever, in the direction FB, and let the two directions EA and FB be comprised in the plane in which the lever can turn; these two right lines, or their productions, must meet in a certain point M, which, from what has been already established, may be assumed as the common point of application of P and Q. This being agreed on, we can determine by the rule of the parallellogram of forces, the resultant of these two forces, of which resultant M will also be the point of application. Now, in order that it may be destroyed, and that the lever may remain in equilibrio, it will be necessary that its direction should pass through the prop C, and this will be sufficient, because by transferring this resultant to this point, which is fixed, it will be destroyed by its resistance. From what has been stated in No. 29, it appears that if from the point C, the perpendiculars CG and CH be let fall, on the directions of the forces P and Q, we shall have in the case of an equilibrium

$$P , Q .: CH : CG ,$$

and conversely, the equilibrium will subsist, when this proportion obtains. Consequently, if we denote the perpendiculars CG and CH by $p$ and $q$, the equation of equilibrium will be

$$Pp = Qq$$

By the *moment of a force with respect to a point*, is meant the product of this force into the perpendicular let fall from this point on its direction. Hence the condition of equilibrium in the case of the lever consists in this, that the moments of the power and resistance, taken relatively to the prop or point on which the lever turns, are equal; these two forces tending besides to make the lever turn in opposite directions.

If the lines CG and CH be supposed to be invariably attached to the lever, G and H may be taken for the points of application of the forces P and Q, and whatever be the figure

ECF of the lever, it may be replaced by the bent lever GCH (fig. 11.) The perpendiculars CG and CH, are termed *the arms of the lever*. The condition of equilibrium does not depend at all on the magnitude of the angle GCH, this indeed is evident *à priori*.

In fact, if from the point C as centre, and with a radius equal to CH, we describe an arc of the circle HH′, which is supposed to be invariably attached to the lever, and if at the point H′ two forces equal to Q be applied, acting in opposite directions along the parts H′B′ and H′B″ of the tangent at this point, it is evident that the force Q acting in the direction H′B″, will be destroyed by the force Q acting in the direction HB, for as the two forces tend to make the system revolve in opposite directions, there is no reason why it should obey one in preference to the other. Hence the second of these two forces will be replaced by the force Q acting in the direction H′B′, and the angle GCH will be changed into the angle GCH′, which may be greater or less, without the equilibrium being deranged.

By this change, the angle of the two arms may become 180° or zero, the lever will then be strait; the power and resistance will be parallel forces, acting in the same or in contrary directions; and in order to an equilibrium, it is always necessary that their intensities should be in the inverse ratio of the lengths of the arms of the lever.

43. If R denotes the resultant of the two forces P and Q, which meet at the point M (fig 10), and $m$ the angle AMB comprised between their directions, we shall have (No. 29)

$$R^2 = P^2 + Q^2 + 2PQ \cos m,$$

and the value of R will make known the load which the prop sustains in the case of equilibrium The direction of the force R applied to this point, will be along the line CD, the production of MC. In figure 10, the point C is supposed to be situated between E and F, the points of application of the

power and of the resistance. In figure 12, the contrary is the case, but the same reasoning is applicable to both cases; they differ from each other in this only, that in the first case, the forces P and Q act on different sides of the lever, and the angle AMB is acute, while in the second, they act on the same side, and the angle AMB is obtuse.

If, the three points E, F, C, remaining the same, the point of concourse of the three forces P, Q, R, be supposed infinitely distant, these forces will become parallel. In the case of fig 10, as the angle $m$ becomes then infinitely small, we have $\cos m = 1$, and consequently

$$R = P + Q$$

In the second case, it is the supplement of the angle $m$ which becomes infinitely small, hence we have $\cos m = -1$, and

$$R = Q - P,$$

P being supposed $\angle$ Q. Consequently, the resultants of two parallel forces is equal to their sum, when they act in the same direction, and to their difference, when they act in opposite directions, and in the last case, the resultant acts in the direction of the greater force. In these two cases, the components P and Q are in the inverse ratio of their distances from the resultant.

This being so, if a common perpendicular to the three forces be drawn, and if $a$ denotes GH the part of this line (fig. 13 and 14) comprised between the two components P and Q, and $x$ the distance CH of the resultant R from the component Q, which is supposed to be the greater, we shall have

$$P \quad Q : x \quad a \mp x,$$

the superior or inferior sign is to be taken, according as P and Q act in the same (fig. 13), or in opposite directions (fig. 14). Hence we obtain,

$$P : Q \pm P \quad x \quad a,$$

and, consequently,

$$x = \frac{a\text{P}}{\text{Q} \pm \text{P}};$$

by means of which, *the position* of the resultant, of which the *value* is Q ± P, will be determined.

44. When the forces P and Q act in opposite directions, and differ very little from each other, their resultant, which always acts in the direction of the greater, will be situated at a very great distance from the given forces. But when they are rigorously equal, this distance becomes infinite, which indicates that two equal and parallel forces, acting in opposite directions, cannot be replaced by a single force, and in point of fact, there is no reason why this unique force should act in one direction in preference to the other.

Two such forces acting at the extremities of the same line GH, (fig. 15,) will cause this line to revolve about its middle point K, which effect cannot evidently be produced by the action of a single force They may be replaced in an infinite variety of different ways by *two* other forces, which present the same case, for their action is not at all changed, by applying, for example, to the points G and H along GE and HF the productions of the line GH, equal forces of any magnitude whatever; now the resultant of the forces acting in the directions GA and GE, and that of the forces acting in the directions HB and HF, will be also equal and parallel forces, acting in opposite directions along the lines GC and HD, and these resultants will replace the primitive forces, which act along GA and HB. If P denotes the common magnitude of these two forces, and *a* their mutual distance, each of these two quantities will be changed by the operation that has been indicated, but their product $a$P will remain constant, as will be proved immediately.

45. In fine, this particular case is the only one in which a system of any number of forces P, P′, P″, &c., existing in the same plane, and acting on material points connected together

in an invariable manner, cannot be reduced to a single force. For whether the two forces P and P′ meet in a point, or are parallel to each other, they may be reduced to a single force Q, by the rule of the parallelogram of forces, or by that of the preceding number. In the same manner, this first resultant Q and P″ the third given force, may be reduced to a single force Q′, then, the second resultant Q′, and P‴ the fourth force to a single force Q″; and so on until, all the given forces are reduced to two only, which may themselves be reduced to a single force R, unless they fall under the case of exception of which we have been treating.

In the general case, this force R is the resultant of the given forces P, P′, P″, &c., and if to the components a force R′ equal and contrary to R be joined, there will be an equilibrium in the system. The magnitude of R, and its position in the plane of the given forces does not depend at all on the order in which these forces are taken, in the successive reductions which have been indicated, for if, in changing this order, we should arrive at S a force different from R in magnitude and direction, one of these two forces taken in an opposite direction would constitute an equilibrium with the other, which is impossible. In order to an equilibrium of the forces P, P′, P″, &c., when they are applied to a lever situated in their plane, it is necessary, in the first place, that they should be reducible to one sole force; for if they were reduced to S and S′, two parallel forces not reducible to a single force, and if S′ was the nearer to the prop, S′ may be resolved into two parallel forces Q and Q′, acting in the same direction, of which the first will be directly opposed to S, and the second will pass through the prop, each of these two forces will be less than S′ or S, the force Q′ will be destroyed, and there will only remain the force S — Q, which will cause the lever to turn in the direction of S($a$). The given forces being reduced to a unique force R, it is moreover necessary for the equilibrium of the lever, that this force should pass through the prop This

condition will be expressed by an equation, by means of the following theorem which we proceed to demonstrate.

46. Let us first only consider two forces and their resultant. The moment of this resultant with respect to a point situated in the plane of the three forces, is equal to the sum or difference of the moments of the two components with respect to the same point, to the difference when the centre of the moments is situated within the angle of the components, or within its vertically opposite angle, and to the sum when this point lies without these two angles. In fact, let P and P′ be these two forces, MA and MA′ (fig. 16 and 17) their directions, Q their resultant acting along MB, C the centre of the moments, $p, p', q$, the perpendiculars C$a$, C$a'$, C$b$, let fall from the point C on the directions of P, P′, Q. Let each of these three forces be resolved into others, in the direction of the line MC and of KMK′ perpendicular to this line, and let the perpendicular components be considered. We have evidently

$$\cos \mathrm{BMK} = \sin \mathrm{BMC} = \frac{q}{c},$$

$c$ denoting the length of the line MC, hence the component of Q in the direction MK will be equal to $\frac{\mathrm{Q}q}{c}$ In like manner, the components of P and P′ perpendicular to MC will be $\frac{\mathrm{P}p}{c}$ and $\frac{\mathrm{P}'p'}{c}$. They act in opposite directions when the line MC traverses the angle AMA′ (fig. 16), and in the same direction, when it falls without this angle. Now, the sum of these components in the second case, and the excess of the greater over the less, in the first, ought to reproduce the component of Q, because Q is the resultant of P and P′, consequently, if the component of P be supposed to be greater than that of P′, we shall obtain, by suppressing the common divisor $c$,

$$\mathrm{Q}q = \mathrm{P}p \pm \mathrm{P}'p',$$

which was required to be proved.

If the point C be supposed to be fixed, and if the perpendiculars C$a$, C$a'$, C$b$, constitute an invariable system, the forces P, P$'$, Q, which may be conceived to act at $a$, $a'$, $b$, the extremities of these lines, can only produce a motion of rotation about the centre of the moments. Now, it appears from the inspection of fig. 17, to which the superior sign of the preceding equation refers, that when the point C falls without the angle AMB, or its vertically opposite, the three forces P, P$'$, Q, tend to cause their points of application to turn in the *same* direction about the point C, on the contrary, when this point falls within one of these two angles, figure 16, which refers to the inferior sign, shows that the forces P and P$'$ tend to cause the points $a$ and $a'$ to turn in opposite directions, and it is likewise evident, that in this case, the resultant Q tends to make its point of application turn in the same direction as the component which has the greater moment. It appears from this remark, that the theorem ~~which~~ which has been now demonstrated implies, that the moment of the resultant of the two forces is equal to the sum or difference of the moments of these two forces, according as the components tend to make their points of application turn in the same or in opposite directions about the centre of the moments, and that the resultant tends to make them turn in the direction of the component, which has the greater moment.

As this theorem obtains for forces of which the directions make any angle whatever, it must obtain also when they become parallel, this is, in fact, what it is easy to infer, from the composition of forces of this kind (No. 43).

47. The advantage of this last mode of stating the problem is, that it can thus be easily extended to any number of forces P, P$'$, P$''$, &c., which act in the same plane. The centre of moments being regarded as a fixed point, about which the forces tend to make the system, whose points of application are connected together in an invariable manner, to turn, the moment of the resultant is equal to the sum of the moments of

the forces which tend to make the system to turn in the same direction as it turns, *minus* the sum of the moments of the forces which tend to make it to turn in a contrary direction.

For greater clearness, let the three first forces $P$, $P'$, $P''$, be supposed to make the system turn in one direction, and all the other forces in a contrary direction. Let the series of reductions of No 45 be resumed, and let $Q$ be the resultant of $P$ and $P'$, and $Q'$ that of $Q$ and $P''$, or of $P, P', P''$ Likewise, let $p, p', p''$, $q$, $q'$, be perpendiculars let fall from the centre of moments on the directions of $P$, $P'$, $P''$, $Q'$, $Q$, by what has been already established, we have

$$Qq = Pp + P'p', \quad Q'q' = Qq + P''p'',$$

and, consequently,

$$Q'q' = Pp + P'p + P''p''$$

In like manner, if we denote by $Q_1$ the resultant of all the other forces $P'''$, $P^{IV}$, &c., by $q_1$, the perpendicular let fall from the centre of moments on its direction, by $p'''$, $p^{IV}$, &c., the perpendiculars let fall from the same point on the directions of $P'''$, $P^{IV}$, &c., we shall have also

$$Q_1 q_1 = P'''p''' + P^{IV}p^{IV} + \&c.$$

Now, $R$ the resultant of all the given forces will be that of the two forces $Q'$ and $Q_1$, hence if $r$ denotes the perpendicular let fall from the centre of moments on the direction of $R$, and if we consider that these forces $Q'$ and $Q_1$ tend to produce a revolution in opposite directions, we shall have

$$Rr = \pm (Q'q' - Q_1 q_1),$$

according as $Q'q'$ will be greater or less than $Q_1 q_1$. In the first case, the force $R$ will tend to cause the line to turn in the same direction as the force $Q'$, and consequently in the same direction as the three forces $P, P', P''$ Let the first case be that which obtains, by substituting for $Q'q'$ and $Q_1 q_1$ their values, we shall then have

$$\text{R}r = \text{P}p + \text{P}'p' + \text{P}''p'' - \text{P}'''p''' - \text{P}^{IV}p^{IV} \text{ \&c.}; \qquad (1)$$

which equation contains the theorem we proposed to establish.

If the centre of the moments be supposed to be at the prop of the lever to which the forces P, P′, P‴, &c., are applied, it is necessary in order to insure an equilibrium in this lever, that we should have

$$\text{P}p + \text{P}'p' + \text{P}''p'' - \text{P}'''p''' - \text{P}^{IV}p^{IV} - \text{\&c} = 0, \qquad (2)$$

since, in this case, these forces must have a resultant which should pass through the prop (No. 45), and for which we have therefore $r = 0$.

48. Equation (1) can be rendered more general, by supposing that by resolutions and recompositions of the forces P, P′, P″, &c., they are transformed into other forces S, S′, S″, &c., which taken together are equivalent to the given forces. If $s, s', s''$, &c., denote the perpendiculars let fall from the centre of moments on the directions of S, S′, S″, &c., we shall find by the same mode of reasoning, as in the preceding number,

$$\text{S}s + \text{S}'s' + \text{S}''s'' + \text{\&c.} = \text{P}p + \text{P}'p' + \text{P}''p'' - \text{P}'''p''' - \text{P}^{IV}p^{IV} - \text{\&c.}, \quad (3)$$

in which equation the moments of the forces S, S′, S″, &c., which tend to turn in the same direction as P, P′, P″, should be affected with the sign +, and the moments of the forces which tend to turn in the same direction as P‴, P$^{IV}$, &c., with the sign −. The particular case in which the forces P, P′, P″, &c., are irreducible to a single force, is comprised in this last equation. Let then S and S′ be two equal and parallel forces, not directly opposed to each other, and let $h$ denote their mutual distance. If the centre of the moments is situated between their directions, we shall have $s + s' = h$, they will tend to produce a revolution in the same direction about this point, therefore, their moments must be affected with the same sign, and there will result

$$\text{S}s + \text{S}'s' = \text{S}h$$

If, on the contrary, the centre of moments does not exist between S and S′, and if we suppose $s > s'$, we shall have $s - s' = h$, these two forces will tend to produce a revolution in opposite directions, the moment of S must be affected with the sign + and the moment of S′ with the sign —; and there will result

$$Ss - S's' = Sh.$$

Consequently, equation (3) will always become

$$Sh = Pp + P'p' + P''p'' - P'''p''' - P^{IV}p^{IV} - \&c.$$

As the second member of this equation consists entirely of given quantities, it follows that if the values of S and $h$ undergo any change, their product must remain constant, as has been already stated. It appears likewise from this last equation, that when its second member vanishes, the given forces cannot fall under the case of exception, in which they are irreducible to a single force, it follows, therefore, that equation (2) expresses at the same time that the forces P, P′, P″, &c., have an unique resultant, and also that this resultant passes through the centre of moments, consequently this equation is necessary and sufficient to insure an equilibrium of the lever, of which this centre is the prop The resultant R which is obtained by the series of reductions indicated in No. 45, will express the pressure or load which the lever will have to support; when it vanishes, the forces P, P′, P″, &c., will be in equilibrio in their plane, without the aid of this fixed point.

49. The condition of equilibrium in the lever may also be expressed by an equation analogous to formula ($i$) of No. 39

For example, let M, M′, M″, (fig. 18,) be the points of application of three forces P, P′, P″, which act on the lever ECF in the directions MA, M′A′, M″A″, comprised in its plane. Let the lever turn by an infinitely small quantity about its prop C, in such a manner that M, M′, M″, may assume the positions $m, m', m''$ By the definition of No 39, the infinitely small

arcs $Mm$, $M'm'$, $M''m''$, which may be assumed to be right lines, will be the virtual velocities of $M, M', M''$, the points of application of the three forces, $P, P', P''$. Let the perpendiculars $ma$, $m'a'$, $m''a''$, be let fall from $m$, $m'$, $m''$, on the lines $MA$, $M'A'$, $M''A''$, or on their productions; $Ma$ will be the projection of $Mm$ on the direction of the force $P$, which tends to make the lever turn in the direction of rotation that ensues, $M'a'$ and $M''a''$ will be the projections of $M'm'$ and $M''m''$ on the productions of the two other forces $P'$ and $P''$, which tend to make the lever turn in the opposite direction. On this account, the first of these projections may be considered as a positive, and the two others as negative quantities. Let these three quantities be denoted by $\pi, \pi', \pi''$. It is evident, in virtue of the principle of virtual velocities, that the sum of the given forces multiplied respectively by the projections, thus defined, of the virtual velocities of their points of application, vanishes in the case of equilibrium, and conversely, the equilibrium obtains when this sum is cypher, so that the equation of equilibrium of the lever is

$$P\pi + P'\pi' + P''\pi'' = 0, \tag{4}$$

and, in fact, it is easy to show that it coincides with that which has been deduced from the consideration of moments.

For this purpose, let $p, p', p''$, denote the perpendiculars $CG$, $CG'$, $CG''$, let fall from the point $C$ on the directions of the forces $P, P', P''$, let $c, c', c''$, be the distances $CM$, $C'M'$, $C''M''$, of their points of application from the point $C$, and $\gamma, \gamma', \gamma''$, the virtual velocities $Mm$, $M'm'$, $M''m''$. As the infinitely small arc $Mm$ may be considered as coincident with its tangent, the triangles $Mma$ and $CMG$ have their sides perpendicular to each other and are similar, hence we have

$$Ma \,.\, Mm :: CG \,.\, CM,$$

and because

$$Ma = \pi, \quad Mm = \gamma, \quad CG = p, \quad CM = c,$$

we obtain from it

$$\pi = \frac{p\gamma}{c}$$

In the same manner we shall have

$$\pi' = -\frac{p'\gamma'}{c'}, \quad \pi'' = -\frac{p''\gamma''}{c''},$$

$\pi'$, $\pi''$, being by hypothesis negative quantities. Moreover, the form of the lever being supposed invariable, the three arcs M$m$, M$'m'$, M$''m''$, described in the same time, correspond to the same angle, and if they be divided by their respective radii CM, C$'$M$'$, C$''$M$''$, we shall have three equal ratios. Denoting by $\theta$, the common magnitude of these ratios, there will consequently result

$$\frac{\gamma}{c} = \frac{\gamma'}{c'} = \frac{\gamma''}{c''} = \theta,$$

and, therefore,

$$\pi = p\theta, \quad \pi' = -p'\theta, \quad \pi'' = -p''\theta.$$

Now, if these values be substituted in equation (4), and if $\theta$, the factor common to all its terms, be suppressed, it will become

$$\text{P}p - \text{P}'p - \text{P}''p'' = 0,$$

which is in fact, the equation of equilibrium that we have been considering. Conversely, if this last equation be multiplied by $\theta$, it will be changed into equation (4). The reasoning will be precisely the same, whatever be the number of given forces P, P$'$, P$''$, &c., and the direction in which they tend to make the lever to turn.

## CHAPTER III.

### OF THE COMPOSITION AND EQUILIBRIUM OF PARALLEL FORCES.

50. It appears from No. 43, that the composition of parallel forces may be deduced from the rule of the parallelogram of forces, by considering the point of application at an infinite distance, but by means of this same rule, we may also obtain the resultant of two parallel forces, by another way which it will be useful to know.

Let P and Q be the two components acting at the points E and F of the inflexible line EF, along the parallel directions EA and FB, either in the same direction as is represented in (fig 19), or in opposite directions as in (fig 20). No change will be produced in this system of forces, by applying to the extremities of this line, equal forces respectively denoted by S and acting in opposite directions, along its productions, EC and FD Let the force P′, which is supposed to act in the direction of EA′, that is comprised within the angle AEC, be the resultant of the forces P and S applied to the point E, in like manner, let the force Q′ acting in the direction of the line FB′, which is comprised within the angle BFD, be the resultant of the forces Q and S, then if the case of No. 44, in which the given forces P and Q are equal and act in opposite directions, be excepted, the two lines EA′ and FB′ will not be parallel. Consequently, if K their point of intersection be supposed to be invariably attached to the line EF, it may be assumed as the common point of application of the two forces P′ and Q′ (No. 41). Through this point K, let the lines E′F′ and KH′ be drawn parallel to the line EF and to the direction

of the forces P and Q, and if then each of the forces P′ and Q′ be decomposed in the direction of these parallels, it is evident that we shall find again the components S and P directed along KE′ and KH, and the components S and Q directed along KF′ and KH (fig. 19), or along KF′ and KH′ (fig. 20). Therefore, we shall have the same four forces as before, but all applied to the same point K. And if the two forces S be suppressed, there will remain the two forces P and Q acting in the direction of the same line KH in the case of fig. 19, or along this line and its production KH′ in the case of fig. 20, in which it is supposed that Q is the greater of the two given forces. Hence, the resultant of these two forces will be parallel to them; and if it be denoted by R, we shall have

$$R = Q \pm P,$$

according as they act in the same or in opposite directions. In order to determine the point O, where its direction cuts the line EF or its production, let E′ and F′ be the intersections of the lines AE and BF with the line E′F′, then the two quadrilaterals EE′KO, FF′KO will be parallelograms, and if their diagonals KE and KF be taken to represent the resultants P′ and Q′, we shall have

$$S : P :: EO : KO,$$

$$S : Q :: FO : KO,$$

for the ratios of the components. Hence we infer

$$P : Q :: FO : EO;$$

by means of which the position of the point O, which may be assumed as the point of application of the resultant R, can be determined. We can likewise infer

$$P : Q \pm P :: FO : EF;$$

$$Q : Q \pm P :: EO : EF;$$

in which the superior signs refer to figure 19, and the inferior

signs to figure 20, therefore, if the preceding value of R be taken into account, we shall have in the two cases,

$$P \quad Q \quad R : FO \quad EO \quad EF;$$

which shows that each of the three forces is proportional to the distance comprised between the points of application of the two others.

This proportion, and consequently the position of the point O, are independent of the angle at which the directions of the given forces are cut by the line EF, which may be any line whose extremities terminate at these two directions.

51. We can now resolve, without any difficulty, all questions which may be proposed on the composition of two parallel forces into one, and on the decomposition of a force into two others parallel to it. However, we will not enter into any details on this subject, neither will we hereafter revert to the particular case of forces which are equal, but not directly opposed to each other, which has been excluded from the preceding demonstration, since it has been sufficiently examined in No. 44. We proceed, therefore, to consider any number whatever of parallel forces, of which one part acts in one direction, and the other part in the opposite direction, and which, while they may or may not exist in the same plane, are supposed to be applied to points connected together in an invariable manner, as for example, to different points of the same solid body.

The magnitude and position in space of the resultant of all the given forces, will be obtained by compounding two of these forces into a single one, then this last and a third into a single one, and so on, until all the forces are exhausted, provided that the two last forces which are considered, do not fall under the case of exception of No. 44. This resultant will be evidently parallel to the common direction of the components; moreover, it will be equal to the sum of those which act in one direction, *minus* the sum of those which act in the oppo-

site direction, and it will act in the direction of the greater sum. If, therefore, the first be considered as positive, and the other as negative quantities (No. 11), and if they be represented by P, P$'$, P$''$, &c, and their resultant by R, we shall have always

$$R = P + P' + P'' + \&c.$$

52. If while the given forces are made to turn about their points of application, their parallelism continues, the resultant of these forces will also turn about one of the points of its direction, for its point of application, which is found by compounding the given forces one after another, in the manner already pointed out, does not at all depend on the common direction of these forces, and consequently, it remains the same, when this direction changes. Thus, for example, let us suppose that the given forces are three in number, namely P, P$'$, P$''$, acting in the direction of the lines MA, M$'$A$'$, M$''$A$''$, (fig. 21) First, let NB be the direction of the resultant of P and P$'$, which resultant will be equal to P + P$'$, let then N$'$B$'$ be the direction of the resultant of P + P$'$ and P$''$, this last force P$''$ being supposed in the figure to act in the opposite direction from P and P$'$, and to be greater than their sum. If now the three forces P, P$'$, P$''$, be supposed to turn about the points M, M$'$, M$''$, retaining their parallelism and the relative direction in which they act, and if M$a$, M$'a'$, M$''a''$, be their new directions, In this new state, the resultant of the forces P and P$'$ will meet the line MM$'$ at the same point N as before, since the position of this point depends solely on the ratio of the components, and not at all on the angle which the line MM$'$ makes with their directions (No. 50), it will now be directed along the line N$b$ parallel to M$a$, and M$'a'$, and it will be still equal to P + P$'$. For the same reason, the resultant of P + P$'$ and P$''$ will meet the production of the line MM$'$ in N$'$ the same point as before, and it will act in the direction of the line N$'b'$ parallel to N$b$, consequently, while the three forces P, P$'$, P$''$, turn about their

points of application M, M′, M″, their resultant will also turn about the same point N′.

53. *The centre of parallel forces* is termed the point in which all the successive directions of the resultant intersect, when its components revolve about their points of application, which are supposed to be invariable.

We shall see in the sequel, of what importance the consideration of the centre of parallel forces, is, especially in all questions respecting the equilibrium and motions of heavy bodies. We can already perceive, that if a solid body is solicited by any parallel forces whatever, and if the centre of these forces is determined and supposed fixed, the equilibrium will obtain in all positions which the body can assume about this point, provided that the given forces continue always parallel and applied to the same points of this body; for then their resultant will constantly pass through the fixed point, which is sufficient in order that it may be destroyed.

The rectangular coordinates of the centre of parallel forces depend, as we now proceed to shew, on the products of these forces multiplied by the coordinates of their points of application. As these products occur in a great number of cases, a particular denomination has been given to them, the product of a force into its distance from a plane, is termed *the moment of the force with respect to this plane*. Thus, P being the intensity of a force applied to a point of which the coordinates are $x$, $y$, $z$, the products P$z$, P$y$, P$x$, will be the moments with respect to the planes of the axes of $x$ and $y$, of the axes of $x$ and $z$, and of the axes of $y$ and $z$. In general, this species of moments has nothing in common with moments which refer to a point, and which have been defined in No. 42. These depend upon the direction of the force, and are independent of its point of application; on the contrary, moments with respect to a plane depend on the position of the point of application of the force, and are independent of its direction. These last are only made use of in the case

of parallel forces; so that they may be either positive or negative, according to the sign of the force and of the coordinates of the point at which it is applied.

54 Let $M$, $M'$, $M''$, &c (fig 22), be the points of application of the parallel forces $P$, $P'$, $P''$, &c, of which it will be unnecessary to indicate the directions. Let the three rectangular axes $Ox$, $Oy$, $Oz$, drawn arbitrarily, be the axes of the coordinates, let $x$, $y$, $z$, be the coordinates of $M$, $x'$, $y'$, $z'$, those of $M'$, $x''$, $y''$, $z''$, those of $M''$, &c., and let these coordinates and forces be supposed to be given quantities, which may be either positive or negative. Likewise, let $Q$, $Q'$, $Q''$, &c., be the projections of the points $M$, $M'$, $M''$, &c, on the plane of the axes of $x$ and $y$, so that we may have

$$MQ = z, \quad M'Q' = z', \quad M''Q'' = z'', \; \&c.$$

Finally, let $x_1$, $y_1$, $z_1$, represent the three coordinates of the centre of parallel forces, of which it is required to find the values.

$P + P'$, which is equal to the resultant of the two forces $P$ and $P'$ will meet at the point $N$, the line $MM'$, or its production, according as these two forces have the same or opposite signs, but in each case we shall have

$$P' . P + P' \;.\; MN \quad MM'.$$

Let $K$ be the projection of $N$ on the plane of the axes of $x$ and $y$. Through the point $M$, let $MGH$ be drawn parallel to the line $QKQ'$, meeting the lines $NK$ and $M'Q'$ at the points $G$ and $H$, so that we may have

$$MG = GK = HQ',$$

we shall also have

$$MN . MM' . . NG . M'H;$$

and from this proportion, combined with the preceding, we obtain($a$)

$$(P + P') \; NG = P'M'H.$$

If to this equation, we add the identical equation

$$(P + P') GK = P\,MQ + P'HQ',$$

there will result

$$(P + P') NK = Pz + P'z'.$$

The resultant of the two forces $P + P'$ and $P''$ will meet in a point $N'$, either the line $NM''$, or its production, according as these two forces have the same or contrary signs, and if $K'$ be the projection of $N'$ on the plane of the axes of $x$ and $y$, we shall find, as in the preceding case,

$$(P + P' + P'') N'K' = (P + P') NK + P''z'';$$

consequently we shall have

$$(P + P' + P'') N'K' = Pz + P'z' + P''z''.$$

In this manner we can continue, until all the given forces $P, P', P''$, &c., are exhausted, and if $R$ is their total resultant, we shall have finally

$$Rz_1 = Pz + P'z' + P''z'' + \&c.$$

Figure 22 supposes that all the points $M, M', M''$, &c., $N, N'$, &c., are situated on the same side of the plane of the axes of $x$ and $y$, or that their ordinates parallel to the axis of $z$, have all the same sign, but it is easy to perceive, that if the preceding equation is true in this case, it will be also true when these ordinates are partly positive and partly negative. In fact, if the plane of the axes of $x$ and $y$ be transferred parallel to itself, to any distance $h$ from its primitive position, and if $Z, Z', Z''$, &c., be the coordinates of $M, M', M''$, &c., and $Z_1$ that of the centre of parallel forces relatively to this new plane, so that we may have

$$Z_1 = z_1 - h,\ \ Z = z - h,\ \ Z' = z' - h,\ \ Z'' = z'' - h,\ \&c.;$$

and if from the preceding equation the identical one

$$Rh = Ph + P'h + P''h + \&c.,$$

be subducted, there will result

$$RZ_1 = PZ + P'Z' + P''Z'' + \&c.,$$

in which equation the ordinates $z, z', z''$, &c., may be either positive or negative.

It appears, therefore, that in all cases, the moment of the resultant of any number of parallel forces with respect to a plane arbitrarily selected, is equal to the sum of the moments of these forces, with respect to the same plane.

55.) If the moments with respect to the three planes of the coordinates, be taken successively, we shall have by the preceding notations,

$$\left.\begin{aligned} Rx_1 &= Px + P'x' + P''x'' + \text{\&c.}, \\ Ry_1 &= Py + P'y' + P''y'' + \text{\&c.}, \\ Rz_1 &= Pz + P'z' + P''z'' + \text{\&c.}; \end{aligned}\right\} \quad (1)$$

and since

$$R = P + P' + P'' + \text{\&c.}, \quad (2)$$

the three coordinates of the centre of parallel forces will be completely determined. If through this point a line be drawn parallel to the given forces, in the direction of the sign indicated by R, the direction of the resultant will be obtained. These four equations embrace, in the most general manner, the theory of parallel forces.

The sum of the moments of the forces is zero, with respect to every plane passing through the centre of parallel forces, for, if this plane be assumed to be that of the axes of $x$ and $y$, it is necessary that we should have $z_1 = 0$, and consequently

$$Pz + P'z' + P''z'' + \text{\&c.} = 0,$$

In the particular case in which P, P′, P″, &c., are reduced to two equal forces acting in opposite directions, their sum $R = 0$, hence the values of $x_1, y_1, z_1$, will be infinite. Consequently, the centre of parallel forces is at an infinite distance, or rather it does not exist at all, no more than the resultant.

56. When all the points of application M, M′, M″, &c., of the given forces are situated in the same plane, it is evident, from the nature of the centre of parallel forces (No. 52), that

this centre, if it exists, must also lie in this plane, this is in fact what may be inferred from equations (1) and (2).

Let $a, b, c,$ be three given constants, in this case we shall have

$$z = ax + by + c,$$
$$z' = ax' + by' + c,$$
$$z'' = ax'' + by'' + c,$$
&c.

If these values of $z, z', z''$, &c., be substituted in the third equation (1), it becomes

$$\begin{aligned} \text{R}z_1 = (\text{P}x + \text{P}'x' + \text{P}''x'' + \&c.)\, a \\ + (\text{P}y + \text{P}'y' + \text{P}''y'' + \&c.)\, b \\ + (\text{P} + \text{P}' + \text{P}'' + \&c.)\, c. \end{aligned}$$

By means of the two other equations (1), and of equation (2), the coefficients of $a, b, c,$ may be replaced by $\text{R}x_1$, $\text{R}y_1$, R; and if then, the common factor R be suppressed, there results

$$z_1 = ax_1 + by_1 + c,$$

from which it appears that the centre of parallel forces exists in the same plane as the points M, M′, M″, &c.

This centre may be likewise found, when all these points exist on the same right line, and then the first of equations (1) suffices to enable us to determine its position, provided that this line be assumed to be the axis of $x$. Moreover, if the forces P, P′, P″, &c., are perpendicular to this line, the moments which have been considered at page 76, are the same with those which refer to a point O, the origin of the abscissæ $x$, and the first equation (1) coincides with equation (1) of No. 47. In fact, it is easy to perceive, that among the given forces P, P′, P″, &c., those which tend to produce a revolution about the point O, in the same direction as the resultant R, are all forces of which the sign is the same as that of their distances $x, x', x''$, &c., from this point, and that those which

tend to produce a revolution in the opposite direction, are forces of which the sign is contrary to that of these same distances, consequently, the moments of the first must be added, and those of the second subtracted, agreeably to what has been stated in the number already cited

57. The equations of equilibrium of the parallel forces $P, P', P''$, &c, may be easily deduced from the theorem which has been just established.

If there is no fixed point in the system, it is necessary, in order to an equilibrium, that, if one of these forces be abstracted, for example the force $P$, $R'$ the resultant of all the others should be equal and directly opposed to $P$. Since then the forces $P$ and $R'$ are equal and directly opposed to each other, they must be equal and affected with contrary signs, or in other words, we must have $P + R' = 0$. But $R'$ is the sum of the components $P'$, $P''$, &c., it therefore follows, that

$$P + P' + P'' + \&c. = 0, \qquad (a)$$

which is the first equation of equilibrium. In order to express besides, that the forces $P$ and $R'$ are directly opposed to each other, let $\alpha, \beta, \gamma$, be the three coordinates of the centre of parallel forces $P'$, $P''$, &c, so that we may have

$$R'\alpha = P'x' + P''x'' + \&c.,$$
$$R'\beta = P'y' + P''y'' + \&c,$$
$$R'\gamma = P'z' + P''z'' + \&c.,$$

then this centre being the point of application of their resultant $R'$, it is necessary that it should lie on the direction of the force $P$, in order that $R'$ may be directly opposed to this force, or, which comes to the same thing, this centre and $M$ the point of application of the force $P$ should exist on the same line parallel to the common direction of the given forces. If therefore, for greater simplicity, the plane of $x$ and $y$ be assumed perpendicular to this direction, it is necessary that these two points should exist on the same perpendicular to this

plane, they will then have the same projection on this plane; consequently, their coordinates parallel to the axes of $x$ and $y$ will be the same, so that we shall have

$$\alpha = x, \quad \beta = y,$$

hence, if $x$ and $y$ be substituted in place of $\alpha$ and $\beta$ in the first two of the preceding equations, there results, because $R' = -P$,

$$\left.\begin{array}{l} Px + P'x' + P''x'' + \&c. = 0, \\ Py + P'y' + P''y'' + \&c. = 0, \end{array}\right\} \quad (b)$$

which equations indicate that the sum of the moments of all the forces $P$, $P'$, $P''$, &c., is cypher, relatively to the planes of the axes of $x$ and $z$, and of the axes of $y$ and $z$, parallel to their direction.

Thus it appears, that in order to an equilibrium of these forces, it is necessary that equations (a) and (b) should obtain simultaneously Conversely, when these three equations are satisfied, the equilibrium has place, for we shall have, in virtue of these equations,

$$R' = -P, \quad R'\alpha = -Px, \quad R'\beta = -Py,$$

and, consequently,

$$\alpha = x, \quad \beta = y,$$

so that this resultant will be equal, and directly contrary to the force $P$ which has been omitted It is not necessary, in the preceding proof, that the two planes relatively to which the sum of the moments of the given forces is zero, should be perpendicular the one to the other, it is sufficient if they are parallel to the direction of these forces, and it is easy to perceive, that if this condition is satisfied with respect to two planes($b$) parallel to this direction, it will be equally so with respect to all others.

Hence we may infer, that in order to the equilibrium of a system of parallel forces applied to a solid body entirely free,

it is necessary and it suffices—first, that the sum of these forces be equal to zero; secondly, that the sum of their moments be nothing, with respect to any two planes parallel to their common direction When all the forces exist in the same plane, this second condition will be already satisfied with respect to this plane, and it suffices that it be so likewise with respect to some one other plane.

58. If one of the points of this solid body is supposed to be fixed, it will suffice for the equilibrium of parallel forces, that the sum of their moments should be cypher with respect to two planes passing through this point and parallel to their direction, and it will be no longer necessary that their resultant should be equal to cypher, for then the distances of this resultant from these two parallel planes will be cypher, it will therefore coincide with their intersection, and be destroyed by the resistance of the fixed point($c$).

When this point is the centre of parallel forces, the sum of the moments will be zero with respect to all planes passing through this point; consequently the given forces constitute an equilibrium, whatever be their common direction, which we already know from other considerations (No. 53).

If the solid body is retained by a fixed axis, about which it has solely liberty to turn, it will suffice, in order to the equilibrium of parallel forces applied to its different points, that the sum of their moments should be zero, relatively to the plane drawn through this axis parallel to their direction; for as then their resultant lies in this plane, it will meet the fixed axis, and must be destroyed by its resistance. When the fixed axis is itself parallel to the given forces, the plane in question is indeterminate, consequently the condition of equilibrium does not obtain, which indeed, it is evident of itself, must be the case, since the forces, which are all parallel to a fixed axis, cannot cause a solid body to turn about this line, so that in this case the equilibrium obtains independently of their intensities, and of their distances from this axis.

# CHAPTER IV.

## GENERAL CONSIDERATIONS RESPECTING HEAVY BODIES AND CENTRES OF GRAVITY.

59. The force which precipitates bodies to the surface of the earth when they are let to descend, is termed indifferently *gravity* or *weight*. It acts on all material points, in directions perpendicular to this surface, or in the direction of *vertical lines*. The directions of gravity in different places of the earth, must therefore, when produced, converge towards its centre, because its form is nearly spherical, but considering the magnitude of the radius of the earth, relatively to the dimensions of those bodies, which are generally considered, we may, without sensible error, assume that for all the material points of the same body, gravity acts in parallel directions.

Direct experiment proves, that the intensity of this force at the surface of the earth varies with the *latitude;* and that in the same vertical, it likewise varies with the *elevation* above this surface. But it is necessary that the changes of height and latitude should be very considerable, in order that these variations may become sensible, and they are evidently not so, in the extent of a body of ordinary dimensions.

60. It follows from this that the resultant of the parallel forces, that act on all the points of a heavy body, (which forces are infinite in number,) is independent of its form; this resultant is what is termed the *weight* of the body. In *homogeneous* bodies, the weight is evidently proportional to the volume, but daily experience shows, that bodies of different natures have not the same weight under the same volume;

this may either arise from this, that the attraction of the earth, which is the *principal* cause of gravity, as will appear in the sequel, depends on the nature of the material points on which it acts, or from the circumstance that *heterogeneous* bodies contain, under equal volumes, different quantities of material points which are equally heavy. We will explain in another chapter, how philosophers have inferred from the observed motion of heavy bodies, that it is the second of these cases which obtains in nature.

It results from this, that the weight of any body is in a ratio compounded of its mass and of the intensity of gravity in the place where it is situated. Hence, if we denote this weight by $P$, the mass by $M$, and the measure of gravity by $g$, we have

$$P = gM.$$

This quantity $g$, which is independent of the particular nature of each body, is thus the weight of what has been arbitrarily assumed to represent the unit of mass. In the sequel it will appear how its value has been determined in different points of the earth, by means of the motion of bodies subjected to the sole action of gravity. We may likewise write

$$P = wV,$$

$w$ denoting the weight of the body, which represents the unit of volume, and $V$ its volume. The weight $w$ is what is termed the specific gravity of the body that is considered, a denomination which is evidently improper, for gravity is common to all bodies of different species, and therefore, there ought to be substituted for it, the denomination *specific weight.*

Finally, if $D$ represents the mass, under the unit of volume of the body which is considered, $D$ will be what is termed the *density* of this body, and we shall have

$$M = DV, \quad P = gDV.$$

The preceding are the equations which obtain between the

five quantities, P, $g$, M, D, V, each of which should be expressed numerically, by referring it to a unit of its species.

61 The *gramme*, or unit of weight, is that of a cube of distilled water, of which one side is a centimetre, taken at its *greatest* density, which we know has place at about the temperature of 4° of the centigrade thermometer. This weight varies with the latitude and elevation of the place, in which it exists, however as the weights of other bodies, which it is made use of to ascertain, vary exactly in the same ratio, it follows that the weight of any body expressed in grammes, is every where the same, and that there is no occasion to specify in what latitude, or at what elevation it has been determined. According to the experiments of M Hallstrom, the weight of a cube of distilled water, of which one side is a centimetre, at the temperature zero, is

$$0^{\text{gram}}, 9998918.$$

The density of distilled water at this last temperature, is most commonly assumed as the unit of density. The densities of a great number of substances have been determined by experiment, and expressed in numbers by means of this unit. Thus, for example, the density of mercury at this same temperature is,

$$13{,}5975,$$

and it increases or diminishes by

$$\frac{1}{5550},$$

for each degree of increase or diminution of the temperature. The density of air, taken also at that of melting ice, and when the barometrical pressure is 76 centimetres at the observatory of Paris, is found to be equal to

$$\frac{1}{769{,}4},$$

and for each variation of a degree in the temperature, it varies in a contrary sense, by

0,00375,

like that of every other gas.

As the weight of a column of mercury which expresses the barometrical pressure, varies with the latitude and elevation above the surface of the earth, the density of the air, subjected to a pressure of a given height, varies at the same time. It is on this account, that it is not sufficient merely to assign this height, it is likewise necessary to specify in what place the observation has been made. The ratio of the density of mercury to that of air, corresponding to the preceding numbers, is

10462.

Whenever a phenomenon, such as, for example, caloric, can be ascribed to a material substance, this substance must be acted on by gravity, and the expression *imponderable* should refer solely to matter of which the density is so feeble as to elude all our means of investigation, so that its presence does not increase in an appreciable manner either the weight or mass of the body of which it constitutes a part, however considerable the quantities of it are, that may be supposed to exist in the body

62 As weights are forces with which we are most familiar, and of which the relative values may be determined by means of the balance, with the greatest precision and facility, it is natural for us to make use of them, in comparing forces of different natures. Thus, when the muscular force of an animal, or any other force, acts on a body by the intervention of a chord attached to its surface, we can always conceive that this force is equivalent to a certain determinate weight, or we can even, without changing its direction, substitute for its action, that of this weight, by suspending it at the extremity of the chord, this chord having been previously made to pass over a fixed pulley properly placed.

The weight furnishes us with the most commodious measure

of the mass, in fact, without the aid of gravity it would be extremely difficult to determine the ratio of the masses of two bodies It will appear in the sequel, that this measure may also be strictly inferred from the mutual collision of two bodies, but it is much simpler to substitute for the ratio of the masses, that of the weights, to which it is equal in every part of the earth, in virtue of the equation $P = gM$. Nevertheless, as the weight is only a secondary property of bodies which may be rendered altogether insensible, by transferring them to a sufficiently great distance from the earth, without any change whatever taking place in the masses, we should be able to obtain a conception of the equality and relation of the masses, independently of the consideration of the weight; this point will be adverted to in a subsequent part of this treatise.

63. Since all the points of a heavy body are solicited by parallel forces, it follows, that if it be made to assume successively different positions with respect to the direction of these forces, their resultant will constantly pass through a given point of this body; this point, which in general has been designated the centre of parallel forces (No 53), is in this particular case termed *the centre of gravity*. Its characteristic property in solid bodies, which are subjected to the sole action of gravity, consists in this, that if it be supposed fixed, the body to which it appertains remains in equilibrio in all possible positions about this point, since in all these positions, the resultant of the forces applied to all the points of the body passes through the fixed point.

It also appears, that when a heavy solid body is retained by another fixed point, it is necessary, and it suffices in order to an equilibrium, that the right line which connects this point with the centre of gravity should be vertical, this centre may however exist indifferently, above or below the fixed point. In fact, as the weight of the body is a vertical force applied at its centre of gravity, its direction will coincide in this hypothesis, with the right line which connects this centre with the fixed point, or with its production, consequently this force

will be destroyed by the resistance of the fixed point, just as if it was immediately applied to it

For the same reason, if a heavy solid body be suspended to a fixed point, by means of a thread, of which the inferior extremity is attached to a point of its surface, the direction of this thread will be vertical in the state of equilibrium, and its production will pass through the centre of gravity of the body. This will be also the case when the same body is suspended from the fixed point by attaching any other point of its surface to the inferior extremity of the thread, and the successive productions of the thread drawn in the interior of the body, will intersect in its centre of gravity, this furnishes us with a practical method of determining the position of this centre in a body of any form whatever, whether homogeneous or heterogeneous

In all questions relating to the equilibrium of a solid body, we may abstract from the consideration of the weight of its different parts, provided that there be added to the given forces which act on the body, a force equal to its weight, and applied vertically to its centre of gravity Thus, for example, in the case of the equilibrium of the lever, in the number of given forces of which the sum of the moments is cypher, with respect to the prop, there should be included the weight of the lever, acting in the direction of gravity, at *its* centre of gravity (No. 47).

64. When G and G′, the centres of gravity of the two parts of a body are known, and also their weights $p$ and $p'$, K the centre of gravity of this body may be immediately obtained, for this centre is the point of application on the line GG′, of the resultant of the parallel forces $p$ and $p'$, which act in the same direction at its extremities G and G′, and we consequently have, in order to determine it,

$$\mathrm{GK} \quad \mathrm{GG'} \quad p' \quad p+p'.$$

In like manner, if K and G, the centres of gravity of a body

and of one of its parts, be known, and if P and $p$ be the respective weights of the body and of this part, $G'$ the centre of gravity of the other part may be deduced from them, for this point will be situated beyond the point K on the production of the line GK, and its distance from the point G will be determined by the proportion

$$GG' \quad GK \, . \, . \, P : P - p.$$

If a body is divided into any number of parts of which the weights and the centres of gravity are known, its centre of gravity may be deduced by a series of proportions; but it is more convenient to determine its three coordinates by means of the theorem which has been established respecting the moments of parallel forces (No. 54).

For this purpose, let $p, p', p''$, &c, be the weights of the different parts of the body, and P the total weight, so that

$$P = p + p' + p'' + \&c.$$

Likewise, let $x, y, z$, be the coordinates of the centre of gravity of the part of which $p$ is the weight, $x', y', z'$, those of the centre of gravity of the part of which $p'$ is the weight, &c. All these quantities are supposed to be given, and if $x_1, y_1, z_1$, be the coordinates of the centre of gravity of the entire body, referred to the same axes as the preceding, we shall have, by the theorem just cited,

$$Px_1 = px + p'x' + p''x'' + \&c.,$$
$$Py_1 = py + p'y' + p''y'' + \&c,$$
$$Pz_1 = pz + p'z' + p''z'' + \&c.,$$

by means of which the values of $x_1, y_1, z_1$, can be determined.

65. In these equations, the weights may be replaced by the masses to which they are proportional. Therefore, if $m, m', m''$, &c, denote the masses of the different parts of the body to which the weights represented by $p, p', p''$, &c., are proportional, and if M denotes the entire mass, so that

$$M = m + m' + m'' + \&c.,$$

there will result

$$\left.\begin{aligned} Mx_1 &= mx + m'x' + m''x'' + \&c, \\ My_1 &= my + m'y' + m''y'' + \&c, \\ Mz_1 &= mz + m'z' + m''z'' + \&c.; \end{aligned}\right\} \quad (1)$$

from which it appears, that the centre of gravity is independent of the intensity of gravity, and that it will be always the same point of the body, in different latitudes and at different elevations above the surface of the earth. It was the consideration that this point does not imply the action of gravity, and that it depends solely on the masses and on their respective positions, which induced Euler, and other authors, to term it the *centre of inertia*; however the denomination of centre of gravity has more generally obtained.

If the mass M be supposed to be divided into an infinite number of infinitely small parts $m$, $m'$, $m''$, &c., we may assume whatever point of each of these we choose for its centre of gravity, since the coordinates of all the points of the same element, parallel to the same axis, differ only from each other, by infinitely small quantities. In this case, the second members of equations (1) will consist of an infinite number of infinitely small terms, the sums of which will be definite integrals, extended by the theorem of No. 13 to multiple integrals. Consequently, we can always, by the rules of the integral calculus, determine, exactly or by approximation, the centre of gravity of any body whatever, without knowing that of any of its parts.

In a body of which all the parts are homogeneous, their masses are to each other as their respective volumes; we may, therefore, substitute the volumes for the masses in equations (1), and if the entire volume be represented by V, and the parts corresponding to $m$, $m'$, $m''$, &c., by $v$, $v'$, $v''$, &c., we shall have

$$\text{V} = v + v' + v'' + \&c.,$$
$$\text{V}x_1 = vx + v'x' + v''x'' + \&c.$$
$$\text{V}y_1 = vy + v'y' + v''y'' + \&c ,$$
$$\text{V}z_1 = vz + v'z' + v''z'' + \&c.$$

The point which is determined by means of these equations, is the centre of parallel forces applied to all the points of a body, and proportional to the elements of its volume, this point is termed the *centre of gravity* of the *volume*, although a volume has neither weight nor mass In like manner, the centre of parallel forces applied to all the points of a surface, or a line, and proportional to their elements, is termed the *centre of gravity*. Its coordinates are determined by substituting in the preceding equations, for the volumes V, $v$, $v'$, $v''$, &c , either the areas of the surface and its parts, or the lengths of the line and its parts.

66. The masses M, $m$, $m'$, $m''$, &c , and the mutual distances of their centres of gravity, are connected together by an equation which may be easily deduced from equations (1). For this purpose, let the origin of the coordinates be placed at the centre of gravity of M; and these equations will then become

$$mx + m'x' + m''x'' + \&c. = 0,$$
$$my + m'y' + m''y'' + \&c. = 0,$$
$$mz + m'z' + m''z'' + \&c. = 0.$$

By squaring the first, there results

$$m^2x^2 + m'^2x'^2 + m''^2x''^2 + \&c. = -2mm'xx' - 2mm''xx'' \\ - 2m'm''x'x'' - \&c.$$

If to the two members of this equation, there be added the quantity

$$m\,(m' + m'' + \&c.)\,x^2 + m'\,(m + m'' + \&c.)\,x'^2 \\ + m''\,(m + m' + \&c.)\,x''^2 + \&c.,$$

there will result$(a)$

$$\mathrm{M}\,(mx^2+m'x'^2+m''x''^2+\&c.)=mm'(x-x')^2+mm''(x-x'')^2$$
$$+\,m'm''(x'-x'')^2+\&c.$$

In the same manner, the second and third equations (1) will give

$$\mathrm{M}\,(my^2+m'y'^2+m''y''^2+\&c.)=mm'(y-y')^2+mm''(y-y'')^2$$
$$+\,m'm''(y'-y'')^2+\&c.$$

$$\mathrm{M}\,(mz^2+m'z'^2+m''z''^2+\&c.)=mm'(z-z')^2+mm''(z-z'')^2$$
$$+\,m'm''(z'-z'')^2+\&c.$$

Now, if these three last equations be added together, and if we make

$$\begin{aligned}
&x^2+y^2+z^2=r^2,\\
&x'^2+y'^2+z'^2=r'^2,\\
&x''^2+y''^2+z''^2=r''^2,\\
&\&c.,
\end{aligned}$$

$$\begin{aligned}
&(x-x')^2+(y-y')^2+(z-z')^2=\rho^2,\\
&(x-x'')^2+(y-y'')^2+(z-z'')^2=\rho'^2,\\
&(x'-x'')^2+(y'-y'')^2+(z'-z'')^2=\rho''^2,\\
&\&c.,
\end{aligned}$$

we shall have

$$\mathrm{M}\,(mr^2+m'r'^2+m''r''^2+\&c)=mm'\rho^2+mm''\rho'^2$$
$$+\,m'm''\rho''^2+\&c\,,$$

for the equation which it was proposed to obtain, and in which $\rho$, $\rho'$, $\rho''$, &c., are the mutual distances of the centres of gravity of $m$, $m'$, $m''$, &c., and $r$, $r'$, $r''$, &c., the distances of these points from the centre of gravity of M.

67. We can also infer from equations (1), a remarkable property of the equilibrium of a material point entirely free. It may be stated in the following manner.

Let O be the point in equilibrio (fig. 23); and let the forces which solicit it be represented in magnitude and direction by the lines OA, OA$'$, OA$''$, &c.; if their extremities

A, A′, A″, &c., be the centres of gravity of equal masses, the point O will be the centre of gravity of this entire system. In fact, if equations (1) be applied to these masses, and if their number be supposed to be $n$, we shall have

$$nx_1 = x + x' + x'' + \&c.,$$
$$ny_1 = y + y' + y'' + \&c.,$$
$$nz_1 = z + z' + z'' + \&c.$$

On the other hand, if the angles which the force A makes with three rectangular axes drawn through the point O, be $\alpha, \beta, \gamma$, and if $\alpha', \beta', \gamma'$, be what those angles become relatively to the force OA′, and $\alpha'', \beta'', \gamma''$, what they become relatively to the force OA″, &c., the equations of equilibrium of these forces will be

$$\text{OA} \cos \alpha + \text{OA}' \cos \alpha' + \text{OA}'' \cos \alpha'' + \&c. = 0,$$
$$\text{OA} \cos \beta + \text{OA}' \cos \beta' + \text{OA}'' \cos \beta'' + \&c. = 0,$$
$$\text{OA} \cos \gamma + \text{OA}' \cos \gamma' + \text{OA}'' \cos \gamma'' + \&c. = 0,$$

Now, if the origin of the coordinates be placed at the point O, the coordinates of the points A, A′, A″, &c., will be

$$x = \text{OA} \cos \alpha, \quad y = \text{OA} \cos \beta, \quad z = \text{OA} \cos \gamma,$$
$$x' = \text{OA}' \cos \alpha', \quad y' = \text{OA}' \cos \beta', \quad z' = \text{OA}' \cos \gamma',$$
$$x'' = \text{OA}'' \cos \alpha'', \quad y'' = \text{OA}'' \cos \beta'', \quad z'' = \text{OA}'' \cos \gamma'',$$
&c.,

therefore we shall have, in virtue of the equations of equilibrium,

$$x + x' + x'' + \&c = 0,$$
$$y + y' + y'' + \&c. = 0,$$
$$z + z' + z'' + \&c. = 0;$$

hence we infer

$$x_1 = 0, \quad y_1 = 0, \quad z_1 = 0,$$

for the coordinates of the centre of gravity of equal masses,

consequently, this centre will coincide with the point o, which was proposed to be demonstrated.

68. There are several particular cases in which the centre of gravity is immediately known. Thus, the centre of gravity of a sphere, or of an ellipsoid, is evidently the centre of the figure; that of a parallellopiped is at the intersection of its four diagonals, that of a cylinder whose bases are parallel, at the middle point of its axis. The centre of gravity of a circle, or of an ellipse, is likewise in the centre of the figure, and that of a parallellogram at the intersection of the two diagonals. The centre of gravity of a right line is in the middle point of this line, hence may be inferred, without any difficulty, the centre of gravity of the perimeter of any polygon whatever, either by a series of proportions (No. 64), or by the equations of the moments of parallel forces. It appears in like manner, that when the centres of gravity of a triangle and of a triangular pyramid are known, we can deduce, by one or other of these methods, the centres of gravity of any given polygon, or polyhedron, since they may be respectively decomposed into triangles, or triangular pyramids.

But in general, the determination of centres of gravity requires the application of the integral calculus, and in the following chapter, we propose to furnish all the requisite details for this purpose.

# CHAPTER V.

## DETERMINATION OF CENTRES OF GRAVITY

### *Centres of Gravity of curved Lines.*

69. Let $s$ be the arc of the given curve, terminating at any point M, and measured from a fixed point C. Likewise, let $x, y, z$, be the three rectangular coordinates of M This curve may be considered as a polygon of an infinite number of sides, $ds$ will be the side, or the element, of the curve which corresponds to the point M, and in whatever point the centre of gravity of this element may be, we can assume $x, y, z$, for its three coordinates, as they cannot, in point of fact, differ from those of M, except by infinitely small quantities.

Let $l$ denote the length of the determinate part of the curve, of which it is required to determine the centre of gravity, and let $s_0$ and $s_1$ represent the given values of $s$ which refer to the two extremities of $l$. Let $x_1, y_1, z_1$, be the coordinates of the centre of gravity of this arc $l$, referred to the axes $x, y, z$ By the theorem of No. 13, the sum of the values of each of the products $xds, yds, zds$, in the entire extent of $l$, will be a definite integral taken from $s = s_0$, to $s = s_1$, $x, y, z$, being considered as functions of $s$, depending on the nature of the curve in question. Hence we shall have (No. 65)

$$lx_1 = \int_{s_0}^{s_1} xds, \quad ly_1 = \int_{s_0}^{s_1} yds, \quad lz_1 = \int_{s_0}^{s_1} zds, \qquad (1)$$

for the three equations, by means of which $x, y, z$, can be determined.

Suppose, for example, that the given line is a right one, and that its part $l$ terminates at the point C, so that we have

$s_0 = o_1$, $s_1 = l$. Let $\alpha$, $\beta$, $\gamma$, denote the three angles which this part $l$ makes with the axes drawn through the point C, in the direction of the positive $xs$, $ys$, $zs$ Likewise let $a$, $b$, $c$, be the three coordinates of the point C; for any point M we shall have

$$x = a + s\ \cos\alpha,\quad y = b + s.\cos\beta,\quad z = c + s.\cos\gamma.$$

By substituting these values in equations (1), integrating and then dividing by $l$, there results(*a*)

$$x_1 = a + \tfrac{1}{2}l\cos\alpha,\quad y_1 = b + \tfrac{1}{2}l\cos\beta,\quad z_1 = c + \tfrac{1}{2}l\cos\gamma;$$ +

which indicates that the centre of gravity of the line $l$, is situated at its middle point, as we know ought to be the case.

70. When it is proposed to find the centre of gravity of a plane curve, if its plane be assumed to be that of the axes of $x$ and $y$, the two first equations (1) suffice to enable us to determine the position of its centre of gravity in this plane. Moreover, if the portion $l$ of the curve be symmetrical on each side of the point C, we shall have $s_0 = -\frac{1}{2}l$ and $s_1 = \frac{1}{2}l_1$, the centre of gravity will be situated on the normal raised at the point C; and if this line be taken for the axis of $x$, it will be sufficient to determine the value of $x_1$, which will be furnished by the equation

$$lx_1 = \int_{-\frac{1}{2}l}^{\frac{1}{2}l} xds.$$

The arc of a circle is comprised in this particular case, by assuming for the axis of $x$, the diameter which passes through its middle point. If the origin of the coordinates be at the centre of the circle, and if we denote its radius by $a$, we shall have

$$x = a.\cos\frac{s}{a},$$ †

for the abscissa of any point M, hence we infer(*b*)

$$lx_1 = 2a^2\sin\frac{l}{2a};$$

and if $c$ be the chord of the arc $l$, we shall have

$$c = 2a \sin \frac{l}{2a}, \quad lx_1 = ac,$$

from which it appears that $x_1$, the distance of the centre of gravity of an arc of a circle from the centre of the circle, is a fourth proportional to the radius, the chord, and the arc.

71 As the equation of a plane curve makes known one of the two variables in a function of the other, if the value of $y$ is supposed to be given in a function of $x$, we shall have

$$ds = \sqrt{1 + \frac{dy^2}{dx^2}}\, dx,$$

and if $\alpha$ and $\beta$ denote the values of $x$, which refer to the two extremities of the arc $l$, instead of the preceding equations, we shall have

$$\left.\begin{aligned} l &= \int_\alpha^\beta \sqrt{1 + \frac{dy^2}{dx^2}}\, dx, \\ lx_1 &= \int_\alpha^\beta x\sqrt{1 + \frac{dy^2}{dx^2}}\, dx, \\ ly_1 &= \int_\alpha^\beta y\sqrt{1 + \frac{dy^2}{dx^2}} . dx \end{aligned}\right\} \quad (2)$$

If the given curve be a conic section, we can obtain by the ordinary rules, the values of the integrals contained in the two last equations (2), under a finite form And in the case of the parabola, we can likewise obtain the value of the integral of the first of these equations, so that the two($c$) co-ordinates of the centre of gravity of an arc of a parabola may be always obtained in functions of $\alpha$ and $\beta$ the abscissæ of its extremities By a theorem of Landens, the arc of the hyperbola may be expressed by means of two arcs of the ellipse and of an algebraic part, as to the arc of the ellipse, it must be considered as a function irreducible to other simpler functions; however, by means of very extensive tables of this function, computed by M. Legendre, their numerical values may be computed to a very close approximation. Hence, when the

numerical values of $\alpha$ and $\beta$, and those of the axes of the ellipse or hyperbola are given, it will be easy to compute the value of $l$, and consequently, $x_1$ and $y_1$, the coordinates of the centre of gravity of an arc belonging to one or other of these two curves.

72. Let the arc of the *cycloid* be taken for another example of the application of equations (2). In this curve, the length, the area, the surface, the volume generated by its revolution, and the coordinates of their centres of gravity, can be determined exactly. The drawing of a tangent at any point of this curve is also extremely simple, its evolute is another cycloid, and moreover it may demonstrated, that by a series of successive developments, any curve whatever approximates more and more to coincide with the cycloid, and it rigorously($d$) coincides with it after an infinite number of developments. We shall likewise see in the sequel, that the cycloid is a curve which possesses very remarkable properties with respect to the curvilinear motion of heavy bodies. This singular combination of such a number of curious properties of different descriptions, all appertaining to the same curve, has rendered the consideration of it extremely useful, and of very frequent application in geometry and mechanics Its equation may be obtained in the following manner.

As the cycloid is a plane curve ACB (fig 24), generated by a given point M of the circumference of a circle, while it rolls without sliding on the right line AB, if the generating point commences to move at the point A, when it arrives at the point B of this line, the interval AB will be equal to the circumference of the given circle We may also perceive, that its diameter will be equal to the perpendicular CD, let fall from the summit of the cycloid on AB, and that it divides the curve into two parts symmetrically equal Therefore, if $c$ denotes the radius of the given circle, we shall have

$$\text{AB} = 2\pi c, \quad \text{CD} = 2c.$$

In any position whatever of the circle, let HG be its diameter perpendicular to the base AB, and H its point of contact with this line. From the point M let the perpendiculars MK and MP be let fall on GH and AB, and let

$$\text{AP} = p, \quad \text{PM} = q;$$

we shall have

$$\text{AH} = \text{AP} + \text{MK} = p + \sqrt{2cq - q^2},$$

$$\text{arc MH} = c.\text{arc}\left(\sin = \frac{\sqrt{2cq - q^2}}{c}\right),$$

But as the generating circle rolls without sliding on the line AB, it follows that we have constantly

$$\text{AH} = \text{arc MH},$$

hence the required equation of the cycloid will be

$$p + \sqrt{2cq - q^2} = c.\text{arc}\left(\sin = \frac{\sqrt{2cq - q^2}}{c}\right);$$

for the point of which the coordinates are $p$ and $q$.

By differentiating, we obtain($e$)

$$dp = \frac{qdq}{\sqrt{2cq - q^2}},$$

for its differential equation. From which it appears, that the two chords MG and MH of the generating circle are the tangent and normal to the cycloid at the point M. If the radius of curvature at the same point be determined by the known formulæ, it will be found equal to twice MH; hence it follows, that if MH be produced so that HN = MH, the point N will be the centre of curvature If in the same manner, the line CDE be made double of CD, the point E will be the centre of curvature of the cycloid at C the summit of the cycloid; and hence it is easy to perceive, that ANE, the evolute of the semi-cycloid AMC, is an equal cycloid, reversed in such a manner, that

its summit C is transferred to A, and its origin A to E. It follows also that length of AME, or of AMC, is equal to the right line CDE, and consequently, the entire length of the cycloid is equal to four times the diameter of the generating circle

73. In the different applications of the preceding equation, it will be more convenient to transfer the origin of the coordinates to the summit C (fig. 25), and to take for the axes of $x$ and $y$, the lines C$x$ and C$y$, which are respectively perpendicular and parallel to the base AB. Hence, if from any point M a perpendicular MP be let fall on C$x$, we shall have

$$\text{CP} = x, \quad \text{MP} = y.$$

By comparing these coordinates with the preceding, it appears that

$$p = \tfrac{1}{2}\pi a - y, \quad q = a - x,$$

($a$ denoting the diameter CD of the generating circle.) Hence, if these values be substituted in the differential equation of the cycloid, and if $a$ be put instead of $2c$, it will become

$$dy = \frac{(a-x)\,dx}{\sqrt{ax - x^2}}, \qquad \text{(a)}$$

from which we obtain($f$)

$$ds = \sqrt{\frac{a}{x}}\,dx,$$

and if the integral of this equation be taken so that it may vanish when $x = 0$, there results

$$s = 2\sqrt{ax},$$

for the length of the arc CM, of which the origin is at the summit. At the point A we have $x = a$, which gives, as before, $2a$ for the length of the semi-cycloid CMA It may be remarked here, that the equation $s^2 = 4ax$ of the cycloid is similar to that of the parabola, from which it only differs in this, that the ordinate $y$ is replaced by the arc $s$.

If the two last equations (2) be applied to the determination of the centre of gravity of the arc CM, we shall have

$$s x_1 = \int x \sqrt{\frac{a}{x}}\, dx, \qquad s y_1 = \int y \sqrt{\frac{a}{x}}\, dx,$$

in which the integrals must be taken in such a manner that they may vanish with $x$. By substituting for $s$ its value, there results

$$2 x_1 \sqrt{x} = \int \sqrt{x}\, dx, \qquad 2 y_1 \sqrt{x} = \int \frac{y dx}{\sqrt{x}}.$$

Hence we obtain

$$x_1 = \tfrac{1}{3} x,$$

from which it appears, that the centre of gravity of an arc M′CM, which is symmetrical on each side of the summit C, and which must therefore exist on the line CD, is distant from the point C, by one-third of CP, reckoning from the point C. By integrating by parts, we obtain(*g*)

$$\int \frac{y dx}{\sqrt{x}} = 2 y \sqrt{x} - 2 \int \sqrt{x}\, dy.$$

Therefore, if we substitute for $dy$ its value given by equation (a), we shall have

$$y_1 \sqrt{x} = y \sqrt{x} - \int \sqrt{a - x}\, dx,$$

and, consequently,

$$y_1 = y + \frac{2}{3\sqrt{x}} \left[ (a - x)^{\frac{3}{2}} - a\sqrt{a} \right],$$

and this combined with the value of $x_1$, completely determines the centre of gravity of the arc CM In the case of the semicycloid, we have $x = a$ and $y = \frac{1}{2}\pi a$, hence there results

$$x_1 = \tfrac{1}{3} a, \quad y_1 = a\left(\tfrac{1}{2}\pi - \tfrac{2}{3}\right).$$

74. When a plane curve revolves about a line comprised in its plane, which line may be taken as the axis of the ab-

scissæ, it will generate a surface of revolution, the area of which can be expressed in terms of the length of this curve and of the ordinate of its centre of gravity.

In order to demonstrate this, let $x$ and $y$ be the abscissa and ordinate of M any point of this curve, and $s$ the arc CM terminating at this point, and measured from the fixed point C, the element $ds$ will generate the surface of a truncated cone, and its middle point will describe a circumference equal to $2\pi(y+\frac{1}{2}dy)$, or simply to $2\pi y$, because $dy$ is infinitely small. By the known rule, therefore, $2\pi y ds$ expresses the element of this surface. Hence if $s_0$ and $s_1$ denote the values of $s$, which answer to the two extremities of the generating curve, and S the surface generated, we shall have by the theorem of No. 13,

$$S = 2\pi\int_{s_0}^{s_1} y ds.$$

It should be observed here, that this expression implies that the generating curve is not intersected by the axis of $x$, if it is, then its parts situated on the opposite sides of this axis will describe two different surfaces, of which S expresses only the difference. With this restriction, the expression will always obtain even when the generating curve is one which returns into itself, and in order to apply it to this case, it will be sufficient to substitute for $s_1$ the arc $s_0$ increased by the entire circumference of this curve.

This being established, we obtain by comparing this formula with the third equation (2),

$$S = 2\pi l y_1,$$

which shews that S the surface generated is equal to $l$ the length of the generating curve, multiplied by $2\pi y_1$ the circumference described by the centre of gravity. This theorem will enable us to determine the value of S at once, without any computation, and from the mere inspection of the curve, whenever the centre of gravity of the generating curve is known; it

will not be of any use, if it is necessary to compute the ordinate $y_1$, since this computation would be the same as that of s. Let us suppose, for example, that the generating curve is a circle, if we denote its radius by $a$, the distance of its centre from the axis of rotation by $c$, we shall have, on the supposition that $c$ is not less than $a$,

$$l = 2\pi a, \quad y_1 = c,$$

and, consequently,

$$s = 4\pi^2 ac.$$

When the circle touches the axis of rotation, we shall have $c = a$, and the surface generated will be equivalent to a square, of which the side is equal to $2\pi a$, the circumference of the generating circle($g$).

## II.—*Centres of Gravity of Surfaces.*

75. Let $x$, $y$, $z$, be as before, the coordinates of any point M, and $x_1$, $y_1$, $z_1$, those of the centre of gravity which it is required to determine, let $z$ be considered as a given function of $x$ and $y$, and therefore

$$\frac{dz}{dx} = p, \quad \frac{dz}{dy} = q,$$

then if $\omega$ be the element of the given surface which corresponds to the point M, we shall have (No. 21)

$$\omega = dxdy\sqrt{1 + p^2 + q^2}.$$

In whatever point of $\omega$ the centre of gravity of this element exists, its coordinates differ by infinitely small quantities from $x, y, z$, we may therefore take $\omega x$, $\omega y$, $\omega z$, for the moments of $\omega$ with respect to the three planes of the coordinates, and consequently there will result (Nos. 13 and 65)

$$\lambda = \iint \omega, \quad \lambda x_1 = \iint x\omega, \quad \lambda y_1 = \iint y\omega, \quad \lambda z_1 = \iint z\omega,$$

$\lambda$ denotes the area of the portion of the surface, of which the centre of gravity is required, and the double integrals are supposed to extend to all the elements of $\lambda$.

In the case of a plane surface, if it be assumed to be that of the axes of $x$ and $y$, the quantities $p$ and $q$ will be cypher, and we shall have only to consider the three equations

$$\lambda = \iint dxdy, \quad \lambda x_1 = \iint xdxdy, \quad \lambda y_1 = \iint ydxdy.$$

If $\lambda$ be terminated by the curve ABC (fig. 26), to each abscissa $x$ or OP, there corresponds two ordinates PM and PN, which we shall denote by $y$ and $y'$, and which will be given in functions of $x$, by the equation of this curve; also if $a$ and $\beta$ be the abscissæ OD and OE of the points A and B where the tangents are parallel to the ordinates, then the integrals should be taken first from $y' =$ PN to $y =$ PM, and then from $x = a$ to $x = \beta$; by this means there will result

$$\left.\begin{aligned} \lambda &= \int_a^\beta (y-y')\,dx, \\ \lambda x_1 &= \int_a^\beta (y-y')\,xdx, \\ \lambda y_1 &= \tfrac{1}{2}\int_a^\beta (y^2-y'^2)\,dx. \end{aligned}\right\} \qquad (1)$$

If the area $\lambda$, instead of being circumscribed by the reentrant curve ABC, is comprised between two different curves and between two right lines parallel to O$y$ the axis of the ordinates, the value of $y$ should be deduced from the equation of the superior curve, and that of $y'$ from the equation of the inferior curve, and the distances of these two parallels from the point O should be assumed for $a$ and $\beta$. In the case of most frequent occurrence, O$x$, the axis of the abscissæ, replaces the inferior curve; consequently we shall have $y' = o$, and simply

$$\lambda = \int_a^\beta ydx, \quad \lambda x_1 = \int_a^\beta yxdx, \quad \lambda y_1 = \tfrac{1}{2}\int_a^\beta y^2dx, \qquad (2)$$

by means of which the area and centre of gravity of a portion of a plane surface comprised between a given curve, the axis of the abscissæ and two ordinates of this curve, can be determined. It may be remarked, that equations (1) can be also obtained in the following manner:—Let the area ABC be divided into elements such as MNN′M′ infinitely small and parallel to the axis O$y$. Let $u$ denote the length of the line MN; if through its two extremities there be drawn lines parallel to the axis O$x$, then whether triangles, infinitely small of the second order, be added or subtracted from the element MNN′M′, its magnitude will not be changed, consequently this element will be equal to $udx$. If $v$ is assumed to denote the distance of the middle point of MN from the axis O$x$, $x$ and $v$ may be taken for the two coordinates of the centre of gravity of this element; for it is evident that they only differ from it by infinitely small quantities. Therefore, from the other notations already referred to, we shall have

$$\lambda = \int_a^\beta udx, \quad \lambda x_1 = \int_a^\beta xudx, \quad \lambda y_1 = \int_a^\beta vudx. \tag{3}$$

Moreover, as $y$ and $y'$ denote always the coordinates PM and PN which refer to the same abscissa, we have likewise

$$u = y - y', \quad v = \tfrac{1}{2}(y + y'),$$

from which it appears that these last formulæ coincide with equations (1).

76. For the first example, let the centre of gravity of the triangle ABC (fig. 27) be required.

Let the origin of the coordinates be placed at the vertix C, and let the axis of $x$ be perpendicular to the base AB; if we represent this base by $b$, and the height CD by $h$, and if through any point P in the line CD, there be drawn MN perpendicular to this line, CP and MN will be the variables $x$ and $u$, and we shall have the proportion

$$u : x :: b : h,$$

from which we deduce

$$u = \frac{bx}{h}.$$

We shall have besides, $\alpha = o$, $\beta = h$. By means of these values, the two first equations (3) will give

$$\lambda = \tfrac{1}{2}bh, \quad \lambda x_1 = \tfrac{1}{3}bh^2;$$

from which results

$$x_1 = \tfrac{2}{3}h.$$

It will not be necessary to calculate the value of $y$; for if E is the middle point of AB, and if the line CE be drawn it will bisect all the elements of the triangle parallel to AB, and will, consequently, contain its centre of gravity. Hence, if on CD there be taken a part

$$\text{CF} = \tfrac{2}{3}\,\text{CD} = x_1,$$

and if FG be erected perpendicular to CD, the point G in which it meets CE will be the centre of gravity of the triangle. As the line FG cuts CD and CE into proportional parts, we shall also have

$$\text{CG} = \tfrac{2}{3}\,\text{CE};$$

from which it appears that the centre of gravity of a triangle exists on the line which joins its vertix with the middle point of the base, and its distance from the vertix is two-thirds of this line, and consequently its distance from the middle point of the base is one-third of the same line.

77. This theorem may be also demonstrated without the aid of the integral calculus in the following manner:

In fact, as it has been proved by the decomposition of the triangle ABC (fig. 28) into elements parallel to AB that its centre of gravity exists on the line CD, which connects the vertix C with D, the middle point of this side, so, by decomposing the triangle into elements parallel to the side CA, it may be shown in the same manner, that this centre of gravity exists

also on the line BE drawn from the vertix B to E the middle point of CA; consequently this point must be at G, the intersection of the two lines CD and BE. But, if the line DE be drawn, it will be parallel to CB, since it cuts CA and AB into proportional parts, from which there results

DE CB AD AB 1:2,

DG CG : DE CB · 1.2,

so that DG will be half of CG, and consequently the third of CD, which it was proposed to demonstrate.

It is evident from this, that the three lines which are drawn from the three angles of a triangle to the points of bisection of the opposite sides, intersect in the same point, which is agreeable to a known theorem.

If the vertices A, B, C, of the triangle are the centres of gravity of three equal masses, the centre of gravity of these three bodies will coincide with that of the triangle, for, first, the centre of gravity of the two masses which answer to A and B is found at D, the point of bisection of the line AB; and then the centre of gravity of these two masses and of the third will be G, a point in the line CD, so situated, that GD is half of CG, or a third of CD.

It follows from this, and from the theorem of No. 67, that if there be applied to G the centre of gravity of a triangle, forces represented in magnitude and direction by the lines GA, GB, GC, drawn from this point to the three vertices, these three forces will be in equilibrio.

78. Knowing the centre of gravity of a triangle, it is easy to deduce successively those of a circular sector or segment.

Let CADB (fig. 29) be the sector, and C the centre of the circle If the arc ADB be considered as a portion of a polygon of an infinite number of equal sides, the sector may likewise be decomposed into equal triangular elements which will have all these sides for bases, and their common vertix at the point C.

Let the force which acts on each of these elements be applied at their respective centres of gravity; and as the distance from the point C of each centre of gravity is two-thirds of the radius of the circle, there will result a system of equal and parallel forces applied to all the elements of the arc A′D′B′ described from the point C as centre, and with a radius equal to $\frac{2}{3}$ CD. Consequently, the centre of gravity of the sector will be the centre of these parallel forces, that is to say, the centre of gravity of this arc A′D′B′. Now, if $a$, $l$, $c$ denote respectively the radius CD, the arc ADB and the chord AB, the analogous quantities corresponding to A′D′B′, will be $\frac{2}{3}a$, $\frac{2}{3}l$, $\frac{2}{3}c$, therefore if G be the required centre of gravity, we shall have by the theorem of No 70,

$$x = \frac{2ac}{3l},$$

($x$ being equal to CG).

Now let S, S′, $S_1$, denote the surfaces of the sector CADB, of the triangle CAB, and of the segment ADBE; if their centres of gravity, which will evidently be on the radius CD, terminating at D, the middle of the arc ADB, be G, G′, $G_1$, and if $x$, $x'$, $x_1$, denote the distances of these three points from the centre C, when parallel forces and proportional to S, S′, $S_1$, are applied to them, the first will be the resultant of the two others; therefore, in considering the moments of these forces, we shall have

$$Sx = S'x' + S_1x_1.$$

Besides, we have

$$S = \tfrac{1}{2}al, \quad x = \frac{2ac}{3l}.$$

And if CE, the altitude of the triangle of which the base is AB or $c$, be denoted by $h$, we have likewise

$$S' = \tfrac{1}{2}ch, \quad x' = \tfrac{2}{3}h.$$

Hence, as

$$\text{s}_1 = \text{s} - \text{s}' = \tfrac{1}{2}(al - ch),$$

the equation of moments will become

$$\tfrac{1}{3}a^2c = \tfrac{1}{3}ch^2 + \tfrac{1}{2}(al - ch)\,x_1\,;$$

by means of which, $x_1$ the distance of the centre of gravity of the segment ADBE, from the centre of the circle, can be determined, for since

$$c = 2a\sin\frac{l}{2a}, \quad h = a.\cos\frac{l}{a},$$

we can deduce (*i*)

$$x_1 = \frac{4a^2\sin^3\frac{l}{2a}}{3\left(l - a\sin\frac{l}{a}\right)}.$$

When the arc $l$ is the semi-circumference, we have $l = \pi a$; the sector and the segment coincide, as also the distances $x$ and $x_1$, of which the common value is

$$x = x_1 = \frac{4a}{3\pi}.$$

79. If the three conic sections be taken successively for the curve to which formulæ (2) refer, the integrations can be effected by the known rules, and the values of $x_1$ and $y_1$, the two coordinates of the centre of gravity, may be obtained in a finite form. As this example has been merely adverted to as an exercise of the calculus, we will not enter into any details, but pass on to the determination of the centre of gravity of the area of the cycloid.

Let CPM (fig 25) be the segment of which it is required to determine the centre of gravity, if the abscissa CP and the ordinate PM be denoted by $x$ and $y$, as in equation ($a$) of No. 73, it is necessary that the integrals contained in formulæ (2) should vanish when $x = o$, and these formulæ will become, by integrating by parts,

$$\left.\begin{aligned} \lambda &= xy - \int xdy, \\ \lambda x_1 &= \tfrac{1}{2}x^2y - \tfrac{1}{2}\int x^2dy, \\ \lambda y_1 &= \tfrac{1}{2}xy^2 - \int xydy; \end{aligned}\right\} \quad (4)$$

the new integrals likewise vanishing at the same time as $x$.

In virtue of equation (a), we have

$$\int xdy = \int \sqrt{ax - x^2}\, dx;$$

but if N be the point where the ordinate MP meets the circle described on CD as diameter, this last integral expresses the semicircular segment CNP; therefore, if in order to abridge, the area of this semi-segment be denoted by $\gamma$, we shall have

$$\lambda = xy - \gamma.$$

In the case in which the point M coincides with the point A, we shall have

$$x = \text{CD} = a, \quad y = \text{DA} = \tfrac{1}{2}\pi a, \quad \gamma = \tfrac{1}{8}\pi a^2,$$

and consequently,

$$\lambda = \tfrac{3}{8}\pi a^2.$$

Hence it appears, that the area of the semi-cycloid is triple of that of the semicircle CND, of which the radius is $\frac{1}{2}a$, or in other words, the area of the entire cycloid is three times that of its generating circle. We shall have in like manner,

$$\int x^2dy = \int x\sqrt{ax - x^2}.dx,$$

or, what is the same thing,

$$\int x^2dy = \tfrac{1}{2}a\int \sqrt{ax - x^2}\, dx - \int(\tfrac{1}{2}a - x)\sqrt{ax - x^2}\, dx.$$

The last integral may be obtained at once, and because it must vanish when $x = 0$, we shall have($k$)

$$\lambda x_1 = \tfrac{1}{2}x^2y - \tfrac{1}{4}a\gamma + \tfrac{1}{6}(ax - x^2)^{\frac{3}{2}},$$

by means of which the value of $x_1$ may be obtained from that of $\lambda$.

In the case of the semi-cyloid CAD, in which we have at the same time

$$x = a, \quad y = \tfrac{1}{2}\pi a, \quad \gamma = \tfrac{1}{8}\pi a^2, \quad \lambda = \tfrac{3}{8}\pi a^2,$$

there results

$$x_1 = \frac{7a}{12},$$

for the distance of its centre of gravity from the axis $cy$. Hence the centre of gravity of the area of the entire cycloid is at the distance of seven-twelfths of the height CD from the summit C.

With respect to any *other* segment, such as CMP, the ordinate $y_1$ should also be determined, this, however, requires a much more complicated calculus.

80. In virtue of equation (a), we have

$$\int xydy = \int y\sqrt{ax - x^2}\, dx,$$

and as the value of $y$ given in No. 73 may be written as follows.

$$y = \int \frac{(\frac{1}{2}a - x)\, dx}{\sqrt{ax - x^2}} + \frac{a}{2}\int \frac{dx}{\sqrt{ax - x^2}},$$

if we assume

$$\int \frac{dx}{\sqrt{ax - x^2}} = z,$$

this integral being supposed, as in the other cases, to vanish when $x = 0$, we shall have

$$y = \sqrt{ax - x^2} + \tfrac{1}{2}az;$$

from which results($l$)

$$\int xydy = \tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3 + \tfrac{1}{2}a\int z\sqrt{ax - x^2}\, dx. \tag{5}$$

Because we made

$$\gamma = \int \sqrt{ax - x^2}\, dx,$$

we shall have by partial integration

$$\int z\sqrt{ax - x^2}\, dx = z\gamma - \int \gamma dz. \tag{6}$$

The expression for $\gamma$ may be written as follows,

$$\gamma = \tfrac{1}{4}a^2 \int \frac{dx}{\sqrt{ax - x^2}} - \int \frac{(\frac{1}{2}a - x)^2 dx}{\sqrt{ax - x^2}},$$

and by integrating the second term by parts, there results

$$\gamma = \tfrac{1}{4}a^2 \int \frac{dx}{\sqrt{ax - x^2}} - (\tfrac{1}{2}a - x)\sqrt{ax - x^2} - \int \sqrt{ax - x^2}\, dx;$$

hence we infer $(m)$

$$\gamma = \tfrac{1}{8}a^2 z - \tfrac{1}{2}(\tfrac{1}{2}a - x)\sqrt{ax - x^2},$$

and because $\sqrt{ax - x^2}\, dz = dx$, we shall consequently have

$$\int \gamma dz = \tfrac{1}{16}a^2 z^2 - \tfrac{1}{4}(ax - x^2)$$

If these values of $\gamma$ and $\int \gamma dz$ be substituted in equation (6), there will result $(n)$

$$\int z\sqrt{ax - x^2}\, dx = \tfrac{1}{16}a^2 z^2 - \frac{z}{2}\left(\frac{a}{x} - x\right)\sqrt{ax - x^2} + \tfrac{1}{4}(ax - x)^2,$$

which changes equation (5) into

$$\left.\begin{aligned} \int xydy = \tfrac{1}{8}a^2 x + \tfrac{5}{8}ax^2 - \tfrac{1}{3}x^3 \\ + \tfrac{1}{32}a^3 z^2 - \tfrac{1}{4}az(\tfrac{1}{2}a - x)\sqrt{ax - x^2}. \end{aligned}\right\} \quad (7)$$

By means of this value and that of $z$, namely:

$$z = arc\left(\cos = \frac{a - 2x}{a}\right),$$

the third equation (4) will not contain any unknown quantity, and will, therefore, make known the value of $y_1$, for any segment whatever such as CMP.

In the case of the semi-cycloid CAD, we shall have

$$x = a, \quad z = arc(\cos = -1) = \pi,$$

and formula (7) will be reduced to

$$\int xydy = a^3\left(\tfrac{1}{6} + \frac{\pi^2}{32}\right),$$

and because

$$y = \tfrac{1}{2}\pi a, \quad \lambda = \tfrac{3}{8}\pi a^3,$$

the third equation (4) will give

$$y_1 = \frac{\pi a}{4}\left(1 - \frac{4}{9\pi^2}\right);$$

which, together with the value of $x_1$ given in the preceding number, will completely determine the position of the centre of gravity.

81. Let s be the area of a zone of a surface of revolution, comprised between two planes perpendicular to its axis of figure. The centre of gravity of s will be on this axis, if this axis coincides with that of $x$, and if $x_1$ denotes the distance of this centre from the origin of the coordinates, $\alpha$ and $\beta$ being the distances of the two planes which bound s, from the same origin; the determination of the centre of gravity of this zone will be reduced to that of the value of $x_1$.

Let s be resolved into elements of which each is the surface of a truncated cone described by the infinitely small side of the generating curve, as in No 74; that which corresponds to the point M of this curve, the coordinates of which are $x$ and $y$, will be equal to $2\pi y\sqrt{dx^2 + dy^2}$, its centre of gravity will be also on the axis of $x$, and the distance of this point from the origin of the coordinates can be assumed to be equal to $x$, since it can only differ from $x$ by an infinitely small quantity. This being so, (by Nos 13 and 65,) we shall have

$$\left.\begin{aligned} s &= 2\pi\int_\alpha^\beta y\sqrt{1+\frac{dy^2}{dx^2}}\,dx, \\ sx_1 &= 2\pi\int_\alpha^\beta xy\sqrt{1+\frac{dy^2}{dx^2}}\,dx, \end{aligned}\right\} \quad (8)$$

$y$ being considered as a function of $x$, which is known from the equation of the generating curve.

If, for example, this curve is the arc of a circle, by placing the origin of the coordinates at the centre, we shall have

$$y = \sqrt{a^2 - x^2},$$

($a$ denoting the radius,) from which there results($o$)

$$s = 2\pi a \sqrt{\beta - a},$$

$$sx_1 = \pi a\,(\beta^2 - a^2),$$

and, consequently,

$$x_1 = \tfrac{1}{2}(\beta + a);$$

from which it appears, that the centre of gravity of a spherical zone, is in the middle point of the part of the diameter comprised between the two planes that terminate it, and which is perpendicular to these planes.

82. The cycloid will furnish us with two examples of the application of formulæ (8), by causing the arc CM to turn successively about the axis C$x$, and the axis C$y$.

In the first case, in virtue of equation (a) of No. 73, we shall have

$$s = 2\pi\sqrt{a}\int y\frac{dx}{\sqrt{x}}, \qquad sx_1 = 2\pi\sqrt{a}\int y\sqrt{x}\,dx,$$

the integrals being taken in such a manner, that they may vanish at the point C, where $x = 0$. By partial integration, and taking into account the value of $dy$, furnished by equation (a), there arises

$$s = 4\pi y\sqrt{ax} - 4\pi\sqrt{a}\int\sqrt{a - x}\,dx,$$

$$sx_1 = \frac{4\pi}{3}yx\sqrt{ax} - \frac{4\pi}{3}\sqrt{a}\int x\sqrt{a-x}\,.dx,$$

and, consequently,($p$)

$$s = 4\pi y\sqrt{ax} + \frac{8\pi}{3}\sqrt{a}\,(a-x)^{\frac{3}{2}} - \frac{8\pi}{3}a^2,$$

$$sx_1 = \frac{4\pi}{3}yx\sqrt{ax} + \frac{8\pi}{9}x\sqrt{a}\,(a-x)^{\frac{3}{2}}$$

$$+ \frac{16\pi}{45}\sqrt{a}\,(a-x)^{\frac{5}{2}} - \frac{16\pi}{45}a^3;$$

by means of which the surface generated by the arc CM, which is concave towards the axis of the figure, and the distance of

its centre of gravity from the point C, are determined. When this arc becomes the semicycloid CA, we have $x = a$ and $y = \frac{1}{2}\pi a$, and, consequently,

$$s = 2\pi a^2(\pi - \tfrac{4}{3}), \quad s x_1 = \frac{2\pi a^3}{3}(\pi - \tfrac{8}{15}).$$

In the second case, it is necessary, in order to be able still to apply equation (a) of No. 73, to transpose $x$ into $y$, in formulæ (8), which by this means will become($q$)

$$s = 2\pi \int x \sqrt{1 + \frac{dy^2}{dx^2}}\, dx,$$

$$s y_1 = 2\pi \int xy \sqrt{1 + \frac{dy^2}{dx^2}}\, dx,$$

$y_1$ is the distance from the point C, of the centre of gravity of S situated on the line C$y$, and the integrals are supposed to vanish at the point C, that is to say, when $x = 0$ By means of equation (a), we shall have

$$s = 2\pi \int x \sqrt{\frac{a}{x}}\, dx = \frac{4\pi}{3} x \sqrt{ax},$$

the value of $s y_1$ will be the same as that of $s x_1$ of the first case, and by dividing it by this value of S, the distance from the point C, of the centre of gravity of the surface generated by the arc CM, which is convex towards the axis of the figure, will be obtained. When this arc becomes the semi-cyloid CA, the surface generated will be equal to $4\pi a^2$; in which case also, the value of the distance $y_1$ will be

$$y_1 = \frac{a}{2}(\pi - \tfrac{8}{15}).$$

It may be remarked here, that when the same arc of a curve turns successively about two rectangular axes, which pass through one of its extremities, the value of the second member of the second equation (8) continues to be the same, consequently the distances of the centres of gravity of the two

surfaces which are generated, from this extremity, are in the inverse ratio of the areas of these surfaces.

83. If the curve ABC (fig 26) turns about the axis O$x$, which is comprised in its plane, but does not meet it, its surface will generate a solid of revolution, of which the volume, denoted by V, may be expressed by means of the area of this surface and of $y_1$ the ordinate of its centre of gravity.

For if all the notations of No. 75 be retained, it is easy to perceive that we shall have

$$\mathrm{V} = \pi \int_a^\beta (y^2 - y'^2)\, dx,$$

in fact, the infinitely small slice of this volume, generated by MNN'M' the element of the generating area, will be equal to $\pi y^2 dx - \pi y'^2 dx$ the difference of two cylinders of which the radii are PM and PN, and whose common altitude is $dx$, for we may neglect the infinitely small volumes of the second order, which are generated by the triangles that are added or taken from this element, by drawing through the points M and N lines parallel to the axis O$x$. Now by comparing this expression of V, with the third formula (1) of the number cited, we obtain($r$)

$$\mathrm{V} = 2\pi\lambda y_1;$$

from which it appears, that the volume generated by $\lambda$, the area of a plane curve, is equal to this area multiplied by $2\pi y_1$, the circumference of the circle described by its centre of gravity—a theorem analogous to that of No. 74. By means of this expression the volume V can be determined when the centre of gravity of $\lambda$ is known *à priori*. It will also subsist, when the generating surface, in place of being circumscribed by a reentrant curve, is comprised between two different curves, and between two perpendiculars to the axis of the figure, provided that this axis does not pass between these two plane curves.

If the generating area is a semicircle revolving about its diameter, the distance of its centre of gravity from this axis of rotation will be equal to $\frac{4a}{3\pi}$ (No 78), the radius being de-

noted by $a$; hence the length of the circumference described by this point will be $\frac{8a}{3}$, and as the area of the semicircle is $\frac{1}{2}\pi a^2$, we shall have

$$V = \frac{4\pi a^3}{3},$$

which is, in fact, the volume of the sphere.

If the reentrant curve ABC is an ellipse of which $a$ and $b$ are the two semiaxes, and $c$ the distance of its centre from the axis of rotation; the area $\lambda$ will be, as is well known, equal to $\pi ab$, and as its centre of gravity is evidently the centre of the figure, we shall have $y_1 = c$, hence there will result

$$V = 2\pi^2 abc,$$

whatever may be the inclination of one or other of the axes of the ellipse to the axis of rotation.

84. It is evident that the segment of the solid of revolution comprised between two planes passing through the axis of figure, is to the entire solid, as the angle contained between these two planes is to four right angles, or, which is the same thing, as the arc described between the two planes, by the centre of gravity of the generating area, is to the entire circumference $2\pi y_1$. Therefore, denoting the length of this arc by $l$, and the volume of the corresponding segment by L, we shall have (s)

$$L = l\lambda,$$

$\lambda$ being always the generating area, which, by hypothesis, is not traversed by the axis of rotation.

This formula may be extended in the following manner to other segments which do not belong to solids of revolution.

Let us suppose, in fact, that a plane curve moves without sliding or turning in its plane, and in such a manner that this plane may be constantly perpendicular to a given line, which may be either a plane curve or one of double curvature. In this motion, the same point of this plane will always remain

on the directrix, and the other points will describe curves similar to this line. Let $\lambda$, L, $l$, denote respectively the area of the generating curve, the volume generated by this surface, and the length of the curve described by its centre of gravity. If $l$ was the arc of a circle, L would be a segment of a solid of revolution, but in all cases $l$ may be divided into infinitely small parts, each of which will coincide with the osculating circle that corresponds to it. Let $a$ denote one of these parts, and $v$ the volume of the corresponding segment of L; then if it be supposed that the planes perpendicular to its direction, by which $v$ is terminated, intersect each other in a line, that does not traverse the area of the generating curve, this element $v$ of L will be a segment of a solid of revolution, and by the preceding equation we shall have

$$v = a\lambda.$$

Hence, by taking the sum of all the values of $v$, and observing that the factor $\lambda$ is constant, it will follow that the volume is equal to the product of $l$ and $\lambda$, as in the case of a solid of revolution The rule that results from this equation $\mathrm{L} = \lambda l$, is very useful in practice, and susceptible of a great number of applications, however, we should never forget that it does not obtain when the consecutive generating planes intersect on the surface generated, and form by their successive intersections what is termed by Monge, *arète de rebroussement*

85 The consideration of the centre of gravity furnishes likewise a rule for enabling us to determine the volume of a prism or of a cylinder with any base whatever, which is cut by a plane inclined to this base.

Let $\gamma$ be the area of a section of the cylinder perpendicular to its sides, $\lambda$ the area of the inclined section which terminates it, $\theta$ the angle of these two planes, $\omega$ any element whatever of $\lambda$, $\varepsilon$ its projection on the plane of $\gamma$, or the corresponding element of the area $\gamma$, which is itself the projection of $\lambda$. By the theorem of No. 10 we have

$$\gamma = \lambda \cos\theta, \quad \varepsilon = \omega \cos\theta.$$

This being agreed on, let $\lambda$ be the surface to which the general formulæ of No. 75 refer, and let $\theta$ be the inclination of its plane on that of $x$ and $y$. If the third of these formulæ be multiplied by $\cos\theta$, we shall have, (by making this constant factor to pass under the sign $\iint$) in virtue of the values of $\gamma$ and $\varepsilon$,

$$\gamma z_1 = \iint z\varepsilon.$$

Now, this double integral is the volume of the truncated cylinder comprised between the two sections $\gamma$ and $\lambda$, decomposed into elements infinitely small and perpendicular to $\gamma$, in which, however, it is always implied that these two sections do not mutually intersect, it follows, therefore, that the truncated cylinder is equal to a right cylinder having the same base $\gamma$, and for an altitude, $z_1$ the distance of the centre of gravity of the inclined section from this base.

This theorem is evident in the ordinary case, in which the base of the cylinder is a circle, and the inclined section an ellipse, for if through the centre of this curve a plane be drawn parallel to the base, the volume of the cylinder is not changed, for the segment which is cut off from it is evidently equal to that which is added to it.

If the areas denoted by $\gamma$ and $\lambda$ mutually intersect each other, the volume will consist of two segments, of which the integral $\iint z\varepsilon$ will express the difference and not the sum. When the cylinder is terminated by two inclined sections, whose areas do not intersect, it may be always divided into two parts, of which the common base and perpendicular to the sides of the cylinder does not intersect either the one or the other of these two sections; and as their centres of gravity exist on the same right line perpendicular to this base, it is evident that the entire volume will be equal to the area of this base multiplied by the mutual distance of these two points.

## III.—*Centres of Gravity of Volumes and of Bodies.*

86. The determination of the centre of gravity of a volume depends, in general, on several triple integrals; but there are some bodies for which the position of this centre is determined by simple integrals. It is these which we propose first to consider.

The centre of gravity of a cone or of a pyramid with any base whatever exists on the right line which connects its summit with the centre of gravity of the base, for this line meets all sections parallel to the base, in corresponding points which are their centres of gravity, and which can also be taken for the centres of gravity of the infinitely small elements of this body, that are parallel to its base. Consequently, the line in question contains the centre of gravity of the pyramid or of the cone, and there only remains to determine its position on this line.

Let $l$ and $\mathrm{x}$ be the area of the base and that of a parallel section, and let $h$ and $x$ denote the perpendiculars let fall from the vertix on their planes, it is evident that

$$\mathrm{x} : l \quad x^2 : h^2,$$

and, consequently,

$$\mathrm{x} = \frac{lx^2}{h^2}.$$

Moreover, we may take $\mathrm{x}dx$ for the element of the volume of the cone or of the pyramid, and if $\mathrm{v}$ denotes the entire volume, and $x_1$ the value of $x$ corresponding to the section which contains the centre of gravity, we can obtain, as in the preceding questions,

$$\mathrm{v} = \int_0^h \mathrm{x}dx, \quad \mathrm{v}x_1 = \int_0^h x\,\mathrm{x}dx$$

By substituting for $\mathrm{x}$ its value, and performing the integrations, there results

$$\mathrm{V} = \frac{bh}{3}. \quad \mathrm{V}x_1 = \frac{bh^2}{4},$$

from which we obtain

$$x_1 = \tfrac{3}{4}\,h.$$

But if through the centre of gravity a plane be drawn parallel to the base, it will cut the height $h$ and the line drawn from the summit to the centre of gravity of the base, into proportional parts, it follows, therefore, that the distance of the centre of gravity of a cone or of a pyramid with any base whatever, from the summit, is three-fourths of this line, and therefore one-fourth of it, reckoning from the base.

87. In the case of a triangular pyramid, this theorem may be demonstrated without the aid of the integral calculus. Let ABCD (fig. 30) be this pyramid. Likewise let E and F be the centres of gravity of the faces ACD and BCD, let the lines BF and AE be drawn, their productions will meet in H, the middle point of the side CD, and then in the plane AHB, let the lines AF and BE de drawn intersecting in a certain point G. This point will be the centre of gravity of the pyramid ABCD, for by resolving it into elements parallel to the base ACD, it is evident, as in the preceding number, that its centre of gravity must exist on the right line BE, and by resolving it into elements parallel to BCD, it is likewise evident that this point appertains to the line AF, and as these two lines exist in the same plane they must cut each other, consequently their intersection G will be the required centre of gravity

Now, in the triangle ABH, the line EF is parallel to the base AB, since it cuts the sides AH and BH into proportional parts, that is to say, in the ratio of one to three, reckoning from H; we shall have therefore

FG GA : EF AB EH AH

and, consequently,

FG . GA . . 1 3

so that FG is the third of GA, or the fourth of FA, which was to be proved.

Hence it may be shown that if A, B, C, D, the four summits of the pyramid, be the centres of gravity of equal masses, the point G will be the centre of gravity of these four masses, for it was already shown in (No. 77) that the point F is that of the three masses which are situated in B, C, D; and then the point G so taken that GF is the third of GA, will be the centre of gravity of these three masses and of the fourth.

It follows from this (No. 67) that if to the centre of gravity of the pyramid, forces be applied, represented in magnitude and direction by lines drawn from this point to the four summits, these four forces will be in equilibrio.

88. The centre of gravity of a triangular pyramid being thus determined, it is easy to deduce that of a pyramid or cone of any base whatever, by decomposing this base into a finite or infinite number of triangles, the centre of gravity of this pyramid or cone must exist at the same time in the line drawn from the summit to the centre of gravity of the base, and in that plane parallel to the base, which cuts all lines drawn from the summit to this base, in the ratio of three to four, measuring from the summit, which agrees with the result of No. 86

The centre of gravity of a spherical sector may be also deduced from it. In fact, if this sector be decomposed into an infinite number of pyramids, of which the common summit is at the centre of the sphere, and whose bases are the infinitely small elements of the base of the sector, their centres of gravity must all exist on the base of a concentric sector, the radius of which will be three-fourths of that of the given sector, hence we infer that the centre of gravity of the given sector will be the same as that of the base of the concentric sector, by means of which consideration its position may be determined

Let the spherical sector be generated by the circular sector CADB (fig. 29) revolving about the radius CD, which is drawn

to the middle of the arc AB. The triangle CAB and the circular segment ADB will generate, at the same time, a cone and a spherical segment, and the centre of gravity of this segment can be determined when those of the spherical sector and of the cone are known.

For this purpose, let $V_1$, $V$, $V'$, denote the respective volumes of these bodies, and $x_1$, $x$, $x'$, the distances of their centres of gravity from the point C, we shall have

$$V = V' + V_1, \quad Vx = V'x' + V_1 x_1. \tag{a}$$

Let $a$ denote the radius CD, $c$ the chord AB, and $f$ the sagitta DE of the arc ADB. Relatively to the cone, we shall have ($r$)

$$V' = \tfrac{1}{12}\pi c^2 (a-f), \quad x' = \tfrac{3}{4}(a-f).$$

The base of the spherical sector will be equal to the product of the sagitta and of the circumference of the great circle, or to $2\pi af$, ($s$) and the value of its volume will be the product of this base and of $\frac{1}{3}a$, or $\frac{2\pi a^2 f}{3}$. If from the point C as centre, and with a radius equal to $\frac{3}{4}$CD, an arc of a circle, such as A'D'B', be described, the centre of gravity of the surface generated by this arc will be in the middle point of the sagitta D'E' (No. 81); or, in other words, at a distance from the point C equal to CD' $- \frac{1}{2}$D'E', the value of which is $\frac{3}{4}(a - \frac{1}{2}f)$ ($t$). Therefore, as by what has been stated, this centre of gravity is that of the spherical sector V, we shall have

$$V = \frac{2\pi a^2 f}{3}, \quad x = \tfrac{3}{4}(a - \tfrac{1}{2}f).$$

By substituting these different values in equations (a), it becomes

$$\tfrac{2}{3}\pi a^2 f = \tfrac{1}{12}\pi c^2 (a-f) + V_1,$$
$$\tfrac{1}{2}\pi a^2 f(a - \tfrac{1}{2}f) = \tfrac{1}{16}\pi c^2 (a-f)^2 + V_1 x_1;$$

by means of which the values of $V_1$ and $x_1$ can be determined. If $l$ denote the length of the arc AB, we shall have ($v$)

$$c = 2a \sin\frac{l}{2a}, \quad f = a\left(1 - \cos\frac{l}{2a}\right),$$

hence there results

$$\text{v}_1 = \frac{2\pi a^3}{3}\left(1 - \cos\frac{l}{2a} - \tfrac{1}{2}\sin^2\frac{l}{2a}\cos\frac{l}{2a}\right),$$

$$x_1 = \frac{3a\sin^4\frac{l}{2a}}{8\left(1 - \cos\frac{l}{2a} - \frac{1}{2}\sin^2\frac{l}{2a}\cos\frac{l}{2a}\right)}.$$

When the arc $l$ is the semi-circumferance, it is equal to $\pi a$, and, consequently,

$$\text{v} = \frac{2\pi a^3}{3}, \quad x_1 = \frac{3a}{8}.$$

89. The volume and centre of gravity of every body which is symmetrical with respect to its axis, as for example an ellipsoid, may likewise be determined by simple integrals. Let $x$, $y$, $z$, be the three rectangular coordinates of any point of the surface, let the axis of the figure be taken for that of $x$, and let $\text{x}$ denote the area of the section perpendicular to this line, which corresponds to the extremity of the abscissa $x$. If the volume be decomposed into infinitely small elements perpendicular to the axis of the figure, $\text{x}dx$ may be assumed equal to the volume of any element, and $x$ may be taken as the distance of its centre of gravity from the origin of the coordinates. Therefore, if we denote by $\text{v}$ a slice comprised between two sections corresponding to the given abscissæ $\alpha$ and $\beta$, and by $x_1$ the distance of its centre of gravity from the origin of the coordinates, we shall have

$$\text{v} = \int_\alpha^\beta \text{x}dx, \quad \text{v}x_1 = \int_\alpha^\beta x\text{x}dx.$$

In the case of the ellipsoid, the equation of the surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$a$, $b$, $c$, denoting the three semiaxes. Those of the section X will be ($x$)

$$b\sqrt{1-\frac{x^2}{a^2}},\quad c\sqrt{1-\frac{x^2}{a^2}},$$

hence we shall have

$$\mathrm{X} = \pi bc\left(1-\frac{x^2}{a^2}\right),$$

and, consequently,

$$\mathrm{V} = \pi bc\,(\beta-\alpha)\left(1-\frac{\alpha^2+\alpha\beta+\beta^2}{3a^2}\right),$$

$$\mathrm{V}x_1 = \tfrac{1}{2}\,\pi bc\,(\beta^2-\alpha^2)\left(1-\frac{\alpha^2+\beta^2}{2a^2}\right),$$

from which we deduce

$$x_1 = \frac{3\,(\alpha+\beta)\;(2a^2-\alpha^2-\beta^2)}{4(3a^2-\alpha^2-\alpha\beta-\beta^2)}.$$

If this formula be applied to the spherical segment which has been considered in the preceding number, we should take

$$\beta = a\cos\frac{l}{2a},\quad \alpha = a,$$

which gives

$$x_1 = \frac{3a\left(1+\cos\frac{l}{2a}\right)\sin^2\frac{l}{2a}}{4\left(1-\cos\frac{l}{2a}+\sin^2\frac{l}{2a}\right)},$$

this expression may be shown to coincide with the value of $x_1$ already found, by multiplying both its numerator and denominator by $1-\cos\dfrac{l}{2a}$.

In order to obtain the entire value of the ellipsoid, it is necessary to make $\beta = a$, and $\alpha = -a$, which gives

$$\mathrm{V} = \frac{4\pi abc}{3}.$$

This value of the volume is also furnished by the triple integral $\iiint dx\,dy\,dz$, extended to all the elements of the space terminated by the surface of the ellipsoid, but by making

$$x = ax', \quad y = by', \quad z = cz',$$

the equation of this surface becomes

$$x'^2 + y'^2 + z'^2 = 1,$$

and the triple integral is changed into

$$abc\iiint dx'dy'dz'.$$

This new integral should be extended to all the elements of the space circumscribed by the surface which answers to the preceding equation, consequently it will express the volume of the sphere of which the radius is unity; and as this volume is equal to $\frac{4\pi}{3}$; it follows that $\frac{4\pi abc}{3}$ expresses, as before, the volume of the ellipsoid.

90. Bodies which are symmetrical with respect to an axis, comprise solids of revolution. In what follows respecting such bodies, the axis of the figure will always be taken for that of the abscissæ $x$. If then a solid of this nature be supposed to be generated by a plane area comprised between two given curves, and the perpendiculars to the axis of $x$ which correspond to $x = \alpha$ and $x = \beta$, and if $y$ and $y'$ denote the ordinates of these curves with respect to the same abscissa $x$, it is necessary to make

$$\mathrm{X} = \pi\,(y^2 - y'^2),$$

in the formulæ of the preceding number, this gives

$$\mathrm{V} = \pi \int_\alpha^\beta (y^2 - y'^2)dx, \quad \mathrm{V}x_1 = \pi \int_\alpha^\beta (y^2 - y'^2)\,x dx$$

In the most usual case, namely, that in which the interior curve coincides with the axis of the figure, we shall have $y' = o$, and simply

$$\text{V} = \pi \int_a^\beta y^2 dx, \quad \text{V}x_1 = \pi \int_a^\beta y^2 x dx. \qquad \text{(b)}$$

The cycloid furnishes also, in this case, examples of the application of these formulæ, in which all the integrations may be effected under a finite form.

In the case of the convex solid generated by the revolution of the area CMP (fig. 25) about the axis C$x$, there will result by partial integration

$$\text{V} = \pi x y^2 - 2\pi \int xy dy,$$
$$\text{V}x_1 = \tfrac{1}{2}\pi x^2 y^2 - \pi \int x^2 y dy,$$

the integrals being so taken that they may vanish at the point C, that is, where $x = o$. Therefore, in virtue of equation (a) of No. 73, we shall have ($y$)

$$\text{V} = \pi x y^2 - 2\pi \int y \sqrt{ax - x^2}\, dx,$$
$$\text{V}x_1 = \tfrac{1}{2}\pi x^2 y^2 - \pi \int xy \sqrt{ax - x^2}.\, dx;$$

and the calculations of these values can be effected by means of transformations similar to those of No. 80. In the case of the volume generated by the semicycloid CAB, we obtain

$$\text{V} = \frac{\pi a^3}{3}\left(\frac{9\pi^2}{16} - 1\right), \quad x_1 = \frac{(63\pi^2 - 64)a}{12(9\pi^2 - 16)}.$$

In the case of the convex solid, generated by the revolution of the area CMP about the axis C$y$, it is requisite previously to change $x$ into $y$ and $y$ into $x$ in equations (b); from which there will result

$$\text{V} = \pi \int x^2 dy, \quad \text{V}y_1 = \pi \int x^2 y dy,$$

$y_1$ being the distance of the centre of gravity, which exists on the axis C$y$, from the point C, and the integrals being supposed to vanish at the point C. Therefore, in virtue of equation (a) of the cycloid, we shall have

$$\text{V} = \pi \int x \sqrt{ax - x^2}\, dx, \quad \text{V}y_1 = \pi \int yx \sqrt{ax - x^2}\, dx.$$

The first integral can be obtained without any difficulty, and the second by transformations similar to those of No 80. In the case in which CM is an entire semicycloid, we shall have

$$\text{V} = \frac{\pi^2 a^3}{16}, \quad y_1 = \left(\frac{16}{9} + \frac{\pi^2}{4}\right)\frac{a}{\pi}.$$

91. Let now, $x_1$, $y_1$, $z_1$, be the rectangular coordinates of the centre of gravity of a body of any form whatever, homogeneous or heterogeneous, of which the mass is represented by M. From what has been already stated in No. 65, it appears, that in order to determine these three unknown quantities, M must be divided into indefinitely small parts, consequently the preceding *sums* must be changed into *integrals* in the second members of equations (1) of this number. By this means, we shall have

$$\text{M}x_1 = \int\!\!\int\!\!\int x dm, \quad \text{M}y_1 = \int\!\!\int\!\!\int y dm, \quad \text{M}z_1 = \int\!\!\int\!\!\int z dm; \qquad (1)$$

$dm$ being the differential element of the mass of the body corresponding to the coordinates $x$, $y$, $z$. Naming $\rho$ the density of this same element, and $dv$ its volume, we shall also have

$$dm = \rho dv.$$

The rectangular parallelopiped of which the adjacent sides are parallel to the axes of $x$, of $y$, and of $z$, and respectively equal to the differentials $dx$, $dy$, $dz$, may now be assumed for the element $dv$ of the volume, hence there will result

$$dv = dxdydz.$$

If the body be homogeneous, its density will be constant; and denoting its volume by V, we shall have

$$\text{M} = \rho \text{V};$$

thus equations (1) will become

$$\text{V}x_1 = \int\!\!\int\!\!\int x dv, \quad \text{V}y_1 = \int\!\!\int\!\!\int y dv, \quad \text{V}z_1 = \int\!\!\int\!\!\int z dv. \qquad (2)$$

If the body be heterogeneous, two different cases may occur; either this body will consist of homogeneous parts of a finite

magnitude, and the density will only vary from one part to the other, in which case we can apply equations (2) to each of them, and then the centre of gravity of the entire body can be had from knowing the centres of gravity of all its parts (No. 64). Or the density will vary by insensible degrees in the interior of the body, and then we should make use of equations (1), in which $\rho$ ought to be a given function of $x$, $y$, $z$.

We should however remark, that in the case both of a homogeneous and heterogeneous body, the division of the mass into infinitely small elements of which the densities are the same, or vary only by insensible gradations, supposes that this body is constituted of continuous matter. But this is not the case in nature, in which the bodies are, on the contrary, made up of detached material parts, and separated from one another by empty spaces, which in point of magnitude are comparable to the parts occupied by the matter. In the following chapter we will revert to this observation, and we will show that formulæ (1) and (2) may, notwithstanding, be applied to bodies as they exist in nature, just as if matter experienced no discontinuity in its interior.

(92) It will be sometimes necessary, in order to facilitate the integrations, to employ the polar coordinates of each element $dm$ in place of the coordinates $x$, $y$, $z$. In this case, if $r$ denotes its radius vector, $\theta$ the angle which it makes with the axis of the positive $xs$, and $\psi$ the angle comprised between the plane of these two lines and that of $x$ and $y$, we shall have (No. 9)

$$x = r.\cos\theta, \quad y = r\sin\theta.\cos\psi, \quad z = r.\sin\theta.\sin\psi.$$

We should also, at the same time, express $dv$ by means of the differentials of those new variables $r$, $\theta$, $\psi$. There are general formulæ for the transformation of independent variables in multiple integrals; but the expression of $dv$, namely,

$$dv = r^2.\sin\theta\, dr d\theta d\psi,$$

may be also found directly, as will be shown immediately.

If $\rho dv$ be put in place of $dm$ in equations (1), and if the values of $x$, $y$, $z$, in polar coordinates be also substituted in these equations, they will become

$$\left.\begin{aligned} \text{M}x_1 &= \iiint \rho r^3 \sin\theta \,.\, \cos\theta dr d\theta d\psi, \\ \text{M}y_1 &= \iiint \rho r^3 \sin^2\theta \,.\, \cos\psi\, dr d\theta d\psi, \\ \text{M}z_1 &= \iiint \rho r^3 \sin^2\theta \,.\, \sin\psi\, dr d\theta d\psi, \end{aligned}\right\} \quad (3)$$

to which should be joined the equation

$$\text{M} = \iiint \rho r^2 \sin\theta\, dr d\theta d\psi.]$$

As to the limits of these triple integrals, they will be different according as the origin of the coordinates is placed without or within the body. When this origin is itself one of the points of M, we should first integrate from $r = o$ to $r = u$, in which $u$ denotes a function of $\theta$ and $\psi$ given by the equation of the surface, this being done, we should then integrate from $\theta = 0$ and $\psi = 0$ to $\theta = \pi$ and $\psi = 2\pi$, beginning with either of the angles $\theta$ or $\psi$ as we please. Generally speaking, the limits will be more complicated, when the origin of the coordinates does not belong to the mass M. In this case, let $u$ and $u'$ represent two given functions of $\theta$ and $\psi$, $\omega$ and $\omega'$, two functions of $\psi$, and $a$ and $a'$, two given angles, if the question was respecting a portion of a body comprised, on one part, between two surfaces of which the equations are $r = u$ and $r = u'$; and on the other part, between conic surfaces, which have for a common axis that of $x$, their common summit being also the origin of the coordinates, and for equations $\theta = \omega$, $\theta = \omega'$; finally, between the two planes passing through this axis, and which make the angles $a$ and $a'$, with the fixed plane from which the angle $\psi$ is reckoned. We should integrate first from $r = u$ to $r = u'$, then from $\theta = \omega$ to $\theta = \omega'$, and finally, from $\psi = a$ to $\psi = a'$. For example, let us take for the two first surfaces those of two concentric spheres having their common centre at the origin of the coordinates, and of which the radii are $a$ and $a'$, at the same time, let the bases of the two cones be

circles, or in other words, let the angles $\omega$ and $\omega'$ be constant; moreover, let the density be solely a function of $r$, so that the portion of the body which is considered, appertains to a sphere composed of concentric strata infinitely slender, each of which has the same density throughout its entire extent, and which density varies from one stratum to another, according to a given function of the distance from the centre. By making, in order to abridge,

$$\int_{a}^{a'} \rho r^2 dr = \text{A}, \quad \int_{a}^{a'} \rho r^3 dr = \text{B},$$

and performing the integrations relative to $\theta$ and $\psi$, we obtain ($z$)

$$\text{M} = \text{A}\,(\alpha'-\alpha)\,(\cos\omega - \cos\omega'),$$
$$\text{M}x_1 = \tfrac{1}{2}\text{B}\,(\alpha'-\alpha)\,(\cos^2\omega - \cos^2\omega'),$$
$$\text{M}y_1 = \tfrac{1}{2}\text{B}\,(\sin\alpha' - \sin\alpha)\,(\omega'-\omega-\tfrac{1}{2}\sin 2\omega' + \tfrac{1}{2}\sin 2\omega),$$
$$\text{M}z_1 = \tfrac{1}{2}\text{B}\,(\cos\alpha - \cos\alpha')\,(\omega'-\omega-\tfrac{1}{2}\sin 2\omega' + \tfrac{1}{2}\sin 2\omega);$$

by means of which equations, the values of $x_1$, $y_1$, $z_1$, will be had, which, in this example, cannot be deduced from equations (1).

If the mass M forms a complete ring, so that we have $\alpha' = \alpha + 2\pi$, there will result $y_1 = 0$ and $z_1 = 0$, that is to say, the centre of gravity will be situated, as we know it should be, on the axis of this ring; the value of $x_1$, its distance from the centre of the sphere, of which this ring is a part, will be ($a'$)

$$x_1 = \frac{\text{B}\,(\cos\omega + \cos\omega')}{2\text{A}}.$$

If the sphere be homogeneous, the density $\rho$ being then constant, we shall have

$$\text{A} = \tfrac{1}{3}\rho\,(a'^3 - a^3), \quad \text{B} = \tfrac{1}{4}\rho\,(a'^4 - a^4).$$

When the portion of the ring that is destitute of matter disappears, we shall have $\omega = 0$, and finally, if it changes into a

spherical sector, we will also have $a = 0$; hence there will result

$$x_1 = \frac{3a'}{8}(1 + \cos\omega');$$

this accords with the value of the quantity denoted by $x$ in No. 88, as appears by observing that the value($b'$) of the sagitta denoted by $f$ will be $a'(1 - \cos\omega')$, and that the radius is $a'$.

93. In order to obtain the expression of $dv$, the differential of the volume, in terms of the differentials of the polar coordinates, let M (fig. 31) be supposed to be the point of which the coordinates are respectively $r$, $\theta$, $\psi$, so that O being their origin, OM will be the radius vector $r$, $\theta$ the angle MO$x$ comprised between this radius and a fixed axis O$x$, and $\psi$ the angle which the plane of these two right lines makes with a fixed plane drawn arbitrarily through the second. Let M′ be any point situated on the production of OM, of which let the radius OM′ be denoted by $r'$. From the point O as centre, and in the plane M′O$x$, let the arcs MN and M′N′, comprised between the two lines OMM′ and ONN′, be described, and let the angle NO$x$ be denoted by $\theta'$, finally, let the plane of this angle turn about the axis O$x$, and let $\psi'$ represent, in its new position, the angle which it makes with the fixed plane. In this motion, the area MM′N′N will generate a volume MM′NN′PP′Q′Q, which we shall represent by U. Now, this area being the difference of the two circular sectors M′ON′ and MON, is equal to

$$\tfrac{1}{2}(r'^2 - r^2)(\theta' - \theta).$$

Denoting the perpendicular let fall from its centre of gravity on the axis O$x$ by $u$, $u(\psi' - \psi)$ will be the length of the arc that this centre will describe about this line. Therefore, by the theorem of No. 84, we shall have

$$\text{U} = \frac{u}{2}(r' + r)(r' - r)(\theta' - \theta)(\psi' - \psi).$$

This being established, if the three dimensions of U become infinitely small, and if, in consequence, we make

$$r' - r = dr, \quad \theta' - \theta = d\theta, \quad \psi' - \psi = d\psi,$$

the factor $r' + r$ will, at the same time, be reduced to $2r$; we can also take for $u$ the perpendicular MH let fall from the point M on the axis O$x$, which is equal to $r.\sin\theta$, and which only differs from $u$ by an infinitely small quantity; finally, U will be changed into $dv$, and its value will be

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\psi,$$

which it was proposed to determine.

In fact, it is evident that this volume $dv$ may be considered as a rectangular parallelopiped, the three adjacent sides of which are MM$'$ or $dr$, the infinitely small arc MN, the centre of which is at the point O, and whose length is $rd\theta$, and the infinitely small arc MP, which has its centre at the point H, and for length $r.\sin\theta \, d\psi$.

MNQP, the base of this parallelopiped, is the element of the spherical surface of which the centre is at the point O and the radius equal to $r$. Therefore, representing it by $d\sigma$, we have

$$d\sigma = r^2 \sin\theta \, d\theta \, d\psi, \quad dv = d\sigma dr.$$

Denoting the element of the spherical surface, of which the radius is taken for unity by $d\omega$, we shall also have($c'$)

$$d\omega = \sin\theta \, d\theta \, d\psi, \quad dv = r^2 dr \, d\omega.$$

By integrating this expression of $d\omega$ from $\theta = 0$ and $\psi = 0$ to $\theta = \pi$ and $\psi = 2\pi$, we obtain $4\pi$ for the ratio of the surface of the sphere to the square of its radius, which is, in fact, its known value.

# CHAPTER VI.

## ON THE ATTRACTION OF BODIES.

### I.—*Formulæ relative to any Body whatever, and to a Sphere in particular.*

94. If a material point o (fig. 32) be subjected to the attractions of all the points of a body of any form whatever, by decomposing each of these forces into three others, acting in the direction of rectangular axes drawn arbitrarily through the point o, and then taking the sum of the positive or negative components which act in the direction of each axis, we shall obtain the three components, the resultant of which will express in magnitude and direction the entire attraction exerted on the point o. These three components will be the sum of an infinite number of infinitely small elements, extended to the entire mass of the attracting body, they will be expressed by means of triple integrals, and the calculation of these quantities will be similar to that of the coordinates of the centre of gravity of any body, with which we have been occupied in the last chapter; and, in fact, this is the reason why the subject of the attractions of bodies is introduced in this place.

This question is one which has engaged much of the attention of geometers, both on account of the difficulties which occur in the analytical investigations, as also on account of its connexion with the problem of the figure of the earth, and of the law of gravity at its surface; but in the present treatise we shall not enter on the considerations of these points in any detail, but shall merely restrict ourselves to giving those formulæ which are of most frequent occurrence, and some of their ap-

plications. For their more complete development, the reader is referred to the second volume of the *Mechanique Celeste of Laplace*, and to the memoir of the author on the *Attraction of Spheroids* inserted in the *Connaissance des Tems* for the year 1829.

95. Let O be a fixed point assumed in the interior of the attracting body; through this point let three rectangular axes OX, OY, OZ, be drawn, these will be the axes of the positive coordinates; let $x$, $y$, $z$, represent the coordinates of M, any point whatever of the attracting body, and $dm$ the differential element of its mass corresponding to this point M; likewise, let $\alpha$, $\beta$, $\gamma$, and $\mu$ denote the three coordinates and mass of the point O; and finally, let $u$ be the distance OM, so that

$$u^2 = (\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2.$$

The attraction exerted by $dm$ on $\mu$ will be in the direction of the line OM. If this force be proportional to the product of the two masses, and in the inverse ratio of the square of the distance $u$, we shall have, by denoting it by F,

$$F = \frac{f\mu dm}{u^2};$$

$f$ being a constant coefficient which expresses the intensity of the attractive force, relative to the units of distance and of mass. In order to obtain an accurate notion of this quantity $f$, let us conceive two bodies of any form and dimension whatever, the masses of which are equal, and respectively assumed for unity, and suppose that the attraction does not vary either in intensity or direction throughout the entire extent of these two bodies, so that it must be the same between any two elements whatever of their masses, equal respectively to $dm$ and $\mu$, as between $\mu$ and $dm$ the material points, which we are considering, when their distance OM is equal to unity; the force $f$ is the entire attraction which will then be exercised by one of these two bodies on the other.

The projections of the line OM, on the axes D$x$, D$y$, D$z$, are $\alpha-x$, $\beta-y$, $\gamma-z$; and by dividing them respectively by $u$, the cosines of the angles which determine the direction of the force F will be obtained; hence its three components are

$$\frac{\alpha-x}{u}F, \quad \frac{\beta-y}{u}F, \quad \frac{\gamma-z}{u}F,$$

and, $u$ being considered as a positive quantity, they will tend, according as they are positive or negative, to diminish or increase $\alpha$, $\beta$, $\gamma$, the three coordinates of the point O. Therefore, if we represent the three components of the entire attraction exercised on this point, by A, B, C, we shall obtain, by substituting for F its value, and observing that $\mu$ and $f$ are constant quantities,

$$\left.\begin{cases} A = \mu f \displaystyle\iiint \frac{\alpha-x}{u^3}\,dm, \\ B = \mu f \displaystyle\iiint \frac{\beta-y}{u^3}\,dm, \\ C = \mu f \displaystyle\iiint \frac{\gamma-z}{u^3}\,dm, \end{cases}\right\} \quad (1)$$

these triple integrals being supposed to extend to the entire mass of the attracting body.

Denoting the density of the element $dm$ by $\rho$, and its volume by $dv$, we shall have

$$dm = \rho\, dv.$$

In the general case, $\rho$ will be a given function of the coordinates of the point M; in the case of the homogeneity of the attracting body, it will become a constant quantity; $dv$ will be expressed by means of the differentials of the coordinates of M, and those are always selected which are most proper to facilitate the integrations.

96. The three triple integrals on which the values of A, B, C, depend, may be reduced to one only, by a very simple consideration.

The limits being supposed the same as in these integrals, let

$$\text{T} = \iiint \frac{dm}{u}.$$

Since these limits are independent of the point O, if T be differenced with respect to its coordinates, these differentiations may be effected under the sign $\int$ (No. 14); and as we have also

$$\frac{d\frac{1}{u}}{d\alpha} = \frac{x-\alpha}{u^3}, \quad \frac{d\frac{1}{u}}{d\beta} = \frac{y-\beta}{u^3}, \quad \frac{d\frac{1}{u}}{d\gamma} = \frac{z-\gamma}{u^3},$$

there will result

$$\frac{d\text{T}}{d\alpha} = \iiint \frac{x-\alpha}{u^3}\, dm,$$

$$\frac{d\text{T}}{d\beta} = \iiint \frac{y-\beta}{u^3}\, dm,$$

$$\frac{d\text{T}}{d\gamma} = \iiint \frac{z-\gamma}{u^3}\, dm,$$

by means of which, equations (1) may be changed into

$$\text{A} = -\mu f \frac{d\text{T}}{d\alpha}, \quad \text{B} = -\mu f \frac{d\text{T}}{d\beta}, \quad \text{C} = -\mu f \frac{d\text{T}}{d\gamma}, \qquad (2)$$

so that the calculation of the three components A, B, C, depends only on one integral T. In determining it, it is material to recollect, that the denominator $u$ must have constantly the same sign in the entire extent of the integration, and that it should be positive, if we wish that the components A, B, C, should tend to diminish or increase the coordinates of O, according as their values furnished by equations (2) are positive or negative.

If instead of an attractive, the point O is acted on by a repulsive force, it will be only necessary to change the signs of the values of A, B, C, or, what comes to the same thing, to regard $f$ as a constant negative quantity. In the case in which

the force, whether attractive or repulsive, that acts on the point O, does not vary, as has been supposed, in the inverse ratio of the square of the distance, and that, consequently, the coefficient of $\mu dm$ is, in general, represented by a given function of $u$, which we can denote by $\phi u$, if we assume another function of $u$ as $\Phi u$, such that we may have

$$\frac{d\Phi u}{du} = -\phi u,$$

then this should be substituted in place of $u$, in the expression of T. It may also happen that this force is attractive for one part of the body that acts on O, and repulsive for another part, in which case the function $\phi u$, in which is comprised the coefficient $f$, would change its sign, in the extent of the integral represented by T.

The components of the action exercised on a body of any *form* or *dimension* whatever, may be deduced from the preceding formulæ, by substituting for $\mu$ the differential element of its mass, which corresponds to the coordinates $\alpha$, $\beta$, $\gamma$, and then integrating with respect to these three variables for the entire extent of the mass; from this it appears, that the components of the action which one body exerts on another, will, generally speaking, depend on *sextuple* integrals.

The preceding are the formulæ, by means of which attractions or repulsions are computed, it is, however, necessary, previously to our making any application of them, to explain how they can be adapted to the intimate constitution of bodies, and to examine the difficulty which was adverted to at the end of No. 91.

97. It was already observed (No. 60) that different bodies contain, under equal volumes, unequal quantities of ponderable matter, and as these quantities vary, for the same body, with its temperature and the exterior pressure to which it is subjected, philosophers have been led to regard natural bodies as a collection of material parts not in contact, but separated from

each other by *pores* or spaces destitute of ponderable matter. These material parts are termed *atoms*; in consequence of their extreme minuteness, their dimensions, and those of the pores, elude our senses, and all our means of measuring them. The atoms are considered as indestructible, and the mass, the form, the volume of each of them, as invariable. The dimensions of the pores, on the contrary, vary with the different degrees of heat, which is either introduced into or expelled from the body, and with the pressures to which it is subjected; and as the changes in the volume of a body may be very great, without its mass undergoing any increase or diminution, it follows that the dimensions of the parts void of matter must be comparable, and generally greater than those of the full parts.

Atoms of the same or of different natures combine in different proportions to constitute other parts of bodies, which are likewise inappreciable by the senses, and which are termed *molecules*. Bodies differ from each other in the nature and proportion of the atoms which enter into the composition of each molecule; and the atoms are regarded as invariable and indestructible, as has been already stated, because that when they are reunited in the same proportions, the same bodies, endowed with the same properties, are invariably reproduced.

98. It is evident from this, that the division of the mass into infinitely small elements, and the hypothesis of a density in each element, which does not vary at all in homogeneous bodies, and which, in the case of heterogeneous bodies, varies by insensible degrees, is not the condition of bodies as they actually exist in nature; however this does not prevent us from making use of the formulæ which are founded on this consideration, and even from applying them, when the bodies have been divided into parts of a finite, but still insensible magnitude.

In fact, the molecules are so small, and so near to each other, that a part of the mass of a body, which contains in-

mense numbers of them, may still be supposed extremely small, and its volume regarded as insensible. Let $v$ be the volume of such a part, the magnitude of which is insensible, and which, nevertheless, contains myriads of molecules; likewise, let $m$ be the sum of their masses, and let M denote one of the points of $v$, which can, if we please, be taken for its centre of gravity. If we make

$$\frac{m}{v} = \rho,$$

this ratio $\rho$ will really express the density of the body at the point M, (a) whatever may be in other particulars the masses of the molecules and their manner of distribution, whether regular or irregular, through the extent of $v$. In like manner, denoting the number of the molecules that $v$ contains by $n$, and making

$$\frac{v}{n} = \varepsilon^3,$$

this line $\varepsilon$, of inappreciable magnitude, may be termed the *mean interval of the molecules*, which corresponds to the point M and the density $\rho$. In a homogeneous body, this ratio and this line do not vary with the position of the point M; in a heterogeneous body these two quantities vary by insensible gradations, and may be supposed to be given functions of the coordinates of this point.

This being established, if it were required to know the mass of a body, or more generally, the sum of the extremely small parts of this mass, multiplied by U, a function of the coordinates of M one of its points; V the volume of this body, should be divided into extremely small parts such as $v$, and then the sum of all the products $\mathrm{U}\rho v$, which we shall denote by

$$\Sigma \mathrm{U}\rho v,$$

should be taken, and extended to all the parts such as $v$ of V. It appears from the theorem of No. 13, that if the terms of this

sum are infinitely small, and if their number be infinite, its value will be rigorously equal to the definite integral

$$\int \mathrm{U}\rho dv,$$

extended to the entire volume v, of which $dv$ is the differential element Now, we may conceive in general that the difference between this sum and this integral will diminish more and more according as the parts of the first become smaller, and their number becomes greater, so that the magnitude of $v$ being insensible, but still always distinct from $dv$, we may nevertheless assume, without sensible error, the integral, in place of the sum. There is, however, one exception to this general principle, namely, when U is of the species of functions which vary very rapidly, and at the same time changes its sign in the extent of the integration, this is the case, in point of fact, in the calculation of the forces which arise from molecular attraction and from calorific repulsion, which are only sensible at insensible distances. But for the present, it is sufficient to observe, that this exception has no respect to the formulæ of Nos. 91 and 95, relative to the centres of gravity of bodies, and to attractions varying in the inverse ratio of the square of the distances, and that we can consequently apply them to natural bodies made up of detached molecules.

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99. Let us now revert to the calculation of attractions. If the distance of the point O from the attracted body, is very great relatively to the dimensions of this body, we can, in the expression for T of No 96, develope the quantity $\frac{1}{u}$ in a converging series, arranged according to the powers and products of $x$, $y$, $z$. By making

$$\alpha^2 + \beta^2 + \gamma^2 = \delta^2,$$

we shall then have

$$\frac{1}{u} = \frac{1}{\delta} + \frac{\alpha x + \beta y + \gamma z}{\delta^3} + \frac{3(\alpha x + \beta y + \gamma z)^2 - (x^2 + y^2 + z^2)\delta^2}{2\delta^5} + \&c. \qquad (b)$$

$$T = \frac{M}{\delta} + \frac{A' + B + C}{2\delta^3} - 3 \ldots + \&c$$

If D the origin of the coordinates be in the centre of gravity of the body, we shall have

$$\iiint xdm = 0, \quad \iiint ydm = 0, \quad \iiint zdm = 0,$$

because these integrals, divided by M, the mass of the body, will be the three coordinates of this point (No 91) Therefore if this mass be denoted by M, we shall have

$$\text{T} = \frac{\text{M}}{\delta} + \frac{3}{2\delta^5}\iiint (ax + \beta y + \gamma z)^2\, dm$$
$$- \frac{1}{2\delta^3}\iiint (x^2 + y^2 + z^2)\, dm + \&c.$$

When the distance OD or $\delta$ is so great that this value of T may be reduced to its first term, equations (2) will become

$$\text{A} = \frac{\mu\text{M}fa}{\delta^3}, \quad \text{B} = \frac{\mu\text{M}f\beta}{\delta^3}, \quad \text{C} = \frac{\mu\text{M}f\gamma}{\delta^3}.$$

Now these components are the same as those of a force equal to $\frac{\mu\text{M}f}{\delta^2}$ acting at the point 0, in the direction OD; it follows therefore, that the attractions exerted on the point 0, by a body which is at a considerable distance from it, is very nearly the same, in magnitude and direction, as if M the mass of this body was condensed in its centre of gravity

When the body is a homogeneous sphere, or one composed of concentrical strata, we shall find that all the terms of the value of T, except the first, destroy each other. In order to demonstrate this, it is only necessary to substitute $r, \theta, \psi$, in place of the coordinates $x, y, z$, as in No. 92, by means of which the integrations relative to $\theta$ and $\psi$ can be effected. Therefore, the theorem which has been stated above, will then be altogether exact, if the distance $\delta$ is only so great that the development of $\frac{1}{u}$ may be a convergent series; and in fact, we shall see in the following number, without having recourse to a reduction into a series, that this theorem obtains, what-

ever may be the distance of the point O from the attracting sphere, provided that it is not situated in its interior. It is easy to infer from this, that the attraction of one sphere on another is the same as if the mass of each sphere was condensed in its centre, for, denoting by M and M′ the masses of the two spheres, and by C and C′ their centres, the attraction of M on O, any point of M′, is the same as if the mass M was condensed into the point C, moreover, this attraction of C on all the points, such as O of M′ is equal and contrary to the attraction of all these points, or of M′ on C, which is the same as if the mass M′ was condensed in the point C′, consequently, the attraction of the two spheres is the same as that of two material points situated in C and C′, and of which the masses are M and M′.

100. The attraction exerted on the point O, by a spherical stratum, which is homogeneous and of a constant thickness, will evidently be reduced to a force acting in the direction OD. Hence, if this line be made to coincide with the axis D$x$, the components B and C parallel to the axes D$y$ and D$z$ will vanish, and there will only remain the value of A to compute. In this computation, by making use of the polar coordinates $r, \theta, \psi$, as in No. 92, we shall have, since the axis D$x$ coincides with the line DO,

$$\text{ODM} = \theta, \quad \text{DO} = a, \quad \beta = 0, \quad \gamma = 0;$$

and as DM $= r$ and OM $= u$, there will result

$$u^2 = a^2 - 2ar\cos\theta + r^2.$$

Because the angle $\psi$ is that which the plane ODM makes with a fixed plane passing through the line DO, we shall have (No. 93)

$$dv = r^2.\sin.\theta\, dr\, d\theta\, d\psi,$$

for the element of the volume, and in the expression $\rho dv = dm$ the element of the mass, $\rho$ can be regarded as a constant factor. After having substituted these values in the expression for T of No. 96, we should integrate from $r = b$ to $r = a$, ($a$

and $b$ denoting the exterior and interior radii of the spherical stratum) and from $\theta = 0$ and $\psi = 0$ to $\theta = \pi$ and $\psi = 2\pi$. As the variable $\psi$ does not occur under the sign $\int$, the integration with respect to this variable, is in fact effected by substituting $2\pi$ for the differential $d\psi$. This being so, we shall have

$$\text{T} = 2\pi\rho \int_b^a \left( \int_0^\pi \frac{r \sin \theta d\theta}{\sqrt{a^2 - 2ar \cos \theta + r^2}} \right) r dr.$$

At the limits $\theta = 0$ and $\theta = \pi$, the radical will have for values

$$\pm (a - r), \quad \pm (a + r),$$

but as it expresses the value of $u$, which ought to be always positive (No. 96), it is necessary to assume $a + r$ at the limit $\theta = \pi$, and $r - a$ or $a - r$ at the limit $\theta = 0$, according as the point 0 will be situated within or without the spherical stratum. We shall see immediately how we ought to proceed, when this point appertains to the stratum itself, in which case we have $r > a$ in one part, and $r \angle a$ in another part of this stratum.

The indefinite integral being, relatively to $\theta$,

$$\int \frac{r \sin \theta d\theta}{\sqrt{a^2 - 2ar \cos \theta + r^2}} = \frac{1}{a} \sqrt{a^2 - 2ar \cos \theta + r^2} + \text{const.},$$

when the point is within the stratum, we must have($c$)

$$\int_0^\pi \frac{r \sin \theta d\theta}{\sqrt{a^2 - 2ar \cos \theta + r^2}} = \frac{1}{a} [(r + a) - (r - a)] = 2;$$

consequently, the value of T will not all depend on $a$, and that of A which is deduced from it by means of the first equation (2), will be equal to zero. In the case where the point is without the stratum, we shall have

$$\int_0^\pi \frac{r \sin \theta d\theta}{\sqrt{a^2 - 2ar \cos \theta + r^2}} = \frac{1}{a} [(a + r) - (a - r)] = \frac{2r}{a},$$

and, consequently,

$$T = \frac{4\pi\rho}{a}\int_b^a r^2 dr = \frac{4\pi\rho\,(a^3 - b^3)}{3a},$$

or, what comes to the same thing,

$$T = \frac{M}{a},$$

M being the mass of a spherical stratum, of which the volume is $\frac{4\pi\,(a^3 - b^3)}{3}$ Hence we deduce

$$A = \frac{\mu M f}{a^2}, \tag{3}$$

which is the same force as if the entire mass of this attracting stratum was condensed in its centre.

101. These results may be immediately extended to the case of a spherical stratum of a constant thickness, but composed of other concentrical strata, the density of which varies from one to the other, according to any law whatever, which however, does not change in the extent of the same stratum; for we can determine separately the attractions of these different strata, and then take the sum of all these forces, which in the case of an interior point will be cypher, and in the case of an exterior point will be determined by formula (3); M always expressing the entire mass of the attracting body. Hence we infer

1st. That when the force varies in the inverse ratio of the square of the distance, the attractions exerted by all the points of a spherical stratum of a constant thickness, (which is either homogeneous or composed of concentrical strata, the density of which varies from one to the other according to any given law,) on a point o, existing in the void space circumscribed by this stratum, mutually destroy each other, so that this point will remain in equilibrio, wherever it may be situated in this space.

2ndly. That the attraction of this same stratum, and consequently also, the attraction of an entire sphere exerted on o a point without the sphere, is the same as if the mass of the at-

$T = \frac{4\pi\rho a^3}{3a}$

tracting body was condensed into its centre If the point O makes a part of the attracting stratum, or, in other words, if we have $a > b$ and $a \angle a$, this spherical stratum may be conceived to consist of two others, the exterior and interior radii of one of which will be $a$ and $a$, and those of other $a$ and $b$; as the point O is within the first of these two strata, it will exert no action on this point, and if the mass of the second stratum, without which the point O exists, be denoted by $m$, the attraction of this stratum may be deduced from formula (3), by substituting $m$ in place of M. Therefore, the value of the entire attraction exerted on the point O will be

$$\text{A} = \frac{\mu m f}{a^2}$$

If the spherical stratum changes into a sphere entirely full of matter, and of which the density is every where the same, we shall have

$$m = \frac{4\pi\rho a^3}{3}, \quad \text{A} = \frac{4\pi\mu f\rho a}{3},$$

that is to say, in the interior of a homogeneous sphere, the attraction is proportional to the distance of the attracted point from its centre

The same theorems obtain in the case where the force is repulsive, provided it varies in the inverse ratio of the square of the distances.

102. The equilibrium of the point O, situated within the space bounded by a spherical stratum, and which is attracted or repelled by all its points, may be easily verified.

For this purpose, let this stratum be first supposed infinitely thin, if its thickness be denoted by $\varepsilon$, by decomposing its surface into infinitely small elements, and denoting the area of that which corresponds to the point P (fig. 33) by $\omega$, the corresponding elements of the volume and of the mass of this stratum will be $\varepsilon\omega$ and $\rho\varepsilon\omega$; hence denoting the distance OP by $r$, the value of the force acting in the direction of this line will be

$$\frac{\mu f \rho \varepsilon \omega}{r^2}$$

If we conceive the sides of a cone of which the base is $\omega$ and the summit 0, to be produced through O until they meet the spherical surface again in P′, a second element will be determined on this surface; if we denote it by $\omega'$, and its distance from the summit O by $r'$, the value of the force acting along this line, in an opposite direction from the preceding, will be

$$\frac{\mu f \rho \varepsilon \omega'}{r'^2},$$

now it is easy to show that these two opposite forces are equal to each other, that is to say, that

$$\frac{\omega}{r^2} = \frac{\omega'}{r'^2}$$

In fact, if POQ and P′OQ′ be sections of the two cones made by the same plane passing through their common summit O, then the similar surfaces $\omega$ and $\omega'$ will be to each other as the squares of the homologous lines PQ and P′Q′. Besides, as the triangles POQ and P′OQ′ are similar, we have

$$PQ \quad P'Q' . \quad OP \quad OP',$$

therefore, by squaring the four terms of the proportion, we shall obtain

$$\omega \quad \omega' . . r^2 : r'^2,$$

from which it is evident that $\dfrac{\omega}{r^2} = \dfrac{\omega'}{r'^2}$.

Hence it follows, that the actions exerted on the point 0, by all the elements of the spherical stratum, destroy each other two by two, consequently the total action of this stratum will be cypher, and this will still be the case even when the thickness of the stratum is finite; for in this case it may be decomposed into an infinite number of infinitely thin strata, the action of each of which on the point 0 is cypher.

## II. *Formulæ relative to the Ellipsoid.*

103. When the point 0 (fig. 32) belongs to the attracting mass, the integrations will frequently be facilitated by placing the origin of the *polar* coordinates at this point. The radius vector of any point M will be then $u$, hence denoting, as in No. 93, the element of the spherical surface of which the radius is unity, by $d\omega$, we shall have,

$$dv = u^2 du d\omega, \quad dm = \rho u^2 du d\omega\ ,$$

and if the angles which the line OM makes with the parallels to the axes D$x$, D$y$, D$z$, drawn through the point O, be denoted by $g, h, k$, we shall likewise, agreeably to the notation of No. 95, have

$$\cos g = \frac{x-\alpha}{u}, \quad \cos h = \frac{y-\beta}{u}, \quad \cos k = \frac{z-\gamma}{u},$$

this will change equations (1) of this number into the following($d$),

$$\begin{aligned} A &= -\mu f \iiint \rho \cos g\, du d\omega, \\ B &= -\mu f \iiint \rho \cos h\, du d\omega, \\ C &= -\mu f \iiint \rho \cos k\, du d\omega. \end{aligned}$$

The integrals relative to $u$ should be taken from $u = o$ to $u = r$, denoting by $r$ the radius vector of any point of the attracting surface which terminates the attracting body. If, for greater simplicity, this body be supposed to be homogeneous, these integrations can be effected at once, and there will result

$$\left.\begin{aligned} A &= -\mu f \rho \iint r \cos g\, d\omega, \\ B &= -\mu f \rho \iint r \cos h\, d\omega, \\ C &= -\mu f \rho \iint r \cos k\, d\omega. \end{aligned}\right\} \quad \text{(a)}$$

In order to determine the value of $r$, which should be substituted in these formulæ, let

$$F(x, y, z) = 0,$$

be the equation of the surface of the attracting body, expressed in rectangular coordinates. At any point of this surface, we have

$$x = a + r\cos g, \quad y = \beta + r\cos h, \quad z = \gamma + r\cos k,$$

as is evident from the preceding values of $\cos g$, $\cos h$, $\cos k$, the three coordinates of the point o, of which the values are given, being always $a$, $\beta$, $\gamma$. Hence, if these values of $x, y, z$ be substituted in the preceding equation, there will result in general, two values of $r$, the one positive and the other negative, but the negative value should be rejected, since the radius vector $r$ is a positive quantity, the direction of which is solely determined by the angles $g, h, k$, which may be acute or obtuse. After the substitution of the value of $r$ in equations (a), the double integral should be extended to all the elements, such as $d\omega$ of the spherical surface, described from the point o as centre, and with a radius equal to unity.

104. Let these formulæ be applied to the case of an homogeneous ellipsoid, the equation of whose surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{b}$$

$a, b, c$, denoting the three semi-axes, and D the origin of the coordinates, being at the centre of the figure. If the preceding values of $x, y, z$, be substituted for them in this equation, there results($e$)

$$pr^2 + 2qr = l,$$

by making, in order to abridge,

$$\frac{\cos^2 g}{a^2} + \frac{\cos^2 h}{b^2} + \frac{\cos^2 k}{c^2} = p,$$

$$\frac{a\cos g}{a^2} + \frac{\beta\cos h}{b^2} + \frac{\gamma\cos k}{c^2} = q,$$

$$1 - \frac{a^2}{a^2} - \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2} = l.$$

Consequently we shall have

$$r = \frac{-q \pm \sqrt{q^2+pl}}{p}.$$

Now, as the quantity $p$ is positive, and as the quantity $l$ is also either positive or cypher, (since the point O, which answers to the coordinates $\alpha$, $\beta$, $\gamma$, is situated in the interior of the ellipsoid, or at farthest at its surface,) in order that the radius $r$ may not be negative, the radical must be affected with the sign $+$. Moreover, this radical may be suppressed in formulæ (a). In fact, the corresponding part of the integral contained in A, for example, will be

$$\iint \frac{1}{p}\sqrt{q^2+pl}\cos g d\omega,$$

but for each couple of elements such as $d\omega$, of which the radii are the productions the one of the other, the elements of this double integral mutually destroy each other, for in passing from one of these elements $d\omega$ to the other, each of the three cosines $\cos g$, $\cos h$, $\cos k$, changes its sign, the quantities $p$, $l$, $q^2$, remain the same, hence the values of the coefficient of $d\omega$ under the sign $\int$, are equal, but affected with contrary signs Thus all the elements of the preceding integral destroy each other two by two, and the value of A becomes, having regard to the value of $q$ ($f$)

$$\text{A} = \mu f \rho \left(\frac{\alpha}{a^2}\iint \frac{\cos^2 g}{p} d\omega + \frac{\beta}{b^2}\iint \frac{\cos g \ \cos h}{p} d\omega \right.$$
$$\left. + \frac{\gamma}{c^2}\iint \frac{\cos g \cos k}{p} d\omega\right).$$

Now, the two last of these three integrals are composed of couples of elements which correspond to the same values of $h$ and of $k$, and to values of $g$, which are supplements, the one of the other. Hence, each of these couples of elements will be reduced to cypher, and consequently also, the entire integrals. Therefore, by suppressing these integrals and by

making the values of B and C undergo similar reductions, we shall have simply

$$\text{A} = \frac{\mu f \rho \alpha}{a^2} \iint \frac{\cos^2 g}{p} d\omega,$$

$$\text{B} = \frac{\mu f \rho \beta}{b^2} \iint \frac{\cos^2 h}{p} d\omega,$$

$$\text{C} = \frac{\mu f \rho \gamma}{c^2} \iint \frac{\cos^2 k}{p} d\omega$$

Now let $\theta$ be the angle comprised between the radius OM and the parallel to the axis O$x$, drawn through the point D, and $\psi$ the angle which the plane of these two lines makes with a plane passing through the second and parallel to that of the axes of $x$ and $y$, we shall have (No. 8)

$$\cos g = \cos\theta, \quad \cos h = \sin\theta . \cos\psi, \quad \cos k = \sin\theta . \sin\psi,$$

and at the same time (No. 93)

$$d\omega = \sin\theta d\theta d\psi;$$

from which there will result($g$)

$$a^2 b^2 c^2 p = b^2 c^2 \cos^2\theta + (c^2 \cos^2\psi + b^2 \sin^2\psi)\, a^2 \sin^2\theta,$$

$$\text{A} = \frac{\mu f \rho a}{a^2} \iint \frac{\cos^2\theta \sin\theta d\theta d\psi}{p}.$$

In order to include the directions of all the radii OM, the integrals should be extended from $\theta = 0$ and $\psi = 0$ to $\theta = \pi$ and $\psi = 2\pi$, but because the coefficient of $d\theta$ has the same value for $\theta$ and for $\pi - \theta$, it will be sufficient to integrate from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, and to double the result, and since the coefficient of $\psi$ is the same for $\psi$ and for $\pi \pm \psi$, it will likewise suffice to integrate from $\psi = 0$ to $\psi = \frac{1}{2}\pi$, and to quadruple the result. This being agreed on, if we make

N

$$\phi = \tan\psi, \quad d\phi = \frac{d\psi}{\cos^2\psi},$$

then since

$$\cos^2\psi = \frac{1}{1+\phi^2}, \quad \sin^2\psi = \frac{\phi^2}{1+\phi^2},$$

there results$(h)$

$$\int_0^{\frac{1}{2}\pi}\frac{d\psi}{p} = a^2b^2c^2\int_0^{\infty}\frac{d\phi}{(b^2\cos^2\theta+a^2\sin^2\theta)c^2+(c^2\cos^2\theta+a^2\sin^2\theta)b^2\phi^2}$$

$$=\frac{\pi a^2 bc}{2\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)(c^2\cos^2\theta+a^2\sin^2\theta)}},$$

by means of which the value of A will only depend on the integral relative to $\theta$. If we substitute $\beta$ in place of $a$, and transpose the letters $a$ and $b$, we can deduce B from A without any new computation; and in the same manner, by substituting $\gamma$ in place of $a$, and by transposing the letters $a$ and $c$ C may be deduced from A. In this manner, we shall have finally

$$\text{A}=4\pi\mu f\rho a\int_0^{\frac{1}{2}\pi}\frac{bc\cos^2\theta\sin\theta d\theta}{\sqrt{(b^2.\cos^2\theta+a^2\sin^2\theta)(c^2\cos^2\theta+a^2\sin^2\theta)}},$$

$$\text{B}=4\pi\mu f\rho\beta\int_0^{\frac{1}{2}\pi}\frac{ac\cos^2\theta\sin\theta d\theta}{\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)(c^2\cos^2\theta+b^2\sin^2\theta)}},$$

$$\text{C}=4\pi\mu f\rho\gamma\int_0^{\frac{1}{2}\pi}\frac{ab\cos^2\theta\sin\theta d\theta}{\sqrt{(b^2.\cos^2\theta+c^2\sin^2\theta)(a^2\cos^2\theta+c^2\sin^2\theta)}}.$$

= $\alpha$, A
(*) B, $\beta$
C, $\gamma$

These values of A, B, C, being positive, it follows that each of these three components tends to make the point 0 approach towards the centre of the ellipsoid, the contrary has place when the force is repulsive, in which case we should substitute, in these formulæ, $-f$ instead of $f$.

105. If we denote by $\delta$ a positive constant, and then substitute $(1+\delta)a$, $(1+\delta)b$, $(1+\delta)c$, instead of $a$, $b$, $c$, in formulæ (c), the factor $1+\delta$ will disappear, and the values of A, B, C will remain the same as before. But by this substitution, the ellipsoid is increased by a part comprised between its primitive surface and a similar surface, and as the components A, B, C, do not change, it follows that the action of this additive part on the point O within this part is cypher.

Hence it appears, that a homogeneous stratum, comprised between two similar elliptic surfaces, having the same centre, and their axes in the same directions, does not exert any attractive or repulsive action on a point o situated within the empty space terminated by its interior surface, so that wherever this point may be situated in this space, it will remain in equilibrio, a theorem which includes that which has been already established in the case of a *spherical* stratum.

From this it follows, that the action of an ellipsoid composed of homogeneous matter, on a point 0 of its mass, is reduced to that which is exerted by the part of this mass, that is terminated by the elliptic surface passing through this point, similar and similarly placed to that of the entire body. And it is evident from formulæ (c), that the component of this force, parallel to each of the three axes of the ellipsoid, is proportional to the ordinate of the point o parallel to this axis, and depends solely on this variable(*i*). In the general case, in which the three semi-axes $a, b, c$, are unequal, the integrals relative to $\theta$, which these formulæ contain, can be transformed into elliptic functions, this enables us to compute their numerical values, by means of the tables of M. Legendre. These same integrals may also be obtained in a *finite* form, when two of the constants $a, b, c$, are equal, which is, as we know, the case of an ellipsoid of revolution.

106. Suppose, for example, that we have $c = b$, the form of the integrals relative to $\theta$ will be different, according as the ellipsoid is oblate or prolate, that is, according as $b > a$ or $b \angle a$. If we suppose the first case to obtain, and make, on this hypothesis(*k*),

$$b^2 - a^2 = a^2 e^2, \quad \frac{4\pi\rho a^3 (1 + e^2)}{3} = m;$$

so that the fraction $e$ may be the compression of the ellipsoid, and $m$ its mass. There will result

$$\text{A} = \frac{3\mu f m a}{a^3} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta \sin\theta d\theta}{1 + e^2 \cos^2\theta},$$

and by performing the integration, we shall have

$$\text{A} = \frac{3\mu fma}{a^3 e^3}[e - arc\,(\text{tang} = e)],$$

for the component parallel to the axis of revolution. We shall also have

$$\frac{\text{B}}{\beta} = \frac{\text{C}}{\gamma} = \frac{3\mu fm}{a^3(1+e^2)}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta\sin\theta d\theta}{\sqrt{1+e^2\sin^2\theta}}.$$

As the components B and C are to each other as the ordinates $\beta$ and $\gamma$ of the point 0, it follows that their resultant will act in the direction of the perpendicular, let fall from this point on the axis of revolution. Naming A′ this force, and $a'$ the length of the perpendicular, so that

$$\text{A}' = \sqrt{\text{B}^2 + \text{C}^2}, \quad a' = \sqrt{\beta^2 + \gamma^2},$$

there results, by performing the required integration($l$)

$$\text{A}' = \frac{3f\mu ma'}{2a^3e^3}\left[\text{arc}\,(\text{tang} = e) - \frac{e}{1+e^2}\right].$$

The resultant of the two forces A and B will express, in magnitude and direction, the entire action of the ellipsoid on the point 0.

When $e$ is a very small fraction, these values of A and A′ may be developed into very convergent series arranged according to the powers of $e$. Because that

$$\text{arc}\,(\text{tang} = e) = \frac{e}{1} - \frac{e^3}{3} + \frac{e^5}{5} - \&\text{c.},$$

$$\frac{e}{1+e^2} = e - e^3 + e^5 - \&\text{c.},$$

we shall have($m$)

$$\text{A} = \frac{\mu fma}{a^3}\left(1 - \frac{3e^2}{5} + \&\text{c.}\right),$$

$$\text{A}' = \frac{\mu fma'}{a^3}\left(1 - \frac{6e^3}{5} + \&\text{c}\right).$$

In the case of the sphere, in which $e = 0$, the resultant of A and A′ will be directed towards the centre, and it will have the same intensity as in No 101.

107. The determination of the attraction of an homogeneous ellipsoid on a point without the ellipsoid presents much greater difficulties, but by means of a theorem, for which we are indebted to Mr. Evory, this case may be reduced to that of an interior point, by which we are enabled to express the components of the attraction by simple integrals similar to formulæ (c). The following is a demonstration of this important proposition. By making, in first equation (1) of No. 95,

$$dm = \rho dx dy dz,$$

and observing that $\rho$ is a constant factor, we have

$$A = \mu f \rho \iiint \frac{(a-x)\, dx\, dy\, dz}{[(a-x)^2 + (\beta-y)^2 + (\gamma-z)^2]^{\frac{3}{2}}}$$

If equation (b) is that of the surface, then by substituting $ax'$, $by'$, $cz'$, in place of $x$, $y$, $z$, it will be changed into

$$x'^2 + y'^2 + z'^2 = 1.$$

At the same time the value of A becomes

$$A = \mu f \rho abc \iiint \frac{(a - ax')\, dx'\, dy'\, dz'}{[(a-ax')^2 + (\beta - by')^2 + (\gamma - cz')^2]^{\frac{3}{2}}};$$

and if we denote by $\pm x_1$, the values of $x'$ equal and affected with contrary signs, which are deduced from the preceding equation, the integral relative to $x'$ should be taken from $x' = -x_1$, to $x' = x_1$, which gives

$$A = \mu f \rho bc \left( \iint \frac{dy' dz'}{[(a - ax_1)^2 + (\beta - by')^2 + (\gamma - cz')^2]^{\frac{1}{2}}} \right.$$
$$\left. - \iint \frac{dy' dz'}{[(a + ax_1)^2 + (\beta - by')^2 + (\gamma - cz')^2]^{\frac{1}{2}}} \right).$$

Each of these two double integrals should be extended to

all the elements of the spherical semi-surface of which the radius is unity, and whose centre is at the origin of the coordinates, the product $dy'dz'$ is the projection on the plane of the axes of $y$ and $z$, of any element whatever. Therefore, if $\theta$ denotes the angle which the radius that is drawn to this element, makes with the axis of $x_1$, and if $\psi$ is the angle comprised between the planes of these two right lines and the planes of the axes of $x$ and $y$, the area of this element will be $\sin\theta\, d\theta\, d\psi$, its inclination on the plane of the axes of $y$ and $z$ will be the angle $\theta$, and therefore there will result

$$dy'dz' = \cos\theta \sin\theta\, d\theta\, d\psi,$$

for its projection on this plane. We shall have at the same time

$$x_1 = \cos\theta, \quad y' = \sin\theta\cos\psi, \quad z' = \sin\theta\sin\psi.$$

The limits of the two integrals will be now $\theta = 0$, and $\psi = 0$, $\theta = \frac{1}{2}\pi$ and $\psi = 2\pi$, but if in the second, $\pi - \theta$ is substituted instead of $\theta$, it is easy to see that these two integrals will unite into one, which will have the same limits with respect to $\psi$, and of which the limits relatively to $\theta$ will become $\theta = 0$ and $\theta = \pi$, $(n)$ so that by making, in order to abridge,

$$\text{R}^2 = \alpha^2 + \beta^2 + \gamma^2 - 2(\alpha a\cos\theta + \beta b\sin\theta\cos\psi + \gamma c\sin\theta\sin\psi) \\ + a^2\cos^2\theta + b^2\sin^2\theta\cos^2\psi + c^2\sin^2\theta\sin^2\psi,$$

and considering it as a positive quantity, we shall have simply

$$\text{A} = \mu f\rho bc\int_0^{\pi}\int_0^{2\pi}\frac{\cos\theta\sin\theta\, d\theta\, d\psi}{\text{R}}.$$

The two other components B and C may in like manner be expressed by double integrals.

Let us now consider the attraction of another ellipsoid having the same density $\rho$, the same centre, and its axes in the same direction as those of the first, let $a_1, b_1, c_1$, denote the three semi-axes corresponding to $a, b, c$, and let $\text{O}_1$ be the

point which is subjected to this attraction, $\alpha_1, \beta_1, \gamma_1$, its coordinates, and $A_1$, $B_1$, $C_1$, the components of this force parallel to the three axes of the ellipsoid. If we denote as before the mass of the attracted point by $\mu$, we shall have

$$A_1 = \mu f b_1 c_1 \int_0^{\pi} \int_0^{2\pi} \frac{\cos\theta \sin\theta \, d\theta \, d\psi}{R_1};$$

$R_1$ being what $R$ becomes, when $a, b, c, \alpha, \beta, \gamma$, are changed into $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$, respectively The values of $B_1$ and $C_1$ may be obtained in a similar manner from those of $B$ and $C$. If the two ellipsoids have the same foci, and consequently the same excentricity, we shall have

$$b^2 = a^2 + h, \quad c^2 = a^2 + k, \quad b_1^2 = a_1^2 + h, \quad c_1^2 = a_1^2 + k;$$

$h, k, h-k$, being either positive or negative quantities which express, abstracting from the sign, the squares of the excentricities common to these two bodies Moreover, let us suppose that the point $O_1$ attracted by the second ellipsoid exists on the surface of the first, and the point $O$ attracted by the first, on the surface of the second. By equation (b) and that of the surface of the second ellipsoid, we must have

$$\left.\begin{aligned} \frac{\alpha_1^2}{a^2} + \frac{\beta_1^2}{b^2} + \frac{\gamma_1^2}{c^2} = 1, \\ \frac{\alpha^2}{a_1^2} + \frac{\beta^2}{b_1^2} + \frac{\gamma^2}{c_1^2} = 1. \end{aligned}\right\} \quad (1)$$

Finally, if $p$ and $q$ be two given angles, and if we assume

$$\left.\begin{aligned} \alpha_1 = a\cos p, \quad \beta_1 = b\sin p\cos q, \quad \gamma_1 = c\sin p\sin q, \\ \alpha = a_1\cos p, \quad \beta = b_1\sin p\cos q, \quad \gamma = c_1\sin p\sin q; \end{aligned}\right\} \quad (2)$$

these values will satisfy the two preceding equations, and will establish a particular relation between the coordinates of the points $O$ and $O_1$. If these values of $\alpha, \beta, \gamma$, be substituted in the preceding expression for $R^2$, there results by substituting also the preceding values of $b^2, c^2, b_1^2, c_1^2, (o)$

$$\begin{aligned}\text{R}^2 = a_1^2 + a^2 + h\,(\sin^2 p\cos^2 q + \sin^2\theta\,.\cos^2\psi)\\ + k\,(\sin^2 p\sin^2 q + \sin^2\theta\,.\sin^2\psi)\\ -2\,(a_1 a\cos p\cos\theta + b_1 b\sin p\cos q\sin\theta\cos\psi\\ + c_1 c\sin p\sin q\sin\theta\sin\psi).\end{aligned}$$

Now, it appears without writing down the value of $\text{R}_1^2$, that it will be the same as that of $\text{R}^2$, for it may be deduced from it by the transposition of $a$ and $a_1$, $b$ and $b_1$, $c$ and $c_1$, $h$ and $k$ not undergoing any change, as being quantities common to the two ellipsoids, hence it is evident, that this last formula is not changed by these transpositions Since then $\text{R}_1 = \text{R}$, the values of $\text{A}$ and $\text{A}_1$, will contain the same double integrals, therefore, by eliminating it, we obtain

$$\text{A}_1 bc = \text{A} b_1 c_1.$$

Similar results may be obtained relatively to the other components, so that from the suppositions which have been made about the two attracted points O and $\text{O}_1$, we shall have finally

$$\frac{\text{A}_1}{\text{A}} = \frac{b_1 c_1}{bc},\quad \frac{\text{B}_1}{\text{B}} = \frac{a_1 c_1}{ac},\quad \frac{\text{C}_1}{\text{C}} = \frac{a_1 b_1}{ab}. \tag{3}$$

In order to enable us to state the theorem which these three equations imply, let the two points on the surfaces of the two ellipsoids, of which the coordinates are in the ratio of the semi-axes, to which they are parallel, be termed *corresponding points*. The point $\text{O}_1$ of the surface of the first ellipsoid, of which the coordinates parallel to the semi-axes $a, b, c$, are $\alpha_1, \beta_1, \gamma_1$, will have for its correspondent on the surface of the second ellipsoid, the point O, of which the coordinates parallel to the semi-axes $a_1, b_1, c_1$, are $\alpha, \beta, \gamma$, since by equations (2), we have

$$\frac{\alpha_1}{\alpha} = \frac{a}{a_1},\quad \frac{\beta_1}{\beta} = \frac{b}{b_1},\quad \frac{\gamma_1}{\gamma} = \frac{c}{c_1}.$$

This being established, the following theorem results from equations (3).

If two ellipsoids have the same centre and foci, the attraction in the direction of each axis, which one of these two bodies exerts on a point situated on the surface of the other, is to the attraction of this last on the corresponding point of the surface of the first, as the product of the two other axes of the first ellipsoid to the product of the two other axes of the second.

108. When two ellipsoids have, as has been assumed, the same centre and foci, one of the two falls entirely within the other, consequently, if the point O is external with respect to the first ellipsoid, the point $O_1$, will be internal with respect to the second. In order to determine, by means of the preceding theorem, the attraction of a given ellipsoid on an external point O likewise given, the surface of a second ellipsoid having the same centre and foci as the first, is made to pass through this point, by the formulæ relative to internal points, $A_1$, $B_1$, $C_1$, the three components of the attraction of this second body on the point $O_1$ of the surface of the first, corresponding to the point O, can be determined, then equations (3) will make known A, B, C, the components of the attraction of the given ellipsoid on the given point. Thus the determination is reduced to finding the values of three semi-axes $a_1, b_1, c_1$, of the second ellipse, from knowing those of the first, (which are denoted by $a, b, c$,) and the coordinates $\alpha, \beta, \gamma$, of the given point O

For greater clearness, let $a$ be the least of the three quantities $a$, $b$, $c$, then the quantities $h$ and $k$ of the preceding number will be positive. If we denote the square of $a_1$ by $u$ we shall have

$$a_1 = \sqrt{u}, \quad b_1 = \sqrt{u+h}, \quad c_1 = \sqrt{u+k};$$

and it only remains to determine this unknown quantity $u$, which must be real and positive. Now, in virtue of the second equation (1), we shall have $(p)$

$$a^2 + \frac{\beta^2 u}{u+h} + \frac{\gamma^2 u}{u+k} = u; \qquad (4)$$

this equation which is of the third degree with respect to $u$ must have at least one *real* and *positive* root; for by supposing $u$ to increase from zero to infinity, its first member will be at the commencement greater, and afterwards less than the second, so that there must be at least one positive value of $u$, which renders them equal, moreover, there is *only one*, for if we suppose that there are two, $u$ and $u^1$, then we must have at the same time

$$\frac{\alpha^2}{u}+\frac{\beta^2}{u+h}+\frac{\gamma^2}{u+k}=1,$$

$$\frac{\alpha^2}{u'}+\frac{\beta^2}{u'+h}+\frac{\gamma^2}{u'+k}=1;$$

and by subtracting these equations, the one from the other, and suppressing the factor $u'-u$, which is common to all the terms, there will result

$$\frac{\alpha^2}{uu'}+\frac{\beta^2}{(u+h)(u'+h)}+\frac{\gamma^2}{(u+k)(u'+k)}=0;$$

which is evidently impossible

Therefore, there exists only one ellipsoid, which has the same centre and foci as a given ellipsoid, and which besides passes through a given point, and the quantity $u$, on which its three semi-axes $a_1$, $b_1$, $c_1$, depend, is determined by equation (4), which was proposed to be proved.

109 It may be remarked here, that the theorem of No. 107 is true for all laws of attraction, which are functions of the distance, for the demonstration given above is founded on the form that the expression of $R^2$ assumes, which is found to be identically the same for the two points O and $O_1$, and not on the form of the function of $R$, which expresses the law of attraction.

If the two ellipsoids become concentric spheres, the attraction of each will be same on *all* the points of the surface of the other, so that it will not be necessary that O and $O_1$ should be correspondent points. Denoting the radii of these two spheres

by $a$ and $a_1$, the attraction of the sphere of which the radius is $a$, on a point of the spherical surface of which the radius is $a_1$ by D, and the attraction of the sphere of which the radius is $a_1$ on a point of the spherical surface of which the radius is $a$ by $D_1$, these forces will act in the direction of the radii of the attracted points, and we shall have($q$)

$$D : D_1 : a^2 \; a_1^2,$$

whatever *function* of the distance expresses the law of the attraction.

It is easy to verify this proposition, in the case in which the attraction is in the inverse ratio of the square of the distance. In fact, it appears from the results of No. 101, that if we suppose $a > a_1$, D the attraction of the sphere of which the radius is $a$ on an internal point situated at the distance $a_1$ from its centre, and having a mass equal to $\mu$, will be

$$D = \frac{4\pi\mu f a_1}{3},$$

and $D_1$ the attraction of the sphere of which the radius is $a_1$ on an external point, of which the mass is also $\mu$, and whose distance from its centre is $a$, will have for value

$$D_1 = \frac{4\pi\mu f a_1^3}{3a^2};$$

and from a comparison of these values of D and $D_1$, it appears that they are in the ratio of the squares of the radii $a$ and $a_1$.

# BOOK THE SECOND.

## DYNAMICS.

## FIRST PART.

### CHAPTER I

#### OF RECTILINEAR MOTION AND OF THE MEASURE OF FORCES

##### I *Formulæ of rectilinear Motion*

110 THE simplest motion that can be impressed on a material point, is that in which it moves in a right line, in such a manner that the point describes equal spaces in equal times This *rectilinear* motion is termed *uniform*, and it is made use of in comparing every other description of motion

When the ratio of the spaces described to the times employed in describing them, is continually changing, the motion is said to be *variable*, when this change takes place after finite intervals of time, the motion will be a succession of uniform motions

In any motion whatever, the space described by the moveable, or more generally, its distance from a fixed point assumed on the line that it describes, is a function of the time which has lapsed from a given epoch. Thus, denoting this time by $t$ and this distance by $x$, we shall have in all cases,

$$x = \mathrm{F}t,$$

*

and the various descriptions of motion will differ from each other by the form of this function $ft$. The *variable* $t$ may be either positive or negative; its positive values will refer to epochs posterior to that from which the time is reckoned, and its negative values to anterior epochs.

In uniform motion, if $a$ denotes the space described in each unit of time, and $b$ the distance from the fixed point at the commencement of the motion, that is to say, the value of $x$ when $t = 0$, we shall have, at any instant whatever,

$$x = b + at,$$

for, by the definition of this motion, the space $x - b$ described in the time $t$, must be equal to the constant space $a$ repeated as often as there are units in $t$.

III. Neither time nor space can be defined; however, this is not of any consequence, because, for the purposes of geometry and dynamics, it is sufficient that we are able to measure the dimensions of bodies and the durations of their motions. The measurement of lengths may be easily conceived, being founded on the principle of superimposition; that of time, however, requires some explanation.

It would be to reason in a circle, to say, on the one hand, that uniform motion is that in which the spaces described are proportional to the times, and on the other, that the measure of time is the uniform motion, that is to say, that the time is proportional to the spaces described in this motion. But the notion of equal times and the measure of the time, does not necessarily imply any particular law of motion, and we can consequently suppose them in the definition of uniform motion, and of any other description of motion whatever.

In fact, we may conceive, that bodies perfectly identical move successively one after the other, and that, during the entire continuance of the motion, each of the bodies is precisely in the same state as that which precedes it: it is evident, that all these motions, of which the law is not given, are per-

formed in equal times, and that their number may be made use of to measure the time Thus, for example, if we suppose bodies, such as the preceding, to be heavy, and to be retained by a fixed horizontal axis, and if we make them all to deviate equally from the position of equilibrium, and then remit them to themselves, in such a manner, that the motion of the second may commence when the first has returned to this position, and that of the third as soon as the second has returned in the same way, and so on of the rest; there will be no possible difference whatever between all these successive motions, which will be performed in equal times. It will be proved hereafter, that it is not necessary for this, that *different bodies* should succeed each other, and, that the successive oscillations of the same body, on each side of its position of equilibrium, are also *isochronous*, or of equal duration; but the preceding consideration, which does not suppose the solution of any problem of mechanics, is sufficient for the object which it was adduced to explain.

The invariability of the apparent revolution of the celestial sphere about the earth, has been established by repeated and accurate astronomical observations, and in fact, theory does not indicate any sensible inequality in the rotatory motion of the earth, which is the cause of this apparent motion. The constant duration of this revolution is termed a *sidereal day*, which duration is less than the diurnal revolution of the sun. This last is not exactly the same at all epochs of the year; and it is its mean magnitude which is assumed as the unit of time in ordinary usages, which on that account is called the *mean day*. In this treatise, the division of the day into 24 hours, of the hour into 60 minutes, and of the minute into 60 seconds, will be adopted; so that the second will be the 86400th part of the *mean* day The *sidereal* day contains 86164,09 seconds, hence it follows, that in order to express in sidereal days a portion of time which is given in mean days, it should be multiplied by the ratio of 86400 to 86164,09, or by the constant number 1,0027379.

112. One uniform motion differs from another, by the magnitude of the space described in the unit of time. In *each* uniform motion this constant space is what is termed the *velocity* of the moveable, indeed, strictly speaking, this space is not the velocity itself, but only the measure of the velocity. The velocity of a material point in motion is a quality which inheres in this point, by which it is actuated, and which distinguishes it from a material point at rest; it is not susceptible of any other definition. The velocity, which in uniform motion is expressed by the space described by the moveable in each unit of time, supposes that we assume for the unit of velocity that of the moveable which describes the linear unit in the unit of time.

In any variable motion whatever, the velocity of the moveable varies by infinitely small degrees, and is a function of the time which may be deduced, as will be shown presently, from that which expresses the space described; but, it is necessary to know beforehand, the kind of motion which a material point would assume in consequence of its acquired velocity, if the force which impresses this velocity on it, by its action continued during a certain time, should cease to act, and the body be consequently abandoned to itself.

113 It is in the first place evident, that if the moveable had previously moved in a right line, it will continue to move along this line, for there is no reason why this material point should deviate from the direction in which it moved, to one side in preference to the other. But we cannot affirm, *à priori*, that the velocity which has been impressed on it, will not diminish of itself, and eventually vanish altogether; it is only by experience and induction that this question can be decided.

Now, according as the obstacles to the motion of bodies, such as friction and the resistance of the media which they traverse, diminish in intensity, they are observed to persevere more and more in this state, and, as often as an alteration is

observed to take place in their velocity, we at once perceive that this effect can only be ascribed to a foreign cause. We are therefore led to conclude, that if it was possible for a material point, after being put in motion, neither to be solicited by any other force, nor to meet with any obstacle, its motion would be rectilinear and uniform, that is to say, the simplest of all motions

Thus, for example, if a mass of iron is made to move in a vacuo, on a horizontal plane, without friction, by the sole action of the pole of a magnet, and if the attractive power of this pole was suddenly annihilated by placing in juxtaposition with it an opposite pole of equal power, this mass would still be directed towards this point; but its motion will become uniform, and its velocity will be more or less considerable, according as the attractive force has been permitted to act for a longer or shorter time.

The impossibility which bodies are in, of exciting motion in themselves, or of changing the motion which has been communicated to them, is termed the *inertia* of matter. This term does not imply that matter is incapable of acting; for on the contrary, the cause of the motion of each material point, is always found in the action of other points, but never in itself.

114 At the end of the time $t$, when the distance of the moveable from a fixed point taken on the line that it describes, is $x$, let $v$ be the acquired velocity, that is to say, the velocity of uniform motion, which would have place if, at this instant, the force which acts on the moveable should cease to act. The action of this force continuing, the space $dx$ which the moveable would describe in the instant $dt$, will be described in virtue of this action and of the velocity $v$; $vdt$ will be the value of the part of $dx$ corresponding to this velocity, which would be described with a uniform motion. Therefore, denoting by $\varepsilon$ the part of this space which answers to the action of the force during the instant $dt$, we shall have

$$dx = vdt + \varepsilon$$

Now, as the velocity varies by insensible degrees, and as its variations are solely owing to the action of the force applied to the moveable, it follows that in the time $dt$ this action can only produce a velocity infinitely small; consequently, this same action can only cause to be described a space infinitely small of the second order, and evidently less than that which would be described uniformly by the body, if at the commencement of $dt$, it had received all the velocity which would be produced during this instant.

We may, therefore, neglect $\varepsilon$ relatively to $vdt$ in the preceding equation, and then we shall have

$$v = \frac{dx}{dt},$$

for the expression of the velocity in any motion whatever. If it were required to determine the part $\varepsilon$ of the space described by the moveable in the time $dt$, in virtue of the action of the force which solicits it, the powers of $dt$ superior to the first should be retained. Now, denoting the distance of the moveable from the fixed point at the end of the time $t + dt$ by $x'$, we shall have by Taylor's theorem,

$$x' - x = \frac{dx}{dt}dt + \tfrac{1}{2}\frac{d^2x}{dt^2}dt^2 + \&c.$$

for the complete expression of the space described in this instant $dt$. The first term, equal to $vdt$, is the space due to the velocity acquired at the end of the time $t$, therefore, if the third and higher terms are neglected relatively to the second, we shall have

$$\varepsilon = \tfrac{1}{2}\frac{d^2x}{dt^2}dt^2,$$

or, what comes to the same thing,

$$\varepsilon = \tfrac{1}{2}dvdt,$$

for the part of the space $x' - x$ described by the action of the force. As the velocity which is at the same time produced by

this action is $dv$, it is evident that the space which the moveable would describe uniformly during this time $dt$, if at the commencement, it had received all this increase of velocity, would be equal to the product of $dv$ and $dt$, or to double of the space $\varepsilon$ which it actually describes.

115. When the space described is given in a function of the time, the corresponding velocity may be immediately deduced by means of the equation $v = \frac{ds}{dt}$. For example, as the moveables in Atwood's machine, describe spaces which increase as the squares of the times, i. e., in which $s$ is : : $l$ to $t^2$, it follows that $\frac{ds}{dt}$ is : : $l$ to $t$, hence their acquired velocities must be proportional to the times during which these spaces are described; which, indeed, this machine furnishes us with the means of verifying. (Nos. 400, 401.)

Conversely, if the velocity be given in a function of the time, by the definition of the motion, we can infer from it, by integration, the expression of the space described. Thus, after uniform motion, the simplest is that in which the velocity increases or diminishes by equal quantities in equal times, and which on that account is said to be *uniformly* accelerated or retarded Therefore, if we denote the constant increment, whether positive or negative, of the velocity, in each unit of time by $g$, and by $a$ the velocity of the moveable when $t = 0$, the velocity $v$ at any instant whatever, in this description of motion, will be,

$$v = a + gt;$$

by multiplying by $dt$, and integrating, we shall have

$$x = b + at + \tfrac{1}{2}gt^2,$$

for the distance of the moveable from a fixed point of the line that it describes, $b$ being this distance at the commencement of the time $t$.

When the two constants $a$ and $b$ vanish, we shall have simply

$$v = gt, \quad x = \tfrac{1}{2}gt^2.$$

Therefore, the space described is proportional to the square of the time, and the velocity acquired at the end of any time $t$ is such, that in virtue of this sole velocity, the moveable would describe in a time equal to $t$, a space $vt$ double of that which it actually describes It follows, that if the space described in the first unit of time be known, we can obtain by doubling it, the value of the constant velocity $g$, by which one uniformly accelerated motion differs from any other motion of the same nature.

The motion of heavy bodies which descend in a vacuo is of this description. In the same place, the velocity $g$ is equal for all their points, so that, in fact, all of them describe vertical lines when actuated by a motion of this kind. This velocity varies from one place to another, by accurate experiments, it has been proved, that if the second is assumed for the unit of time, and the metre for the linear unit,

$$g = 9^m,80896$$

at the Observatory of Paris

The force which produces equal velocities in equal times, is considered to be a *constant force;* thus gravity is a constant force, which implies here that it acts with equal intensity on bodies already actuated with any velocities whatever, and not merely as in No 59, that its intensity is the same throughout the entire extent of a body of ordinary dimensions.

116. The laws of equilibrium do not imply any particular relation between the forces and the corresponding velocities; and to resolve the problems of statics, it is sufficient, if the numerical relations of forces, such as they have been defined in No. 5, be known. The laws of motion, on the contrary, depend on the relation which should exist between the magnitudes of the velocities produced by given forces, and this relation, the knowledge of which is indispensable for the solution of the dynamical problems, is the same as that of the forces, as we proceed now to demonstrate

Let $x$ and $v$ denote, as before, the space described and the

velocity acquired at the end of the time $t$, and at this epoch, let two given forces $f$ and $f'$ act simultaneously on the moveable, in the direction of its motion, let $u$ denote the infinitely small velocity that the force $f$ would impress on the moveable, if it acted solely for the infinitely small portion of time $\tau$, and let $u'$ denote that which would be produced by the force $f'$, in the same time, if the force $f$ did not exist. I say, that the circumstance of these forces being impressed simultaneously, will not modify the velocities of which they are separately capable, and that the velocity produced by the force $f+f'$ will be $u+u'$, that is to say, at the end of the time $t+\tau$, the velocity of the moveable will be $v+u+u'$. In fact, the augmentation of the velocity of the moveable can only depend on the time $\tau$, to which it will be proportional, and on the state of this material point, or in other words, on its position and velocity during this same time $\tau$, therefore, it can only be by influencing this state that the action of the force $f'$ can modify the velocity which will be produced by the force $f$. Now, as during the time $\tau$, the distance of the moveable from a fixed point and its velocity, can only vary by infinitely small quantities, which may be neglected relatively to $x$ and $v$, its variations of distances from other fixed or moveable points, from which the forces $f$ and $f'$ may emanate, may in like manner be neglected, consequently, the velocity which the force $f$ will produce during this interval of time $\tau$, cannot be modified in any manner by the simultaneous action of the force $f'$, and the same will be the case, relatively to the velocity produced by the force $f'$, which will not be changed in any respect by the action of $f$. Therefore, the entire velocity impressed on the moveable during the time $\tau$ by the force $f+f'$, will be equal to $u+u'$.

It is likewise evident, that if the force $f$ acts in the direction of the velocity $v$, and the force $f'$ in the opposite direction, the increase of velocity produced by the force $f-f'$ will be equal to $u-u'$.

Whatever be the nature of each of the forces $f$ and $f'$, if

they are capable of producing the same velocities in the same infinitely small portion of time, we may consider them as *equal forces*. If they are applied in opposite directions at the same time, they will produce no change in the velocity of the body, if it is already in motion; and if this material point is at rest, it will continue so, this is conformable to the definition of equal forces given in No. 5.

When the force which acts on the moveable in the direction of the acquired velocity becomes double, triple, quadruple, . . . . the velocity which will be produced in this time $\tau$, will increase in the same proportion. On the other hand, when this force is reduced to a half, a third, or a fourth, the velocity produced will be diminished in the same manner; and generally, the infinitely small velocities produced in equal times, in the same or opposite directions from the acquired velocity, or impressed on a material point at rest, will be as the intensities of the corresponding forces.

It is on this general principle that the measure of forces in dynamics is founded. It is usually presented as an hypothesis, here we give it as a necessary consequence of the circumstance, that the velocities impressed by any force whatever in infinitely small intervals of time, are always infinitely small, and of this, that in the same time, the displacements of the moveables are also infinitely small.

117. If the forces to be compared together are constant forces, so that each of them produces during the entire time of the motion, equal velocities in equal times (No. 115), their intensities will be to each other as the velocities which they impress, in any equal times whatever, on the same material point. When, therefore, these velocities are given by observation, the relation of the forces may be inferred, and conversely, where this relation is known *à priori*, we can assume it for that of the velocities

For example, let $\omega$, $\omega'$, denote the intensities of gravity at two different latitudes, if $g$, $g'$, the velocities acquired in a

second, by bodies which descend vertically in a vacuo, at these places, be known; we shall have

$$\omega : \omega' :: g : g'.$$

The relation between these forces $\omega$, $\omega'$, will be also that of the weights of the same body, or of two homogeneous bodies having the same volume, at these latitudes. It appears from observation, that the velocities produced by the action of gravity increase from the equator to the pole, and that the entire increment is very nearly $\frac{1}{200}$ of the least weight, or of that at the equator. It follows, therefore, that the weight of the same body, transferred from the equator to the pole, will increase by $\frac{1}{200}$, and that, in order there should be an equilibrium between two homogeneous bodies placed in these two places, the volume of the body situated at the equator should exceed by $\frac{1}{200}$ that of the body situated at the pole.

Also if $\omega$ denotes the intensity of gravity in the vertical direction, and $\omega_1$ its component in the direction of a line which makes the angle $a$ with the vertical, then by the rule for the composition of forces we shall have

$$\omega_1 = \omega \cos a,$$

and if $g$, $g_1$, denote the velocities which would be produced in a unit of time by these two constant forces, acting separately on the same material point, the proportion

$$g : g_1 :: \omega : \omega_1,$$

will also give

$$g_1 = g \cos a.$$

If this heavy material point rests on an inclined plane, which makes with the horizontal plane an angle equal to $90° - a$, the force $\omega$ should be decomposed into two others, the one perpendicular to the given plane, and which will be destroyed by its resistance, the other directed parallel to this same plane, which will be the force $\omega_1$. It is this last force, which, abstracting from the effect of the friction of the

moveable against the inclined plane, will produce the motion in a vacuo. This motion being caused by the action of a *constant* force, will be uniformly accelerated, and if $x_1$ and $v_1$ denote the space described and the velocity acquired at the end of the time $t$, we shall have

$$v_1 = g_1 t, \quad x_1 = \tfrac{1}{2} g_1 t^2,$$

in which equations we should substitute for $g_1$ its value given above.

This example points out the necessity of knowing, *à priori*, the ratio of the velocities produced by the action of forces, of which the ratio is known, for if $g_1$ could not be deduced from $g$, the velocity given by the observation, and if it was necessary, in order to apply the preceding equations, to determine also by experiment the value of $g_1$, which answers to each value of the angle $a$, dynamics would be almost entirely reduced to an experimental science.

[118.] In order to measure a variable force, we should consider its effect for an infinitely small portion of time, during which it may be considered as constant. Therefore, in any rectilineal motion whatever, let $\phi$ be the force which acts on the moveable at the end of the time $t$, and which is considered as positive, or negative, according as it acts in the direction of the acquired velocity, or in the opposite direction. This velocity being $v$ at this same instant, it will be $v + dv$ at the end of the time $t + dt$, so that the force $\phi$ will have impressed the velocity $dv$ on the moveable in the instant $dt$. Therefore, if $\omega$ denotes a *constant* known force, capable of impressing a velocity $g$ in a unit of time, and which can consequently impress the velocity $gdt$ in the time $dt$, we shall have

$$\phi : \omega :: dv : gdt,$$

hence we deduce

$$\phi = \frac{\omega dv}{gdt}.$$

When the linear unit and the unit of time are once arbi-

trarily selected, the constant $g$, and the value of $\frac{dv}{dt}$ at the end of a given time may be expressed in numbers. This formula will then make known, at the same instant, the numerical ratio of the force $\phi$ to the known force $\omega$, and if this last be that of gravity, this ratio will be that of the force $\phi$ to the weight of the moveable on which it acts. So that if this material point be acted on by gravity, and solicited by the force $\phi$ in a direction opposite to that of gravity, it will be in equilibrio, if we have $\frac{1}{g}\frac{dv}{dt} = 1$.

The preceding formula will be simplified if $\omega$ and $g$ be taken for the units of their respective species, which will reduce it to

$$\phi = \frac{dv}{dt}.$$

The unit of force will be then the constant force, which would impress on the moveable, in the unit of time, a velocity represented by the linear unit, so that if these two last units are the second and the metre, the unit of force will, by the value of $g$ given in No. 115, be very nearly the *tenth* part of the weight of the moveable.

It may be remarked here, that this measure $\frac{dv}{dt}$ of the variable force $\phi$ is the velocity which would produce in the unit of time a constant force, that would retain during this time the same intensity, as the force $\phi$ during the instant $dt$. Thus, in the motion of a mass of iron towards the pole of a magnet, which we have already taken as an example, No. 113, the force $\phi$ depends on the distance from the pole, and is consequently variable, but if we suppose, that at a given instant, the pole recedes from the moveable, so as that the distance of the one from the other may become constant, the force $\phi$ will become so likewise, and the motion will be changed into one uniformly accelerated, and the increase of velocity in the unit of time

will be the measure of this force at the instant it becomes constant.

With respect to the value of $\varepsilon$ found in No. 114, it is evident, we can also write

$$\phi = \frac{2\varepsilon}{dt^2}.$$

It follows, therefore, from this and the preceding formula, that a force may be expressed, either by the velocity which it produces in an indefinitely small portion of time divided by this time, or by twice the space which it causes to be described, divided by the square of this same time. In motions uniformly accelerated, these two equivalent expressions for the force obtain also, when the time is finite, and not as in the general case, infinitely small

119. From what has been established in the preceding numbers, it appears that the general formulæ of rectilineal motion are

$$\left[x = \text{F}t, \quad v = \frac{dx}{dt}, \quad \phi = \frac{dv}{dt}.\right]$$

They point out the relations which, in any motion whatever, exist between the space described, the velocity acquired, and the force which acts on the moveable, and how these three functions of the time may be deduced the one from other, either by differentiation or by integration.

By eliminating $v$ between the two last, we have

$$\phi = \frac{d^2x}{dt^2};$$

which implies that $t$ is taken for the independent variable, and that its differential $dt$ is constant, an assumption which we shall make through this entire treatise, without having occasion to repeat it again.

We shall also have by the elimination of $dt$,

$$\phi = \tfrac{1}{2}\frac{d.v^2}{dx};$$

which will enable us to determine $v$ when the force $\phi$ is given in a function of $x$, and conversely, this force, when the velocity is known in a function of the space described

In the following chapter, it is proposed to give different applications of these general formulæ.

## II *Measure of Forces having regard to the Masses.*

120. Previously to our shewing how the masses should be taken into account, in the comparison of forces which act on different moveables, an inaccurate expression, which frequently occurs, should be rectified, because it has a tendency to produce a confusion of ideas.

Let us conceive, that a body is placed on a horizontal plane, and that it is retained there without any friction. If it was proposed to make it slide on this plane, it is necessary nevertheless, on account of the inertia of matter, that some effort should be made to effect this, and if to this body a second is attached, then a third, &c, it is necessary, in order to produce the same motion, that a more considerable force should be exerted In each case, a sensation of the effort, which it is necessary to make, will be produced, but it ought not to be inferred from this, that matter opposes any resistance to this effort, and that there exists in bodies, what has been very improperly denominated *a force of inertia* When such an expression is made use of, the sensation that is experienced, and which results from the effort that is made, is confounded with the sensation of a resistance that does not really exist.

When there is a friction of the body against the plane, there is an *actual* resistance to the horizontal motion, and the moveable cannot be displaced on this plane, unless an effort is made superior to this resistance. In like manner, if it was proposed to raise the moveable vertically, there is a resistance to this motion which must be overcome by an effort that sur-

passes it. In the two cases, no motion is produced as long as the effort which is made is not greater than the weight, or than its adhesion to the horizontal plane, but if it be supposed that the body has no weight, or that it does not experience any resistance from friction, it can be put in motion, however feeble the effort that is made, or however great the mass of the body on which it is exerted; and if it is necessary to make a greater effort, in order to communicate the same motion to one body than to another, it may be inferred that the first consists of a greater quantity of matter than the second and if the magnitudes of these efforts could be accurately compared together, their ratio will be that of the masses of these two moveables. It is on a consideration similar to this, that is founded, as we now proceed to show, the measure of the masses deduced from the magnitudes of the forces, which cause them to move, and conversely, the measure of the forces, the masses and velocities being respectively taken into account

121. The masses of two material points, belonging to bodies which may be of different natures, are equal or unequal according as forces, which are assumed to be equal, impress on them in the same time, equal or unequal velocities. Let us suppose for greater clearness, that the forces applied to these two points are vertical, and that when placed on the two dishes of a balance, they are in equilibrio. These forces will be equal on this hypothesis, and this being the case, if the two points are rendered entirely free, and if the same forces excite motion in them, their masses will be equal or unequal, according as the infinitely small velocities with which they are actuated in the first instant, are equal or unequal.

When, in this manner, the masses of different material points are ascertained to be equal, other points of which the masses may have any relation whatever, will be obtained by uniting these together. Thus, denoting the mass of each of the equal points by $\mu$, and by $m, m'$, the masses of two

other points made up of $n$ and $n'$ of the first, $m$ and $m'$ will be to each other as these numbers $n$ and $n'$, and we shall have

$$m = n\mu, \quad m' = n'\mu$$

Now, let $u, v, v'$, be infinitely small velocities, $i$ and $i'$ integral numbers, and

$$v = iu, \quad v' = i'u$$

If the two forces $f$ and $f'$ impress on the masses $m$ and $m'$, the velocities $v$ and $v'$ in the same instant, we shall have

$$f \quad f' \qquad mv \quad m'v'$$

In fact, the force $f$ may be considered as the sum of $n$ equal forces, which would impress the velocity $v$ on each of the $n$ equal points, of which $m$ is composed, so that denoting one of these equal forces by $k$, we shall have

$$f = nk$$

Moreover, let $h$ be the force which would impress the velocity $u$ on each of these equal points, during the same instant, that the force $k$ impresses on it the velocity $v$ These forces acting on the same material point, will be to each other as the velocities $u$ and $v$ (No 116), and, because $v = iu$, there will result($a$)

$$k = ih$$

We shall have also

$$f' = n'k', \quad k' = i'h',$$

$f'$ being considered as the sum of the $n'$ forces $k'$, capable of impressing the velocity $v'$, on each of the equal points of which $m'$ is composed, and $h'$ denoting the force which would impress on each of these same points the velocity $u$. Now, as $h$, $h'$, are forces which are capable of impressing in the same instant, the same velocity $u$ on two points of equal mass, namely, on two points of which the common mass has been represented by $\mu$, it follows from what precedes, that we must have $h' = h$. Then by means of the preceding equations, we shall have

$$f = mh,\quad f' = m'h',$$

and, if we take into account the values of $m, m', v, v'$, there will result the proportion which it was proposed to demonstrate.

122 This being established, let us consider a body of any magnitude and form whatever, all the points of which describe parallel lines with a common velocity, which may moreover vary with the time. Let this body be divided into an infinite number of material points equal in mass, such as they have been just defined The motion of all these points may be ascribed to forces which are equal and parallel throughout the entire extent of the moveable, their resultant for any part of this body, is equal to their sum, and applied to the centre of gravity of this part. The forces corresponding to any two parts will be therefore to each other as their masses; consequently, if $f$ be the entire force which acts on the moveable, $m$ its mass, and $\phi$ the force which answers to a part of this mass, taken for unity, we shall have

$$f = m\phi.$$

With respect to the force $\phi$, it will be proportional to the increase of the velocity of the moveable during an infinitely small portion of time; and if $v$ denotes this velocity at the end of the time $t$, we may assume for its measure, as in No 118,

$$\phi = \frac{dv}{dv}.$$

Hence there will result

$$f = m\frac{dv}{dt},$$

for the expression of the force in any motion whatever, the mass of the moveable being taken into account, and all its points being supposed to be actuated by the same velocity.

This force $f$, which is the resultant or sum of all the infinitely small forces, with which all the points of which the body is composed, are actuated, is termed the *motive force*,

the factor $\phi$ of its value $m\phi$, is called the *accelerating force*, and is no other thing than the motive force referred to the unit of mass

The *motive* force becomes a *pressure*, when the mass on which it acts, rests on a fixed plane perpendicular to its direction. A pressure and motive force differ from one another in the circumstance, that the velocities which a pressure tends to produce, are continually destroyed by the resistance of the fixed plane that supports it, whilst those which are actually produced during each instant by the motive force accumulate in the body, so that after a finite time there results in it a finite velocity. Two pressures are to each other as the masses multiplied by the infinitely small velocities, which they tend to impress on them in the same instant, and which, in point of fact, they would impress on them, if these masses were free.

123. If the motion that is common to all the points of a moveable be a uniformly accelerated one, and if $g$ denotes the increase of velocity which has place in each unit of time, we have

$$\phi = g, \quad f = mg.$$

Likewise, for another constant force $f'$ acting on a mass $m'$, and producing a velocity $g'$, in a unit of time, we shall also have

$$f' = m'g'$$

Now, it appears from observation, that two heavy bodies, whatever difference there may be in the matter of which they consist, acquire the same velocity in falling in a vacuo during the same interval of time. Hence, in the case of gravity, we have $g = g'$, and consequently, the weights $f$ and $f'$ of any two bodies are to each other as their masses $m$ and $m'$, as has been assumed in No 60. The sole fact, confirmed by daily experience, that heterogeneous bodies have equal weights under different volumes is not sufficient to decide the question, whether their masses are equal or unequal, it is necessary to

know besides, whether gravity impresses the same motion on them, to be able to infer from the equality of the weights, the equality of the quantities of matter.

The weight of a heavy body which falls in a vacuo, is its motive force, and the gravity its accelerating force. For the sake of conciseness we give the name of *gravity* or *weight* to the velocity $g$, which, however, is only the measure of this force

124. If given forces act on the surface, or on other parts of a solid body, and if these forces impress on all its points, equal and parallel velocities, they must have a unique resultant, which will coincide, in magnitude and direction, with the motive force, that has been defined above, and the accelerating force is obtained by dividing it, by the entire mass of the body.

If, for example, we conceive a heavy body to descend in air, in water, or in any other fluid, and if it be symmetrical as to its form and density, about a vertical axis, when it is not homogeneous, it is evident, that as every thing corresponds on each side of this axis, all the points of the body will describe vertical lines, in which case, since the body is solid, all its points must, at each instant, be actuated with the same velocity. The resistance of the medium, which acts on the surface of the body, will consequently be reduced to a force acting in the direction of the axis of its figure. Denoting its intensity at any instant whatever by R, the corresponding part of the accelerating force of the body by $\psi$, and its mass by $m$, we shall then have

$$\psi = \frac{\text{R}}{m}.$$

As this force acts, during the descent of the body, in a direction contrary to that of gravity, the entire accelerating force will be $g - \psi$. If the body is projected perpendicularly upward, the two forces would act in the same direction, and the total accelerating force will be negative and equal to $-g-\psi$

The theory of the resistance of fluids is not sufficiently advanced to enable us to determine, *à priori*, the value of R, which will depend on $v$ the velocity of the body, on its form, and on the density and nature of the fluid It is commonly assumed to be proportional to the square of $v$, and to the density of the fluid, so that if this density be denoted by $\rho$, we shall have

$$\text{R} = \sigma\rho v^2,$$

$\sigma$ being a coefficient depending on the form and dimensions of the body, on the nature of the fluid (namely, whether it is liquid or aeriform,) and on its temperature.

In the case of a sphere, the coefficient $\sigma$ is assumed to be proportional to its surface, or to the square of its diameter. So that if we denote its radius by $r$, and its density by D, in which case its mass $m$ will be equal to $\frac{4\pi}{3}\text{D}r^3$, there will result,

$$\psi = \frac{\gamma\rho v^2}{\text{D}r},$$

$\gamma$ denoting a numerical coefficient which will be the same for all spheres, the *actual* value of which must be determined by experiment for each particular kind of fluid As this quantity $\psi$ is of the same nature as $g$, it follows that if $k$ denotes a given velocity, we should have

$$\frac{\text{D}r}{\gamma\rho} = \frac{k^2}{g},$$

in order that the expression of $\psi$ may assume the form

$$\psi = \frac{gv^2}{k^2},$$

conformably to the principle of the homogeneity of quantities (No 23)

125. The same constant force acting successively on different masses, will produce uniformly accelerated motions, in which the accelerating force, or the constant increment of

the velocity in each unit of time, will be in the inverse ratio of the mass

Thus, for example, $f$ being the weight $mg$ of a mass $m$, if this mass be suspended to the extremity of a thread attached at its other end to another mass $m'$ laid on a horizontal plane, it is evident, that if the friction and the weight of the vertical part of the thread are not taken into account, these two masses will both move with the same uniformly accelerated motion produced by the motive force $f$. Therefore, if the accelerating force of this motion be denoted by $g'$, we shall have

$$g' = \frac{f}{m + m'},$$

or, what comes to the same thing, as $f = mg$,

$$g' = g \cos a,$$

in which $a$ denotes an angle, such that

$$m = (m + m') \cos a.$$

Consequently, the motion in question will be the same as that of a heavy body on an inclined plane, which makes the angle $a$ with the vertical (No 117).

All bodies being moveable and susceptible of acquiring velocities which are in the inverse ratio of their masses, when they are subjected, during the same time, to the action of the same force, it follows, that no body can be considered as really *fixed*, those which are said to be so, are bodies which have very great masses relatively to those, on which the motive forces which are applied to them, depend, and which consequently, only receive extremely small velocities from the action of these forces At the surface of the earth, bodies attached to this surface, constitute one mass with that of the terrestrial globe; and in fact, if $m'$ in the preceding example, be assumed to express this mass, it is evident that the velocity $g'$ which will be impressed on it, in the unit of time,

by a weight $mg$ corresponding to a mass $m$ of ordinary magnitude, may be considered as altogether insensible

126. It is usual to term the product of the mass of a body by its velocity, its *quantity of motion*. This expression will, agreeably to custom, be still made use of, though it would be more correct to substitute that of the *quantity of velocity*, since it is the velocity that inheres in the body, whereas the motion is only a subsequent effect.

No force whatever can produce *instantaneously* a finite quantity of motion. The shock of a solid body in motion against a solid body at rest can impress on this last, in a space of time which, though very short, is not infinitely small, a velocity which may be sometimes very great, and, during this interval of time, the two bodies do not suffer any sensible displacement Though they may be supposed to be ever so hard, they are always susceptible of some compression, however little it may be, thus the velocity is transmitted from the one to the other by infinitely small degrees, and if the elasticities of the bodies is not taken into account, then mutual action ceases, when their velocities become equal.

This rapid transmission of velocity, without any sensible displacement of the masses, is what is termed a *percussion* or *impulsion*, it is equivalent, as appears, to a motive force acting for a very short time, with a very great intensity By considering the percussion in this manner, as the sum of the infinitely small actions of a motive force, it can be shown, that it is resolvable into two other percussions acting in given directions, by the rule of the composition of forces, as also each of these successive actions If, for example, there be directed against the back of a wedge, a normal percussion which we shall denote by P, it is resolvable into two other percussions perpendicular to its two *faces*, and if Q, Q′, represent these two components, K, K′, the lengths of the faces to which they refer, and H that of the back of the wedge, it is easy to perceive that by the rule adverted to, we shall have

$$Q : P :: K : H,$$
$$Q' : P :: K' : H,$$

hence we deduce

$$Q = \frac{PK}{H}, \quad Q' = \frac{PK'}{H},$$

Thus, supposing that this percussion P arises from a mass $m$ which strikes the back of the wedge with a velocity $a$, its two faces, or rather the fixed obstacle against which they are directed, will be in the same circumstances, as if they were struck perpendicularly by the same mass $m$, actuated by velocities proportional to their lengths, and expressed by $\frac{Ka}{H}$ and $\frac{K'a}{H}$.

127. If a solid body at rest is struck at the same time, in opposite directions, by two other bodies, of which the masses are $m$ and $m'$, and the velocities $v$ and $v'$, then if these three bodies are symmetrical about the same axis as to their form and density, and if all the points of the two last move parallel to this line, their percussions on the intermediate body will constitute an equilibrium, when the quantities of motion $mv$ and $m'v'$ are equal, that is to say, these quantities of motion will in a very short portion of time be transmitted into the intermediate body, and will mutually destroy each other's effect, without this body being at all displaced

The equilibrium will equally obtain, if the intermediate body be suppressed, and if the transmission of velocity takes place at once from one body to the other. Thus, two solid bodies which move towards each other in opposite directions, will, if we abstract from a consideration of the elasticity, be reduced to a state of quiescence, when their masses are in the inverse ratio of their velocities, and conversely, the products of the masses and velocities are equal, when there is an equilibrium in the impact of two solid bodies

In such a case, the two bodies are supposed to be, as was stated, symmetrically disposed about the same right line, and the directions of the velocities of all their points are assumed

to be parallel to this line, which is the one that passes through the centres of gravity of the two masses. The condition of equilibrium in the impact of these bodies, is consequently, the equality of their quantities of motion, or the equation

$$mv = m'v',$$

$m$ and $m'$ being their masses, and $v$ and $v'$ their velocities. In a subsequent part of this treatise, we will determine the motions which have place after the impact, when these conditions relatively to the magnitudes and direction of the velocities, and with respect to the form of the bodies, are not satisfied, and also when their elasticity is taken into account.

It results from this law of equilibrium in the impact of two bodies, that percussion should furnish the most direct means of measuring the mass of bodies. A known velocity $a$ being impressed on all the points of a body, the mass of which is taken for unity, if we could determine exactly the velocity $v$ with which all the points of another body should be actuated in order to constitute an equilibrium with the first, when they impinge on one another, moving in opposite directions, the numerical value of the mass of the second will then be the ratio $\frac{a}{v}$, but it is nearly unnecessary to state, that this means is impracticable, and that we should always refer to the weights, when it is required to measure their masses.

It also follows, that two percussions exercised on a solid body must be deemed equivalent when they produce equal quantities of motion; so that in the example of the preceding number, the back and two faces of the wedge will experience the same effects, or will be struck with the same energy, if for the mass $m$ and the velocity $a$, there be substituted a mass $m'$ and a velocity $a'$, such that we may have $ma = m'a'$.

128. When two percussions, arising from velocities which are in the inverse ratio of the masses, are simultaneously exerted on the two dishes of a balance, there will be an equilibrium between them; in this case the balance supplies the

place of the intermediate body which was considered in the preceding number. For example, this will be the case of two heavy bodies, of which the masses are $m$ and $m'$, and which impinge at the same instant on these two dishes, having respectively acquired velocities $v$ and $v'$ which are such that we have $mv = m'v'$

If the mass $m$ is at rest in one of the two dishes, its weight will exercise a pressure which will be generally overcome by the percussion of the other mass; but it is not accurate to say, as is generally done, that this will be always the case, however great the pressure may be, or however small the percussion In fact, we can substitute for the percussion of $m'$ a motive force acting on one of the two dishes, without sensibly deranging it, during an extremely short space of time, such as $\tau$. Denoting by $m'udt$ the infinitely small quantity of velocity, which this variable force is capable of producing during the instant $dt$, the product $m'\int_0^\tau udt$ will be the quantity of velocity which will be communicated to the balance during the time $\tau$. During this same time, the weight of $m$ will produce a quantity of motion represented by $mg\tau$, $g$ denoting the gravity In order to an equilibrium in the system, it is necesssary that the integral $\int_0^\tau udt$ should be the entire velocity $v'$ with which the mass $m'$ is actuated at the instant the percussion commences, so that there may not remain any degree of velocity when the shock has ceased, and this being so, it is sufficient that the quantities of motion $mg\tau$ and $m'\int_0^\tau udt$ impressed in opposite directions on the balance while the shock is going on, should be equal to each other Hence the condition of this equilibrium will be expressed by the equation

$$m'v' = mg\tau,$$

and according as we have $m'v' > mg\tau$, or $m'v' \angle mg\tau$, the per-

cussion will prevail over the pressure, or the pressure over the percussion. Now, although the time $\tau$ may be extremely small, this last case is possible, the mass $m$ being supposed to be sufficiently great with regard to $m'$ in order that it should be impossible, it is necessary that the duration of the percussion be infinitely small, which is not the case in nature.

As in dynamics there is a constant application of the principles that have been explained in this chapter, it is of great consequence to have accurate notions of them, before we proceed to the resolution of the different problems relative to the motion of bodies

# CHAPTER II

## EXAMPLES OF RECTILINEAR MOTION.

129. It appears from No 119, that the equations of rectilinear motion are

$$v = \frac{ds}{dt}, \quad \phi = \frac{dv}{dt}, \quad \phi = \frac{d^2x}{dt^2}, \qquad (1)$$

(the last of which may be deduced from the two others ;) in these expressions $x$ denotes the distance of the moveable at the end of the time $t$ from a fixed point in the line which it describes, $v$ denotes the velocity which it has acquired, and $\phi$ the force that solicits it, which is positive or negative according as it acts in the same or the contrary direction from that of the velocity $v$. These equations are applicable not only to an isolated material point, but also to a solid body of any magnitude whatever, of which all the points describe parallel right lines, and which consequently will be endowed with a motion common to all its points ; $\phi$ in this case will be the accelerating force, equal to the motive force divided by the mass of the moveable.

The value of $\phi$ is supposed to be given in each problem, and the question will be to deduce from the preceding equations, the expressions of $v$ and $x$ in functions of $t$. They will contain two constant arbitraries, the values of which can be determined from those of $x$ and $v$ at the commencement of the motion, and which must be given in each example. In the problems which follow, the time will be always reckoned from this commencement, so that the *given* values of $x$ and $v$ answer to $t = o$.

The integration will not be generally possible in a finite form, except when $\phi$ depends only, as in the following ex-

amples, on one of the quantities $t$, $v$, $x$ When the given value of $\phi$ contains all the three, or even two, the values of $x$ and $v$ can only be expressed by a series.

130 Let the force $\phi$ be first supposed to be constant, then if it was required to determine the vertical motion of a body, which descends in a vacuo, in virtue of the force of gravity, we shall have

$$\frac{d^2x}{dt^2}=g,$$

$g$ denoting this force, hence we deduce,

$$v=gt, \quad x=\tfrac{1}{2}gt^2,$$

and, consequently,

$$v^2=2gx,$$

$x$ being the distance from the point of departure of the moveable, and the initial velocity being supposed to be cypher, so that we have $x=0$ and $v=0$, when $t=0$

If $a$ denotes the velocity acquired in descending through the height $h$, we shall have

$$a=\sqrt{2gh},$$

which is a convenient expression for the velocity in terms of the height, through which a heavy body should fall to acquire this velocity, and of the constant velocity $g$ The time of falling through this height being denoted by $\theta$, we shall also have

$$\theta=\frac{a}{g}=\sqrt{\frac{2h}{g}}, \quad h=\tfrac{1}{2}g\theta^2=\frac{a^2}{2g}$$

If the body is projected vertically upwards, the equation of its motion in a vacuo will be

$$\frac{d^2x}{dt^2}=-g,$$

$g$ being the same constant velocity as in the preceding case, since the action of gravity on bodies in motion is supposed to be independent of the direction in which they move, as well as of the magnitude of their velocity.

If the initial velocity be denoted by $a$, we shall have

$$v = a - gt, \quad x = at - \tfrac{1}{2}gt^2,$$

which express respectively, the velocity and space described in any instant whatever. It is evident, that the moveable will continue to ascend until this velocity vanishes. Therefore, if $\theta'$ denotes the time during which it moves until $v$ vanishes, and $h'$ the height to which it will attain, we shall have

$$\theta' = \frac{a}{g}, \quad h' = \frac{a^2}{2g};$$

and as these values coincide with those of $\theta$ and $h$ of the preceding case, it follows that a heavy body projected vertically upwards with a velocity $a$, ascends in a vacuo to a height, from which, if it fell, it would acquire this same velocity, and the time it takes to attain this height is equal to that of its fall, $h$ is commonly called the height due to the velocity $a$, and conversely, $a$ the velocity due to the height $h$

131 Whether the body ascends or descends, it will be sufficient in order to obtain the equations of its motions on an inclined plane, to substitute in the preceding equations $g.\cos\alpha$ in place of $g$, $\alpha$ denoting, as in No 117, the complement of the inclination of the given plane to a horizontal plane.

Hence, in the case of descent, we shall have

$$v = gt\cos\alpha, \quad x = \tfrac{1}{2}gt^2\cos\alpha, \quad v^2 = 2gx\cos\alpha,$$

now if $l$ be the length of the inclined plane, and $h$ its height, we have

$$h = l\cos\alpha,$$

therefore, if $k$ denotes the velocity acquired in falling through $l$ the entire length of the plane, we shall have also

$$k^2 = 2gl\cos\alpha = 2gh,$$

which shows that this velocity is the same, as if the body had fallen through the vertical height $h$. If ABC (fig. 34) be the

circumference of a circle situated in a vertical plane, and if AB represents its vertical diameter, we can determine, by means of the preceding equations, the time which a heavy material point would take to describe the chord AC, which is terminated at the superior extremity of this diameter. For if from the point C the perpendicular CD be let fall on AB, we shall have, in this case,

$$AC = l, \quad AD = h,$$

and if the required time be denoted by $\theta$, then

$$l = \tfrac{1}{2} g\theta^2 \cos\alpha = \frac{g\theta^2 h}{2l},$$

but by a known property of the circle, we have

$$l^2 = hb,$$

$b$ denoting the diameter AB, hence we can obtain

$$\theta = \sqrt{\frac{2l^2}{gh}} = \sqrt{\frac{2b}{g}}$$

But this time is that of the fall through the vertical height $b$, consequently it follows, that the chord AC will be described in the same time as the diameter AB. The same result would be obtained, if the time of describing the chord CB which is drawn to the lower extremity of the vertical diameter AB be required, for this time is also equal to that of describing the vertical diameter. As then it appears, that the time of describing any chord drawn to either extremity of the vertical diameter, is *always* the same, and independent of the length of the chord, it will be the case, when the chord becomes indefinitely small, which arises from this, that then the component of the gravity in the direction of the infinitely small chord is no longer a finite quantity.

132. Let now the motion of a solid body, which descends or which is projected upwards, in a resisting medium, and of which all the points describe right lines, be considered. In

order that the accelerating force may depend solely on the velocity, the density of the medium is supposed to be every where the same.

In the case of a descent, we shall have

$$\phi = g - \frac{gv^2}{k^2},$$

the resistance being supposed to be proportional to the square of the velocity (No. 124), and $k$ denoting a constant and given velocity As this value of $\phi$ is a function of $v$, we must make use of the second equation (1), and shall obtain from it ($a$),

$$gdt = \frac{k^2dv}{k^2-v^2} = \frac{k}{2}\left(\frac{dv}{k+v} + \frac{dv}{k-v}\right).$$

By integrating and supposing that the initial velocity is nothing, so that $v = 0$ when $t = 0$, there results

$$gt = \tfrac{1}{2}k \log \frac{k+v}{k-v},$$

and, conversely,

$$\frac{k-v}{k+v} = e^{\frac{-2gt}{k}};$$

hence we obtain ($b$)

$$v = \frac{k\left(e^{\frac{gt}{k}} - e^{\frac{-gt}{k}}\right)}{e^{\frac{gt}{k}} + e^{\frac{-gt}{k}}}. \qquad (2)$$

In these and similar expressions, $e$ denotes the base of the Naperian system of logarithms, and $log$ a logarithm of this species, however in formulæ, in which the base of these logarithms does not occur, the letter $e$ will be employed to represent other quantities. Its value computed to an accurate approximation is

$$e = 2{,}7182818,$$

and that of the constant modulus by which the Naperian

logarithm of any number should be multiplied, in order to obtain the vulgar logarithm of this number, is

$$0,4342945.$$

Since $dx = vdt$, we shall have by integrating and supposing that $x = 0$ when $t = 0(c)$,

$$x = \frac{k^2}{g} \log \tfrac{1}{2}\left(e^{\frac{gt}{k}} + e^{\frac{-gt}{k}}\right) \tag{3}$$

We have also

$$gdx = \frac{k^2 vdv}{k^2 - v^2},$$

and, consequently$(d)$,

$$x = \frac{k^2}{2g} \log \frac{k^2}{k^2 - v^2}, \tag{4}$$

is the value of $x$, considered as a function of $v$

133 The preceding formulæ enable us to solve the problem completely It appears from a consideration of them, that the time being supposed to increase continually, the motion approaches more and more towards uniformity, and that it becomes sensibly uniform when $gt$, the velocity arising from the action of gravity, is very great relatively to $k$ In fact, if $e^{\frac{-gt}{k}}$, which in this case is a very small fraction, be neglected, we have$(e)$

$$v = k, \quad \phi = 0, \quad x = kt - \frac{k^2}{g} \log 2.$$

As the resistance of the fluid is a force which acts on the surface of the body, the motive force that results from it is independent of the mass, and will be the same, whether the body consists of a very dense matter, or whether the interior matter be taken away altogether, and nothing left but a very slender envelope Now, as the accelerating force is equal to the motive force divided by the mass of the body, it follows that the first of these two forces will, every thing else being

the same, be in the inverse ratio of this mass, and consequently ($f$) $k$ will be in the direct ratio of its square root. This is the reason why the final motion in a resisting medium is more rapid for those heavy bodies, whose density is greater, the form and extent of surface being supposed to remain the same. When the density of the medium is inconsiderable relatively to that of the body, the quantity $k$ is very great, and in this case the motion will not approach to one of uniformity until after the lapse of a considerable length of time. As long as the velocity $gt$ does not become very considerable, we obtain in converging series

$$\tfrac{1}{2}\left(e^{\frac{gt}{k}}-e^{\frac{-gt}{k}}\right)=\frac{gt}{k}+\frac{g^3t^3}{1.2.3k^3}+\&c.,$$

$$\tfrac{1}{2}\left(e^{\frac{gt}{k}}+e^{\frac{-gt}{k}}\right)=1+\frac{g^2t^2}{2k^2}+\frac{g^4t^4}{1.2.3.4k^4}+\&c.$$

$$\log\tfrac{1}{2}\left(e^{\frac{gt}{k}}+e^{\frac{-gt}{k}}\right)=\frac{g^2t^2}{2k^2}-\frac{g^4t^4}{1.2.3.2k^4}+\&c. \quad (g)$$

and formulæ (2) and (3) become

$$v=gt-\frac{g^3t^3}{1.3k^2}+\&c.,$$

$$x=\tfrac{1}{2}gt^2-\frac{g^3.t^4}{1.2.3.2k^2}+\&c.$$

They are reduced, as ought to be the case, to those which a motion uniformly accelerated would give, when the density of the medium entirely vanishes, in which case the quantity $k$ becomes infinite.

134. In the case in which the body is projected perpendicularly upward, we have

$$\phi=-g-\frac{gv^2}{k^2}$$

If its superior surface is the same as its inferior surface, the constant $k$ will be the same as in the case of descent; but if

these two portions of its surface are different, the values of $k$ will be so likewise, for example, if the projectile was a cone of which the base was horizontal, the quantity $k$ would in its ascensional motion be greater or less than in the descent, according as its summit was situated above or beneath its base If for greater clearness the body be supposed to be a homogeneous sphere, of which the radius is $r$, D its density, $\rho$ that of the medium, then we shall have

$$k^2 = \frac{\mathrm{D}r}{\gamma\rho},$$

$\gamma$ being a constant arbitrary depending on the nature of the medium, (namely, whether it be liquid or aeriform,) and on its temperature If the preceding value of $\phi$ be substituted in the second equation (1), there will result

$$\frac{k\,dv}{k^2 + v^2} = -\frac{g\,dt}{k},$$

which, by integrating, and denoting the initial velocity of the body by $a$, gives(h)

$$\operatorname{arc}\left(\operatorname{tang} = \frac{v}{k}\right) = \operatorname{arc}\left(\operatorname{tang} = \frac{a}{k}\right) - \frac{gt}{k}$$

The value of $v$ which results from this expression may be written under the form

$$v = \frac{k\left(a \cos\frac{gt}{k} - k \sin\frac{gt}{k}\right)}{a \sin\frac{gt}{k} + k \cos\frac{gt}{k}}$$

By multiplying this expression by $dt$, and integrating a second time, so that when $t = 0$, $x$ may be also equal to cypher, there results(i)

$$x = \frac{k^2}{g}\log\left(\frac{a}{k}\sin\frac{gt}{k} + \cos\frac{gt}{k}\right)$$

We shall also have

$$g\,dx = -\frac{k^2 v\,dv}{k^2 + v^2},$$

and, consequently($k$),

$$x = \frac{k^2}{2g} \log \frac{k^2 + a^2}{k^2 + v^2}.$$

If we make $\frac{1}{k} = \alpha$, and then suppose $\alpha = 0$, in order that these formulæ may be applicable to the case of a body moving in a vacuo, they will assume the form $\frac{0}{0}$, and by the known rule for determining the value in these cases, we find, as we ought,

$$v = a - gt, \quad x = at - \tfrac{1}{2}gt^2,$$

a result which we would arrive at by expanding these formulæ into a series as in the preceding number($l$).

135. If $h$ denotes the greatest height to which the body can attain, and which corresponds to $v = 0$, we shall have

$$h = \frac{k^2}{2g} \log \frac{k^2 + a^2}{k^2}.$$

Likewise if $\theta_1$ denotes the time it takes to attain this height, its value will be ($m$)

$$\theta_1 = \frac{k}{g} \text{ arc } \left(\text{tang} = \frac{a}{k}\right)$$

After having attained this height, the body will fall back, and its motion will be expressed by the formulæ of No 132 If $a'$ denotes its velocity, when it will have fallen back the entire height $h$, we shall have, by means of equation (4),

$$h = \frac{k^2}{2g} \log \frac{k^2}{k^2 - a'^2},$$

and by making this value of $h$ equal to the preceding, there results

$$\frac{k^2}{k^2 - a'^2} = \frac{k^2 + a^2}{k^2},$$

and, consequently,

$$a'^2 = \frac{a^2 k^2}{a^2 + k^2},$$

hence it follows, that $a'$ is less than $a$, so that the velocity of the body, when it returns to the point of departure, is less than its initial velocity.

Also, if $\theta'$ be the time of the entire descent, in which case $v = a'$, we shall have

$$\theta' = \frac{k}{2g} \log \frac{k + a'}{k - a'},$$

which, by substituting for $a'$ its value, becomes

$$\theta' = \frac{k}{2g} \log \frac{\sqrt{a^2 + k^2} + a}{\sqrt{a^2 + k^2} - a},$$

it is evidently different from the value of $\theta$, the time of ascent. We obtain a simpler expression, by multiplying both the numerator and denominator of the fraction comprised under the logarithm, by $\sqrt{a^2 + k^2} - a(n)$, by which means we get,

$$\theta' = \frac{k}{g} \log \frac{k}{\sqrt{a^2 + k^2} - a};$$

and if $\theta$ denotes the entire time $\theta' + \theta_1$, which the body takes in ascending and descending, we shall have

$$\frac{g\theta}{k} = \text{arc} \left(\text{tang} = \frac{a}{k}\right) + \log \frac{k}{\sqrt{a^2 + k^2} - a}$$

If the body be a bullet shot into the air by a cannon directed vertically upward, we can, notwithstanding the rapidity of this motion, determine the time $\theta$ with some precision, and if we knew likewise $a$ the velocity of projection, the preceding equation would enable us to determine the value of $k$ with respect to $r$, the radius of the bullet. Also, from a consideration of the expression of $k^2$ given in the preceding number, it appears, that if $k'$ denotes what $k$ becomes, with respect to another bullet consisting of the same kind of matter, and of which the radius is equal to $r'$, we shall have$(o)$

$$k' = k\sqrt{\frac{r'}{r}}$$

136. If the effect of the force of gravity is not taken into account, the solution of the problem presents a remarkable singularity, when the resistance of the medium is supposed to be proportional to a power of the velocity which is less than unity

Suppose, for example, that

$$\phi = -2g\sqrt{\frac{v}{k}},$$

$g$ and $k$ representing, as before, the force of gravity and a given constant velocity The equation of motion will be

$$\frac{dv}{dt} = -2g\sqrt{\frac{v}{k}},$$

from which, if the value of $gdt$ be deduced, we obtain, by integrating and denoting the initial velocity by $a$,

$$gt = \sqrt{k}\,(\sqrt{a} - \sqrt{v}),$$

and, consequently,

$$v = \left(\sqrt{a} - \frac{gt}{\sqrt{k}}\right)^2$$

Multiplying the members of this equation by $dt$, and integrating a second time, so that we may have $x = 0$ when $t = 0$, we obtain for the value of the space passed over at a given instant$(p)$,

$$x = \frac{a\sqrt{ak}}{3g} + \frac{1}{3gk}\left(gt - \sqrt{ak}\right)^3$$

It appears from the value of $v$, that the velocity diminishes from the commencement of the motion until the instant in which $t = \dfrac{\sqrt{ak}}{g}$, at this instant, the velocity vanishes, immediately after, the motion continues in the same direction as before, and the velocity increases indefinitely But as the velocity vanishes at a certain instant, the accelerating force vanishes at the same time, consequently the body should stop at this instant and remain at rest Now, it ought to be re-

marked, that the equation of the motion admits a particular solution $v = 0$, so that its complete solution is the sum of its integral and of this equation $v = 0$, it follows, therefore, that the problem is resolved from $t = 0$ to $t = \frac{\sqrt{ak}}{g}$ by the integral of the equation of motion, and beyond this value of $t$, by the particular solution. During the first interval of time, the body describes, with a motion continually retarded, a line equal to $\frac{a\sqrt{ak}}{3g}$, at the extremity of which it stops and remains at rest.

This example, which is purely hypothetical, suffices to show how necessary it is to take into account particular solutions of the differential equations of motion, if there are such; it appears, however, from the expressions for the forces in functions of the acquired velocity and space passed over, which have place in nature, that no such case actually occurs.

137 We now propose to give some examples of motions, in which the accelerating force varies with the space passed over

The simplest case is, that of a material point attracted towards a fixed centre in the direct ratio of the distance from this point, which is supposed to exist on the right line that the moveable describes. Let $z$ be this distance at the end of the time $t$, and let the accelerating force at a given distance $a$ be supposed equal to the gravity $g$, by the given law, we shall have

$$\phi = \frac{gz}{a},$$

for its value at any instant whatever If $x$ is the space described at this same instant, and if $c$ be the distance of the point from the centre of attraction at the commencement of the motion, which is supposed to be directed *towards* this centre, we shall have

$$x = c - z, \quad v = -\frac{dz}{dt},$$

and the third equation (1) will become

$$\frac{d^2z}{dt^2} = \frac{-gz}{a}.$$

Its complete integral is

$$z = \text{A} \cos t\sqrt{\frac{g}{a}} + \text{B} \sin t\sqrt{\frac{g}{a}};$$

A and B denoting two constant arbitraries

If the initial velocity of the moveable is supposed to be nothing, we shall have at the same time

$$t = 0, \quad z = c, \quad \frac{dz}{dt} = 0,$$

hence we deduce

$$\text{A} = c, \quad \text{B} = 0,$$

and, consequently,

$$z = c \cos t\sqrt{\frac{g}{a}}.$$

From this formula it appears, that the distance $z$ is nothing, or that the point will reach the centre of attraction at the end of a time which is independent of $c$ the distance($q$) of its point of departure, and equal to $\frac{1}{2}\pi\sqrt{\frac{a}{g}}$, after that, it will perform oscillations on each side of this centre, of which the constant amplitude and duration will be respectively equal to the distance $c$ and the time

$$\tfrac{1}{2}\pi\sqrt{\frac{a}{g}}.$$

138.) For another example, let the motion of a heavy body in a vacuo be considered, the height from which the body falls being sufficiently great to require, that during its fall, the variation of gravity should be taken into account.

Let BAE (fig 35) be a vertical great circle of the earth, D the point from which the body departs in this plane, M its position at the end of the time $t$, on the right line DC, which is drawn from D to C the centre of the earth, and meets its surface in A Let CA the radius, be represented by $r$, the height AD by $h$, DM the space passed over by the body by $x$, and CM, its distance from the centre C, by $z$, so that we may have

$$z = r + h - x$$

The accelerating force $\phi$ will be the gravity at the point M, if $g$ denotes this gravity at the surface of the earth, that is to say, at the point A, and if its intensity be supposed to vary in the inverse ratio of the square of the distance from the centre C, we shall have

$$\phi . g : r^2 . z^2;$$

hence we obtain

$$\phi = \frac{gr^2}{z^2},$$

by means of which the third equation (1) will become

$$\frac{d^2x}{dt^2} = \frac{gr^2}{(r + h - x)^2}.$$

If its two members be multiplied by $2dx$, and then integrated, we obtain

$$\frac{dx^2}{dt^2} = 2gr^2\left(\frac{1}{r + h - x} - \frac{1}{r + h}\right),$$

the constant arbitrary being determined by making $\frac{dx}{dt} = 0$, when $t = 0$; by means of this expression, the velocity acquired by the body, at any distance such as $x$ from the point of departure, can be determined At the point A, where $x = h$, this velocity will be(1)

$$2gh\sqrt{\frac{r}{r + h}},$$

and consequently less, as it ought to be, than if the intensity of gravity was the same, through the entire height $h$, as at the surface From the preceding equation we obtain(s)

$$\sqrt{\frac{2gr^2}{r+h}}\,dt = \frac{(r+h-x)\,dx}{\sqrt{(r+h)x-x^2}}.$$

Now, from a comparison of this differential equation with the equation (a) of No. 73, it appears that if a semi-cycloid DOC is constructed, which may have its summit at D, its origin at the point O, situated on the line OC which is perpendicular to the right line CD, and the diameter of the generating circle equal to CD, then if through the point M, MN is drawn perpendicular to the line DC, and meeting the cycloid in N, we shall have

$$\text{MN} = t\sqrt{\frac{2gr^2}{r+h}},$$

so that MN, the ordinate of the point N, makes known $t$, the time in which the abscissa DM is described, and conversely By integrating and observing that $x = 0$, when $t = 0$, we shall have, in a finite form($t$)

$$t\sqrt{\frac{2gr^2}{r+h}} = \sqrt{(r+h)x-x^2} + \tfrac{1}{2}(r+h)\,\text{arc}\left(\cos = \frac{r+h-2x}{r+h}\right)$$

When the height $h$, and consequently the distance $x$, are very small with respect to $r$, this formula differs very little from that which is furnished when the gravity is supposed to be constant In fact, since

$$\text{arc}\left(\cos = \frac{r+h-2x}{r+h}\right) = \text{arc}\left(\sin = \frac{2\sqrt{(r+h)x-x^2}}{r+h}\right);$$

when the sine is very small, we may substitute it in place of the arc, which renders the second term of the second member of the preceding equation equal to the first. We may also substitute the radius $r$ in place of $r+h-x$, and, conse-

quently, reduce their sum to $2\sqrt{rx}$; by this means, the formula in question will become

$$t\sqrt{\frac{2gr^2}{r+h}} = 2\sqrt{rx},$$

or simply, by neglecting $h$ relatively to $r$,

$$x = \tfrac{1}{2}gt^2$$

As an example for the calculus, we shall suggest, before we quit this subject, but without entering into any details, the case in which the moveable is projected perpendicularly upwards, and we now proceed to the consideration of the last case of rectilineal motion that we propose to give, namely, that of a material point attracted to two fixed centres situated on the line which it describes

139. Let A and B (fig 36) be the two centres of attraction, M the position of the body at the end of the time $t$, and D its point of departure Suppose, for greater clearness, that the motion takes place between the two centres of attraction and from A towards B, let

$$\text{DM} = x, \quad \text{AM} = z, \quad \text{AD} = a, \quad \text{BM} = c - z,$$

so that $x$ is the space described, $z$ the distance of the body from the point A, $a$ the initial distance, and $c$ the length of the line AB. If the attractions be supposed to vary in the inverse ratio of the squares of the distances, and if the intensities of the forces which emanate from the centres A and B, are denoted at the unit of distance, by $a^2$ and $b^2$, we shall have $\frac{a^2}{z^2}$ and $\frac{b^2}{(c-z)^2}$, for their respective intensities, when the body is at M The accelerating force $\phi$ will be the excess of the second force which tends to augment the space $x$, over the first which tends to diminish it, therefore, because $dz = dx$, the third equation (1) becomes

$$\frac{d^2z}{dt^2} = \frac{b^2}{(c-z)^2} - \frac{a^2}{z^2}, \qquad \text{(a)}$$

and $\frac{dz}{dt}$ will express the velocity of the body at the point M.

By multiplying equation (a) by $2dz$ and integrating, we obtain,

$$\frac{dz^2}{dt^2} = \frac{2b^2}{c-z} + \frac{2a^2}{z} - \gamma, \tag{b}$$

$\gamma$ being a constant arbitrary. In order to determine it, let $k$ be the initial velocity which answers to $z = a$, then

$$\gamma = \frac{2b^2}{c-a} + \frac{2a^2}{a} - k^2,$$

and if this equation be taken from the preceding, there will result

$$\frac{dz^2}{dt^2} = k^2 + 2b^2\left(\frac{1}{c-z} - \frac{1}{c-a}\right) - 2a^2\left(\frac{1}{a} - \frac{1}{z}\right), \tag{c}$$

this equation will determine the velocity of the body, in any position whatever between the points A and B

140 There exists on the right line AB, a certain point C, in which the two forces of attraction are equal, so that if a body be placed there, or arrives at it without any acquired velocity, it will remain in equilibrio. Denoting the distance AC by $h$, we have

$$\frac{b^2}{(c-h)^2} = \frac{a^2}{h^2}$$

From this equation two values of $h$ may be deduced, of which one belongs to the point C situated between A and B, and the other to a point in the production of AB, in the direction of the centre of the least attraction. The first of these two values is

$$h = \frac{ac}{a+b}.$$

Let $f$ denote the least initial velocity which must be impressed on the body in order that it may reach the point C, so

that when it attains to this point, its velocity may vanish, we shall have at the same time

$$k=f, \quad z=h, \quad \frac{dz}{dt}=0,$$

and by virtue of equation (c) and of the value of $h$, there will result($u$)

$$f^2=\frac{2b^2}{c-a}+\frac{2a^2}{a}-\frac{2(a+b)^2}{c} \qquad \text{(d)}$$

If the initial velocity be less than $f$, the body will fall back on A; if it is greater, it will pass beyond the point C, and will fall on B. In the case of $k=f$, the body will take an infinite time to reach the point C, because that at an infinitely small distance from this point, it will only be actuated by an infinitely small velocity, and solicited by a force which is equally so.

141 If A and B are the centres of two spheres, which are either homogeneous, or composed of concentrical strata, we may suppose that the attractions which have been considered are those of these two spheres, and then $a^2$ and $b^2$, the intensities at the unit of distance, will be to each other as their masses (No 101). Supposing, for example, that A is the centre of the moon, and B that of the earth, if the non-sphericity of the earth is not taken into account, we shall have

$$a^2=\frac{b^2}{75},$$

for the mass of the moon deduced from its action in raising the waters of the sea, is $\frac{1}{75}$ of that of the earth. Hence we shall have($v$)

$$h=\frac{c}{1+\sqrt{75}}=(0{,}10352)c,$$

so that the distance of the point which is equally attracted by the earth and its satellite the moon, from the moon, is very nearly the tenth part of their mutual distance.

Let $r$ be the radius of the earth, then $c$, the distance of

the moon from the earth, is about $60r$, and if the body begins to move from the surface of the moon, we shall have at the same time, by the known ratio of the radius of the moon to that of the earth, $a = \frac{3r}{11}$. By means of these values of $r$ and $a$, and of $a = \frac{b}{\sqrt{75}}$, equation (d) becomes

$$f^2 = (0{,}044894)\frac{2b^2}{r},$$

and since if $g$ denotes the attraction of the earth at its surface, then

$$b^2 = gr^2,$$

is the expression for this force at the unit of distance, hence if we make

$$(0{,}044894)\,r = r',$$

there will result

$$f^2 = 2gr'.$$

Now, the attraction $g$ may be assumed equal to the weight of which it constitutes a principal part, consequently, $f$ is the velocity acquired in falling through the height $r'$, and since

$$g = 9^m{,}80896, \quad \pi r = 20000000^m,$$

its value is

$$f = 2368^m$$

As the atmosphere of the moon is not such, that its resistance can diminish the velocity of bodies projected from its surface, it follows, that if the earth and moon were at rest, a body projected from the surface of the moon, towards the earth, with a velocity greater than 2368 metres in a second, will pass beyond the point of equal attraction, and at length fall on the surface of the earth In the motion of the moon about the earth, the right line AB drawn from one centre to the other always meets the surface of the moon(u) in the same point, which must be the point D from which the body would be projected in the direction DB, but, during a

second, the point D traverses on the circle described about the centre of the earth, a length of about $1000^m$, consequently, the absolute velocity of the body will be, in magnitude and direction, the resultant of a velocity in the direction of DB, and of a velocity of $1000^m$ in a second, perpendicular to DB This being so, the body will not remain on the moveable line AB, but will describe a curve(x) in space, so that the preceding formulæ are not applicable to its motion, neither will it fall on the surface of the earth, as it would do, if the moon was immoveable

142 If the equation (b) be resolved with respect to $dt$, we obtain

$$dt = \frac{\sqrt{cz - z^2}\, dz}{\sqrt{2a^2c - (2a^2 - 2b^2 + c\gamma)z + \gamma z^2}}$$

The integral of this formula may be always expressed by means of elliptic functions, so that when tables of these functions are constructed, the time which corresponds to a given distance such as $z$, may be computed, and reciprocally But, independently of the cases in which one of the two attractions is supposed to vanish, there are also others in which the integral of the preceding formula may be obtained in a finite form These cases obtain when the quantity comprised under the radical is a perfect square, this requires that we should have

$$(2a^2 - 2b^2 + c\gamma)^2 = 8a^2c\gamma;$$

from which equation we can obtain

$$\gamma = \frac{2}{c}(a \pm b)^2.$$

If this value be put equal to that of $\gamma$ of No. 139, there results(y)

$$k^2 = \frac{2b^2}{c-a} + \frac{2a^2}{a} - \frac{2(a \pm b)^2}{c}$$

One of these two values of $k^2$ is that of $f^2$, the other is

evidently much greater. It follows, therefore, if even neither of the two quantities $a$ or $b$ is equal to cypher, that the time can be expressed in a finite form as a function of $z$, when the body is actuated by the least velocity $f$ with which it can reach the point C, and also when a certain velocity greater than this is impressed upon it($z$)

By substituting the double value of $\gamma$ in the expression for $dt$, there results

$$\sqrt{\frac{2}{c}}\,.\,dt = \frac{\sqrt{cz - z^2}\,dz}{ac - (a \pm b)z},$$

which formula may be rendered rational, and integrated without difficulty, by the ordinary rules. The differential of $dt$ must be always positive; the differential of $dz$ is positive while the body advances from D to B, and negative when it returns towards A. In the first case, therefore, the sign of the radical $\sqrt{cz - z^2}$ must be the same as that of the denominator $ac - (a \pm b)z$, and, in the second case, it must be affected with a contrary sign.

143. Suppose that $b = 0$, or $c = \infty$, the body will then only be subjected to the attraction of the centre A. The equation (c) will be reduced to

$$\frac{dz^2}{dt^2} = k^2 - 2a^2\left(\frac{1}{a} - \frac{1}{z}\right), \qquad \text{(e)}$$

the value of $dt$ which can be obtained from this, may be integrated in a finite form, and will make known $t$ in a function of $z$.

If we make $\frac{dz}{dt} = 0$, we shall have

$$\frac{2a^2}{a} - k^2 = \frac{2a^2}{z},$$

by means of which, the distance $z$ at which the body will be arrested, can be obtained In the case of $2a^2 = k^2a$ this distance will be infinite, which denotes that the body will never be arrested in its motion. This is also the case, when $2a^2 \angle k^2a$,

from which there results for $z$ a negative value that cannot belong to any point of the indefinite line DB, in the direction of which the body has been projected. In these two cases, the motion of the body approaches more and more to uniformity, according as the distance of the body from A increases. When the distance $z$ becomes very great, and the motion sensibly uniform, its velocity, as determined by equation (e), will be very nearly equal to $\sqrt{k^2 - \frac{2a^2}{a}}$, or to $\sqrt{k^2 - 2ga}$, on the supposition that $a^2 = ga^2$, that is to say, on the supposition that the body is projected from the surface of a sphere, of which the radius is equal to $a$, and of which the attraction is equal to $g$. This shows that the diminution of the initial velocity $k$ will be so much the greater, as this force and this radius are more considerable.

## CHAPTER III

### OF CURVILINEAR MOTION

### I *General Formulæ of this Motion*

144 In curvilinear motion, the curve described by the moveable is termed the *trajectory* of this material point. At the end of any time $t$, let M (fig. 37) be the position of the moveable. If $s$ denotes the arc CM of the trajectory comprised between the moveable and a fixed point C arbitrarily taken on this same curve, $s$ will be a function of $t$, so that, in any curvilinear motion whatever, we shall have

$$s = \text{F}t.$$

If at the same instant, $x, y, z$ denote the three rectangular coordinates of the moveable, these variables will be also functions of $t$, and we shall in like manner have

$$x = ft, \quad y = f't, \quad z = f''t$$

When these three equations are known, we can deduce from them, by the elimination of $t$, the two equations in $x$, $y$, $z$, of the trajectory By means of the equations of this curve, $s$ can be determined in a function of one of the three coordinates, and, consequently, in a function of $t$, in this way the law of the motion on the trajectory will be obtained Each of the three preceding equations is that of the rectilineal motion of the projection of the moveable on one of the axes of the coordinates, it follows, therefore, that the complete determination of the curvilinear motion of a material point, is reducible to that of three rectilineal motions, which will be the motions

of its projections on $ox$, $oy$, $oz$, the three axes of the coordinates When these three motions are uniform, that of the moveable will be also uniform and rectilinear, and conversely($a$)

145 During the instant $dt$, the moveable will describe $ds$ the element of its trajectory if in this infinitely small interval of time, the action of the forces which solicit it be neglected, its motion during this interval may be considered as uniform and rectilinear Therefore, denoting the velocity acquired at the end of the time $t$ by $v$, we shall have

$$v = \frac{ds}{dt}$$

If these forces cease to act at the instant in question, the moveable will continue to move with this velocity, and along MT the production of the element $ds$, that is to say, along the tangent to the trajectory, since, in consequence of the inertia of matter, it cannot then change, either the direction of its motion, or the magnitude of its velocity (No 113) Hence, a material point which describes any curve line whatever, may be considered as being actuated at each instant, by a velocity in the direction of a tangent to this curve, and expressed by the ratio of its differential element to the element of the time

If, at the end of the same time $t$, $p$, $q$, $r$ denote the velocities of the projections of the moveable on the three axes of $x$, $y$, $z$, we shall likewise have, in these three rectilinear motions,

$$p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}, \quad r = \frac{dz}{dt}$$

Now if $a$, $\beta$, $\gamma$ be the angles which the tangent to the trajectory, or the direction of the velocity $v$, makes with the parallels to the axes of $x$, $y$, $z$, we have (No 17)

$$\cos a = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds},$$

hence we obtain(*b*)

$$p = v\cos\alpha,\quad q = v\cos\beta,\quad r = v\cos\gamma, \tag{1}$$

and also at the same time,

$$p^2 + q^2 + r^2 = v^2.$$

As the time $t$ goes on continually increasing, its differential is always positive. The velocities $p, q, r$ are positive or negative, according as the coordinates $x, y, z$ increase or decrease. In equations (1), the velocity $v$ may be regarded as a positive quantity The direction of this velocity, or the part MT of the tangent to the trajectory, along which it will be directed, will then be determined by the signs of $p, q, r$, which will indicate whether the angles $\alpha, \beta, \gamma$ are acute or obtuse In the equation $v = \frac{ds}{dt}$, the velocity $v$ is considered to be positive or negative, according as the arc $s$ increases or decreases.

The *components* of $v$ the velocity of a material point, are $p, q, r$, the velocities of its three projections on the rectangular axes, and each of these three components is what is meant by *the velocity of the moveable* parallel to the axis to which it refers. From a comparison of equations (1) with those of No 31, it appears that this composition of velocities is performed according to the same rules as that of forces Hence it follows, that if through the point M any line MA be drawn, which makes with the parallels to the axes of $x, y, z$, drawn through this same point, the angles $a, b, c$, which may be either acute or obtuse, the general expression for the component of the velocity $v$ parallel to this line MA, will be

$$p\cos a + q\cos b + r\cos c.$$

The quantity of motion (No. 126) of an isolated material point, and that of a body all whose points are actuated by equal and parallel velocities, can be decomposed into other quantities of this nature, and these may be reduced to one,

according to the same rules as the velocities which they have for factors.

146 Let $p+p'$, $q+q'$, $r+r'$, be what the three components of the velocity of the moveable parallel to the axes of $x$, $y$, $z$, become at the end of the time $t+dt$, so that $p'$, $q'$, $r'$ may represent the infinitely small increments of the velocity, which have place in these directions, during the instant $dt$. The increment of velocity in the direction of the line MA will be

$$p' \cos a + q' \cos b + r' \cos c$$

Now, whatever be the quantities $p'$, $q'$, $r'$, if we make

$$u^2 = p'^2 + q'^2 + r'^2,$$

$u$ being considered as a positive quantity, we can always find three angles $\alpha'$, $\beta'$, $\gamma'$, either acute or obtuse, such that

$$p' = u \cos \alpha', \quad q' = u \cos \beta', \quad r' = u \cos \gamma';$$

by means of which, the increment of velocity in the direction MA will become

$$u\,(\cos a \cos \alpha' + \cos b \cos \beta' + \cos c \cos \gamma').$$

Moreover, we know that the quantity between the brackets is the cosine of a certain angle $\sigma$ Therefore, the increment in question is equal to $u \cos \sigma$, consequently, $u$ is its greatest value, and it answers to the direction of the right line MA, for which the angles $a$, $b$, $c$ are the same as $\alpha'$, $\beta'$, $\gamma'$, and thus renders the coefficient of $u$ equal to unity. In any other direction whatever, the increment of the velocity will be equal to the greatest value $u$, multiplied by the cosine of the angle $\sigma$, which this direction makes with that of the greatest value; hence it follows, that it will vanish with respect to all directions perpendicular to those of its greatest value.

Whatever be the variation of the velocity of the moveable in magnitude and direction, during the instant $dt$, there is always a certain direction, for which the increase of velocity

is a *maximum*, and to which this property belongs, namely that for all directions perpendicular to this one, the velocity is neither increased nor diminished

147 The direction of a force which acts on a material point in motion, is the right line along which it increases or diminishes the acquired velocity, and perpendicularly to which it does not produce any change Thus, when we say that the weight of a body which moves in any direction whatever is vertical, like that of a body at rest, it is meant by this, that this force increases the vertical velocity, and does not produce any change whatever in the horizontal velocity

This being agreed on, let U, U′, U″, &c, denote the intensities of the different forces which at the end of the time $t$ act on the material point, the curvilinear motion of which is considered, $a, b, c$, $a', b', c'$, $a'', b'', c''$, &c, the angles which their given directions make with parallels to the axes of $x, y, z$, and X, Y, Z, the sums of their components in the directions of these axes, we shall have (No 32)

$$\text{X} = \text{U}\cos a + \text{U}'\cos a' + \text{U}''\cos a'' + \&c\,,$$
$$\text{Y} = \text{U}\cos b + \text{U}'\cos b' + \text{U}''\cos b'' + \&c\,,$$
$$\text{Z} = \text{U}\cos c + \text{U}'\cos c' + \text{U}''\cos c'' + \&c$$

Let now $u, u', u''$, &c, be the infinitely small velocities which these forces U, U′, U″ would produce during the instant $dt$, in their respective directions, on the supposition that each of them acted by itself on the moveable actuated by the velocity $v$ It is evident, as in No 116, that the circumstance of the forces acting *simultaneously* will, in no respect, influence either the magnitudes or directions of the velocities, which are actually produced, consequently, if we still denote by $p', q', r'$, the infinitely small quantities by which $p, q, r$, the velocities of the projections of the point, are increased in the instant $dt$, these quantities will be the sums of the components of $u, u', u''$, &c., in the directions of these three axes, so that we shall have

$$p' = u \cos a + u' \cos a' + u'' \cos a'' + \&c\,,$$
$$q' = u \cos b + u' \cos b' + u'' \cos b'' + \&c.,$$
$$r' = u \cos c + u' \cos c' + u'' \cos c'' + \&c$$

Now, since it appears from No. 118, that the measure of any force is the velocity which it is capable of producing, it is evident that

$$u = \text{U}dt, \quad u' = \text{U}'dt, \quad u'' = \text{U}''dt, \ \&c\,,$$

hence, if the values of $p'$, $q'$, $r'$, be compared to those of X, Y, Z, there results

$$p' = \text{X}dt, \quad q' = \text{Y}dt, \quad r' = \text{Z}dt,$$

which shews that the increment of the component of the velocity in the direction of each axis, in the instant $dt$, is the velocity produced during this instant, by the entire component, in the direction of this same axis, of the given forces, which act on this material point.

It is because the forces are proportional to the velocities which they impress on the moveable in an infinitely small portion of time, (which infinitely small velocities are the same, whether these forces act separately, or simultaneously,) that this result obtains It likewise follows, that if three forces not comprised in the same plane are applied to the moveable, and that if, on the directions of these three forces U, U′, U″, there be taken, reckoning from their point of application, right lines of a finite magnitude, which are to each other as the corresponding velocities $u$, $u'$, $u''$, the resultant of these forces will be represented in magnitude and direction by the diagonal of a parallelopiped of which these three lines are the adjacent sides, and its magnitude will be to that of each of these forces as the diagonal is to the corresponding side

148 If the forces which act on the moveable are independent of its velocity and of its position in space, the motions of its three projections on the axes of the coordinates will be independent of each other, so that its projection on each axis

at the end of any time whatever will be found at the same point, and it will have the same velocity as if the forces and the velocities parallel to the other two axes were cypher. In general, this will not be the case when the given forces vary in magnitude or direction, either with the position of the moveable, or with its acquired velocity, its velocity and position can, however, be always determined at each instant, in the following manner Since all the forces which act on the moveable are reducible always to one, let U, which is capable of producing the velocity $u$, be this unique force, and let $\varepsilon$ be the space which it causes the moveable to describe in the instant $dt$, in its direction, independently of $v$, the velocity of this material point at the end of the time $t$. By what has been stated in No 114, we shall have

$$\varepsilon = \tfrac{1}{2} u dt$$

But, in virtue of this acquired velocity $v$, and of the action of the force U, or of its components, the spaces traversed by the projections of the moveable on the axes of $x, y, z$, during the instant $dt$, will be

$$pdt + \tfrac{1}{2}p'dt, \quad qdt + \tfrac{1}{2}q'dt, \quad rdt + \tfrac{1}{2}r'dt,$$

therefore, because$(c)$

$$p' = u \cos a, \quad q' = u \cos b, \quad r' = u \cos c,$$

and in consequence of equations (1) and the value of $\varepsilon$, we shall have

$$\begin{aligned} x' - x &= \omega \cos \alpha + \varepsilon \cos a, \\ y' - y &= \omega \cos \beta + \varepsilon \cos b, \\ z' - z &= \omega \cos \gamma + \varepsilon \cos c; \end{aligned}$$

$\omega$ being the space which the moveable describes in the instant $dt$ in virtue of the velocity $v$ solely, and $x', y', z'$, its three coordinates at the end of the time $t + dt$, which were $x, y, z$, at the end of the time $t$

This being established, let M (fig. 37) be the point of the

trajectory of which $x$, $y$, $z$, are the three coordinates, and MT the direction of the velocity $v$. Likewise let MA be that of the force U. If on MA, MT, there be taken the lines MH and MK equal to $\varepsilon$ and $\omega$, and if the parallelogram MHM′K, of which these lines are adjacent sides, be completed, the extremity M′ of this diagonal will be, in virtue of the preceding equations, the point of which the coordinates are $x'$, $y'$, $z'$, or the position of the moveable at the end of the time $t+dt$(d).

Naming $v'$ the velocity of the moveable at the point M′, which velocity will be directed along MT′ the production of the line MM′, its value will be the component of $v$ in the direction of MM′, increased by the velocity produced in this direction, by the action of the force U during the instant $dt$. As the space $\varepsilon$ is infinitely small, relatively to $\omega$, it follows that the angle TMM′ is also infinitely small, therefore if we neglect quantities infinitely small of the second order(e), the component of $v$ will be the velocity $v$ itself. Moreover, if $\delta$ denote the angle AMM′, which the direction of the force U makes with the side MM′ of the trajectory, $u \cos \delta$ will be the increment of the velocity which will be produced by the action of this force, hence there will result

$$v' = v + u \cos \delta$$

If $v'dt = \omega'$, and if on M′T′ we take a part M′K′ equal to $\omega'$, and if M′A′ be the direction of the force which acts on the moveable when it arrives at M′, by taking on this line a part M′H′ equal to the space that this force causes the moveable to describe in an instant $dt$, and completing the parallelogram M′H′M″K′, M″ the extremity of the diagonal will be a third point of the trajectory. By commencing this series of constructions at the point of departure of the moveable, where it is necessary to know its velocity in magnitude and direction, it is evident that all the points of its trajectory can be successively determined, whether it be a plane curve, or one of double curvature; and also, at the same time, the velocity

with which it is actuated at each of these points. If the intervals of time which have been supposed infinitely small and denoted by $dt$, are only extremely small, a series of points will be obtained, which will be the summits of a polygon, and these will differ so much the less from the trajectory as its sides are smaller If the velocity on each side be regarded as constant, and if for its value there be assumed the semi-sum of the velocities with which the body is actuated at the two extremities, the time employed in describing any portion of the polygon may be computed, consequently we can in this manner determine, to any required degree of accuracy, the curves described by the moveable, and also its velocity and position at any given instant on this curve. But it is preferable to make the values of the coordinates of the moveable in functions of the time, depend on differential equations, which can be afterwards integrated, in these cases in which it is possible.

149 These differential equations of curvilinear motion are an immediate result of the principles established in No 147

In fact, the components of the velocity of the moveable, parallel to the axes of the coordinates $x, y, z$, being $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, at the end of any time $t$, their increments, during the instant $dt$, will be $d\,\frac{dx}{dt}, d.\frac{dy}{dt}, d.\frac{dz}{dt}$, and as each of them arises from the component of the force, which acts at this instant on the moveable, parallel to the corresponding axis, it is evident that if X, Y, Z, denote the components of this force parallel to the axes of the coordinates $x, y, z$, we shall have

$$d.\frac{dx}{dt} = \mathrm{X}dt, \quad d\,\frac{dy}{dt} = \mathrm{Y}dt, \quad d\,\frac{dz}{dt} = \mathrm{Z}dt,$$

or, what is the same thing,

$$\frac{d^2x}{dt^2} = \mathrm{X}, \quad \frac{d^2y}{dt^2} = \mathrm{Y}, \quad \frac{d^2z}{dt^2} = \mathrm{Z} \qquad (2)$$

The problem in each case will depend on the integration of these three equations of motion, and the process of the preceding number may be considered as a general method of approximation, in order to this integration Their integrals will contain six constant arbitraries, which can be determined by means of the three coordinates of the moveable at the commencement of the motion, and of the three components of the initial velocity, that is to say, by means of the values of the six quantities $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, which will be given in the case of $t = 0$. These integrals, and their first differentials, will then make known the position of the moveable at any instant whatever, and also its velocity in magnitude and direction. If the time $t$ be eliminated between them, the two equations of the trajectory will be obtained If it is known beforehand, that this curve is plane, its plane may be assumed for that of the axes of $x$ and $y$ for example, which will reduce the three preceding equations to the two first.

150 At the end of the time $t$, let $a, b, c$, be the coordinates of a second material point, whose position it is proposed to compare with that of the first The axes of the coordinates being those of $x$, $y$, $z$, let

$$x = a + x', \quad y = b + y', \quad z = c + z';$$

the variables $x'$, $y'$, $z'$, will make known at each instant the position of the first point relatively to the second, and by equations (2), we shall have

$$\frac{d^2x'}{dt^2} = \text{X} - \frac{d^2a}{dt^2}, \quad \frac{d^2y'}{dt^2} = \text{Y} - \frac{d^2b}{dt^2}, \quad \frac{d^2z'}{dt^2} = \text{Z} - \frac{d^2c}{dt^2},$$

by means of which they can be determined in functions of the time.

When the motion of the second point is not known, but there are only given A, B, C, the components of the force which solicits it, parallel to the axes of the coordinates, we shall have

$$\frac{d^2a}{dt^2} = \text{A}, \quad \frac{d^2b}{dt^2} = \text{B}, \quad \frac{d^2c}{dt^2} = \text{C},$$

and there will result

$$\frac{d^2x'}{dt^2} = \text{X} - \text{A}, \quad \frac{d^2y'}{dt^2} = \text{Y} - \text{B}, \quad \frac{d^2z'}{dt^2} = \text{Z} - \text{C},$$

for the equations of the relative motion of the first point If the force of which A, B, C, are the components, acts at the same time on the two moveables, these components will also occur in the value of X, Y, Z, and will disappear from these last equations. This will be the case, for example, with respect to bodies moving at the surface of the earth, the positions of which are referred to determinate points of this surface. the forces relative to these points, which arise from the diurnal motion of the earth, do not occur in the equations of the different motions which are considered on its surface, and they should not at all be taken into account in the formation of these equations

However, it is not to be understood by this, that the motions which are observed are altogether independent of the velocity of rotation of the earth, for it influences in a small degree the intensity of gravity, and, consequently, the vertical motions. Moreover, when a body falls from a considerable height, the velocity of rotation with which it is actuated at the point of departure, is somewhat greater than that of the velocity at the foot of the vertical drawn through this point, hence it is easy to perceive, that the moveable must deviate a little from this line, and meet the earth at a small distance from its lower extremity This deviation, which has been actually observed, proves by direct experiment the motion of the earth about its axis($f$). The motions which are independent of this rotation are those which are produced by the shock of two bodies, and also those which arise from the muscular action of men and other animals

151. Equations (2) are those of the motion of a material point entirely free, but it is easy to extend them to a material

point constrained to move on a given surface For this purpose, it is only necessary, as in the case of equilibrium (No 36), to join to the given forces which act on the moveable, a force of unknown magnitude representing the resistance of the surface. This force will be normal to the given surface If N denotes this force, and $\lambda$, $\mu$, $\nu$, the angles which it makes with the productions of the coordinates $x$, $y$, $z$, the equations of motion will then become

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \text{X} + \text{N}\cos\lambda,\\ \frac{d^2y}{dt^2} &= \text{Y} + \text{N}\cos\mu,\\ \frac{d^2z}{dt^2} &= \text{Z} + \text{N}\cos\nu\end{aligned}\right\} \quad (3)$$

If the equation of the given surface be $\text{L} = 0$, and, if for conciseness, we make

$$\text{V} = \pm\left(\frac{d\text{L}^2}{dx^2} + \frac{d\text{L}^2}{dy^2} + \frac{d\text{L}^2}{dz^2}\right)^{-\frac{1}{2}},$$

we shall have (No 21), at the same time

$$\cos\lambda = \text{V}\frac{d\text{L}}{dx}, \quad \cos\mu = \text{V}\frac{d\text{L}}{dy}, \quad \cos\nu = \text{V}\frac{d\text{L}}{dz}$$

If after having substituted these values in equations (3), the product NV be eliminated between them, the two equations which result, combined with $\text{L} = o$, will enable us to determine $x$, $y$, $z$, in functions of $t$. Then from one of equations (3), or from any combination of these equations, the value of NV may be deduced, and as N must be always a positive quantity, the sign of this value will indicate that of V, by means of which the normal force and the direction in which it acts can be completely determined N

If the moveable is constrained to move on two given surfaces, or on their curve of intersection, it may likewise be considered as entirely free, if with the given forces there be

combined two unknown forces N, N′, normal to these surfaces; let $\lambda$, $\mu$, $\nu$, be the angles which determine the direction of the first with respect to the axes of $x, y, z$, and $\lambda'$, $\mu'$, $\nu'$, those which refer to the second, then there will result

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \text{X} + \text{N}\cos\lambda + \text{N}'\cos\lambda',\\ \frac{d^2y}{dt^2} &= \text{Y} + \text{N}\cos\mu + \text{N}'\cos\mu',\\ \frac{d^2z}{dt^2} &= \text{Z} + \text{N}\cos\nu + \text{N}'\cos\nu',\end{aligned}\right\} \quad (4)$$

for the equations of the motion. If $\text{L} = 0$ is the equation of the surface of which N expresses the resistance, and if $\text{L}' = 0$, be that of the surface to which N′ refers, the values of $\cos\lambda$, $\cos\mu$, and $\cos\nu$ will be the same as in the preceding case, and those of $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$, may be deduced from them by changing V and L into V′ and L′. When both the one and the other are substituted in equations (4), the products NV, N′V′ can be eliminated, and the equation which results from this elimination, combined with the given equations $\text{L} = 0$, $\text{L}' = 0$, will enable us to determine the values of $x, y, z$, in functions of $t$. This being done, we can deduce from any two of equations (4) the values of NV and N′V′, the signs of which will make known those of V and V′, and by this means, the normal forces N and N′ can be determined, and also the directions in which they act, their resultant will be expressed in magnitude and direction, by the resistance of the curve on which the moveable is constrained to move.

152 A more simple form can be assigned to equations (4) in the following manner, let $m$ be the mass of the moveable, and $m$P the pressure, which in its state of motion it exercises on the curve it is constrained to describe, if $\omega$, $\omega'$, $\omega''$, be the angles which the direction of this force makes with the productions, on the positive side, of $x, y, z$, the coordinates of this point, the resistance which the curve opposes to the mo-

tion of the moveable, considered as an accelerating force, will be equal and contrary to P, by joining it to the given forces X, Y, Z, which act on the moveable, we shall have, in place of equations (4),

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X - P\cos\omega, \\ \frac{d^2y}{dt^2} &= Y - P.\cos\omega', \\ \frac{d^2z}{dt^2} &= Z - P\cos\omega'' \end{aligned}\right\} \quad (5)$$

The *direction* of the force P is not given *à priori*, it is only known that it is normal to the given curve, from which it follows that the cosine of the angle comprised between this direction and the tangent to the trajectory must be equal to zero, this gives

$$\frac{dx}{ds}\cos\omega + \frac{dy}{ds}\cos\omega' + \frac{dz}{ds}\cos\omega'' = 0. \quad (6)$$

Moreover, the angles $\omega, \omega', \omega''$ are connected together by the equation

$$\cos^2\omega + \cos^2\omega' + \cos^2\omega'' = 1$$

$P, \omega, \omega', \omega''$ can be eliminated between these equations, for by adding together equations (5), after having first multiplied them by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ respectively, and then taking into account equation (6), and making, in order to abridge,

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = \phi,$$

we shall have

$$\frac{dxd^2x + dyd^2y + dzd^2z}{dsdt^2} = \phi.$$

If the identical equation

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = \frac{ds^2}{dt^2},$$

be differentiated, and then divided by $2ds$, it will appear that the first member of the preceding equation is the same thing as $\frac{d^2s}{dt^2}$, therefore we shall have simply

$$\frac{d^2s}{dt^2} = \phi \qquad (7)$$

The force $\phi$ is the sum of the components of the given forces acting along the tangent to the trajectory, which components can be regarded as positive or negative, according as they tend to increase or diminish $s$, the arc described by the moveable. Consequently, equation (7) indicates, that in curvilinear, equally as in rectilinear motion, the force which acts on the moveable in the direction of its motion, is equal to the second differential coefficient of the space described, and because $v = \frac{ds}{dt}$, we may also state, that it is equal to the first differential coefficient of the acquired velocity This equation being independent of the resistance of the curve, obtains equally in a motion entirely free, and in that of a material point constrained to exist on a given curve, but it is principally in the case of a material point which moves on a given curve, that this equation can be useful. The values of $x, y, z$, in functions of $s$, may be deduced from the equations of this curve, and after having substituted them in equation (7), it only remains to integrate this equation of the second order between $s$ and $t$. The two constant arbitraries which are introduced in this integration, can be determined by means of the values of $s$ and $\frac{ds}{dt}$ when $t = 0$, that is to say, by means of the initial position and velocity of the moveable When the three coordinates $x$, $y$, $z$, shall have been determined in functions of $t$, by means of the integral of equation (7), combined with the two given equations of the trajectory; equations (5) will enable us to determine at any instant whatever, the three components of the pressure P, to which the curve, on which

the moveable is constrained to move, is subjected. In the following chapter there will be given a simpler determination of this force both in magnitude and direction

## II. *Principal Consequences of the preceding Formulæ.*

153 When the moveable is solicited by a force directed towards a fixed centre, the three first integrals of equations (2) can be obtained at once For this purpose, let the origin of the coordinates be placed at this point, let the radius vector represent, in magnitude and direction, the force which solicits the moveable, and let the parallelopiped, of which this radius is the diagonal, and the axes of $x, y, z$, its three adjacent sides, be constructed The three coordinates $x, y, z$, of the moveable, will be equal to these three sides respectively, and will represent the three components of the given force, so that we shall have

$$X : Y : Z :: x : y : z,$$

hence we deduce

$$Xy = Yx, \quad Zx = Xz, \quad Yz = Zy.$$

On the other hand, equations (2) can be replaced by the following

$$\left.\begin{aligned} yd^2x - xd^2y &= (Xy - Yx)\,dt^2, \\ xd^2z - zd^2x &= (Zx - Xz)\,dt^2, \\ zd^2y - yd^2z &= (Yz - Zy)\,dt^2, \end{aligned}\right\} \quad (a)$$

but, in virtue of the preceding equations, their second members vanish, and as their first members are the differentials of $ydx - xdy$, $xdz - zdx$, $zdy - ydz$, we shall obtain by integrating

$$\left.\begin{aligned} ydx - xdy &= cdt, \\ xdz - zdx &= c'dt, \\ zdy - ydz &= c''dt, \end{aligned}\right\} \quad (b)$$

$c, c', c''$, being three constant arbitraries.

154. The theorem contained in these first integrals of the

equations of the motion, implies, that the area described during each instant $dt$, by the radius vector of M, the projection of the moveable on the plane of $x$, $y$, is constant and equal to $\frac{1}{2}cdt$, and the same is likewise the case for the projections of the moveable on the planes $x, z$; $y, z$, for at the end of the time $t$, let AMB (fig 38) be the projection of the trajectory of the moveable on the plane of the axes of the coordinates $x$ and $y$, OP and MP, the abscissa $x$, and the ordinate $y$, C being the point where this curve intersects the axis $Oy$, let $u$ denote the sector COM, $p$ the area COPM, $q$ the triangle OPM, we shall have

$$u = p - q, \quad q = \tfrac{1}{2}xy.$$

If M′ is the projection of the moveable at the end of the time $t + dt$, MOM′ will be the area described by the radius vector of this projection during the instant $dt$, likewise this will be the differential of $u$, or of $p - q$, and because

$$dp = ydx, \quad dq = \tfrac{1}{2}xdy + \tfrac{1}{2}ydx,$$

we shall have

$$du = \tfrac{1}{2}(ydx - xdy),$$

hence it appears that the first equation (b) indicates, that the area described during the instant $dt$, by the radius vector of M, the projection of the moveable, is constant and equal to $\frac{1}{2}cdt$, therefore, also, the area described during $t$ any time whatever, is proportional to this variable and equal to $\frac{1}{2}ct$. The areas described in this same time, by the radii vectores of the projections of the moveable, on the planes of $x$ and $z$, and of $y$ and $z$, will be likewise equal to $\frac{1}{2}c't$, $\frac{1}{2}c''t$

It may be therefore inferred, that when a material point is subjected to the action of a force constantly directed towards a fixed centre, the areas described about this point by the radius vector of its projection on any plane whatever, passing through this same point, are proportional to the time employed to describe them.

Conversely, when this property obtains with respect to three rectangular planes drawn through the centre of the areas, it may be inferred that the force, or the resultant of the

forces which solicit the moveable, is constantly directed towards this fixed centre

In fact, if equations (b) are given, we shall obtain by differentiating them,

$$yd^2x - xd^2y = 0, \quad xd^2z - zd^2x = 0, \quad zd^2y - yd^2z = 0,$$

consequently, in virtue of equations (a), which are those of any motion whatever, we shall also have

$$\mathrm{X}y = \mathrm{Y}x, \quad \mathrm{Z}x = \mathrm{X}z, \quad \mathrm{Y}z = \mathrm{Z}y,$$

therefore, the forces X, Y, Z, will be to each other as $x$, $y$, $z$, the coordinates of the moveable, which is all that is required in order for us to be assured that their resultant must be constantly directed towards the origin of the coordinates In fine, this force may be either attractive or repulsive, that is to say, it may act along either the radius vector of the moveable, or along its production

155 When a material point is subjected to the action of a force directed towards a fixed centre, it is evident that its trajectory is a plane curve, since there is no reason why it should deviate towards one side rather than the other of a plane passing through the direction of its initial velocity and through the fixed centre This may likewise be inferred from equations (b), for if we multiply them by $z$, $y$, $x$ respectively, and then divide them by $dt$, there arises from their addition

$$cz + c'y + c''x = 0$$

As this plane may be assumed for that of $x$ and $y$, it follows, that the area described by the radius vector itself of the moveable, in the plane of its trajectory, will be proportional to the time, and moreover, the preceding theorem is reducible to this proportionality. In fact, if it obtains for the area described on the plane of the trajectory, it will equally obtain for the area described by the radius vector of the projection of the moveable on every other plane, for this other area is no other than the projection of the first on this plane, and

we know (No 10) that the projection of a plane bears a constant proportion to the projected area

156. The infinitely small area MOM′ may be also expressed in polar coordinates. For this purpose, let $r$ denote the radius vector OM, $\theta$ the angle MO$x$ which it makes with the axis of $x$. With the point O as centre, let there be described the arc of the circle OMN, which cuts at the point N the radius vector OM′ corresponding to the angle $\theta - d\theta$, the length of which arc will be $rd\theta$ The circular sector MON will be equal to $\frac{1}{2}r^2d\theta$, and may be assumed for the area MOM′, by neglecting the infinitely small area MNM′ of the second order. Consequently, we should have

$$ydx - xdy = r^2d\theta,$$

an equation, which may in fact be verified, by means of the values

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and of their differentials,

$$dx = \cos\theta dr + r\sin\theta d\theta, \quad dy = \sin\theta dr - r\cos\theta d\theta,$$

because that of the angle $\theta$ is in this case $-d\theta$. In this manner, the first equation (b) will assume the form

$$r^2d\theta = cdt,$$

which is that under which it is usually employed.

The element of the curve may be also expressed in polar coordinates. Denoting the arc CM by $\sigma$, and this element by $d\sigma$, we shall have at the same time

$$\text{MM}' = d\sigma, \quad \text{MN} = rd\theta, \quad \text{NM}' = dr,$$

by considering MNM′ as a rectilineal triangle, right angled at N, we may therefore conclude

$$d\sigma^2 = dr^2 + r^2d\theta^2,$$

which we might also deduce from the formula

$$d\sigma^2 = dx^2 + dy^2,$$

by means of the preceding values of $dx$ and $dy$.

It may be observed on this occasion, that in a plane trajectory, the components of the velocity of the moveable in the direction of MO′ the production of the radius vector MO, and in the direction of the perpendicular to this radius, are respectively expressed by

$$\frac{dr}{dt}, \quad \frac{rd\theta}{dt},$$

for the angle O′MT which this production makes with the tangent MT, is the complement of the angle M of the triangle M′MN, by this triangle therefore we have

$$\cos \mathrm{O'MT} = \frac{dr}{d\sigma}, \quad \sin \mathrm{O'MT} = \frac{rd\theta}{d\sigma},$$

and by multiplying this cosine and this sine by the velocity $\frac{d\sigma}{dt}$ directed along MT, we shall have the components in question It is frequently convenient to employ them. They differ from $\frac{dx}{dt}, \frac{dy}{dt}$ the components of the same velocity in this, that the directions of these last are fixed, while those of the preceding vary with the position of the moveable

The velocity $\frac{d\theta}{dt}$ with which the radius vector OM describes the angle COM reckoned from a fixed right line, is termed the *angular* velocity of the moveable It may be obtained, as is evident, from $\frac{rd\theta}{dt}$, its velocity perpendicular to OM, by dividing it by the length of this radius

(157.) Let us now revert to the differential equations of motion If equations (5) of No. 152 be multiplied by $dx$, $dy$, $dz$, respectively, and then added together, there will be obtained by taking into account equation (6) of the same number, and observing that

$$\left.\begin{aligned} \frac{dxd^2x + dyd^2y + dzd^2z}{dt^2} &= \tfrac{1}{2} d\, \frac{ds^2}{dt^2} = \tfrac{1}{2} d\, v^2, \\ \tfrac{1}{2} d\, v^2 &= \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz. \end{aligned}\right\} \quad (c)$$

If the expressions of the given forces X, Y, Z, do not involve the time $t$ or the velocity $v$, explicitly, and if, $x$, $y$, $z$, being regarded as independent variables, this formula is an exact differential, which consequently implies that

$$\text{X}dx + \text{Y}dy + \text{Z}dz = d.\text{F}(x, y, z),$$

in which F denotes a given function, then by integrating equation (c) we obtain

$$v^2 = 2\text{F}(x, y, z) + \text{C},$$

in which C expresses a constant arbitrary. In order to eliminate it, let $a$, $b$, $c$, $k$ be the initial values of $x$, $y$, $z$, $v$, we shall have

$$k^2 = 2\text{F}(a, b, c) + \text{C},$$

and by taking this equation from the preceding, there results

$$v^2 = k^2 + 2\text{F}(x, y, z) - 2\text{F}(a, b, c) \qquad \text{(d)}$$

As this result is independent of N, the resistance of the curve, which resistance is equal and contrary to the force P, that occurs in the equations from which it has been obtained, it follows that it has place equally in the motion of a material point entirely free, and in the motion on a given curve or surface

The immediate consequence of this equation (d) is, that the velocity is constant and the motion uniform, as often as the moveable is not solicited by any given force, for then the function F is cypher, and we have $v = k$, whether the motion is performed on a given curve or surface, or whether the moveable is entirely free

This equation shews us moreover, that in the hypothesis which has been made respecting the nature of the forces X, Y, Z, the increment of the square of the velocity of the moveable, in passing from one point to another, is always the same, whatever be the curve described, and depends solely on $a$, $b$, $c$, $x$, $y$, $z$, the coordinates of the two extreme points When this curve is given, or when the moveable is only constrained to

move on a given surface, $k$ should be assumed as the velocity of the moveable, which is tangential to this curve or this surface. If the percussion impressed on the moveable at the commencement of its motion has not this direction, it can be resolved into two other forces, the one normal, the other tangential, the first will be destroyed by the resistance of the given curve or surface, so that it is the second which will produce the velocity $k$, and determine its direction.

c denoting a constant arbitrary, the equation

$$\mathrm{F}(x, y, z) = \mathrm{c},$$

will be that of a surface which will be reached with equal velocities by all moveables subjected to the action of the same forces, which commence to move from a point of which $a$, $b$, $c$ are the coordinates, with the same velocity $k$, in different directions. When, for example, these moveables are only acted on by the force of gravity, this equation is that of a horizontal plane.

In the case of a given curve, if the values of $x$, $y$, $z$, be deduced from its equations in functions of the arc $s$, there will result by substituting them in equation (d), and putting $\frac{ds}{dt}$ in place of $v(g)$,

$$dt = \mathrm{s}\, ds,$$

where s is a given function of $s$, consequently, in this case, the determination of the time as a function of the space described, will be reduced to the integration of a given differential. But the supposition on which equation (d) is founded, and consequently, this equation, has not place when the moveable experiences any resistance in its motion through a medium, for this is a force dependent on the velocity, neither will it obtain, when the question is respecting the motion of a material point, which is attracted or repelled by other points that are themselves in motion; a circumstance which will introduce the time explicitly in the values of X, Y, Z. In these two cases, if the trajectory be a given curve, equation (c) ought to be employed,

in which $\frac{ds}{dt}$ should be substituted in place of $v$, by which means this equation will be changed into equation (7) of No. 152.

158. The formula $\text{X}dx + \text{Y}dy + \text{Z}dz$ will be an exact differential as often as the moveable being attracted or repelled by fixed centres, the intensities of these forces are expressed by functions of the distance from the centres from which they emanate.

In fact, let $e, f, g$ be the three coordinates of one of the fixed centres, referred to the same axes as $x, y, z$, and $r$ the distance of the moveable from this point, we shall have

$$r^2 = (e - x)^2 + (f - y)^2 + (g - z)^2;$$

and the cosines of the angles which this line $r$ makes with the axes drawn through the moveable in the directions of the positive $x$s, $y$s, $z$s, will be the ratios of $e - x, f - y, g - z$, to $r$. If $\text{R}$ denotes the attractive force directed from the moveable towards this fixed centre, the expressions of its three components will be

$$\frac{\text{R}(e - x)}{r}, \quad \frac{\text{R}(f - y)}{r}, \quad \frac{\text{R}(g - z)}{r},$$

and consequently, the part of $\text{X}dx + \text{Y}dy + \text{Z}dz$ which arises from $\text{R}$, will be

$$\frac{\text{R}}{r}[(e - x)\,dx + (f - y)\,dy + (g - z)\,dz].$$

But by differentiating the value of $r^2$ we obtain

$$rdr = -(e - x)\,dx - (f - y)\,dy - (g - z)\,dz;$$

which reduces the preceding quantity to $-\text{R}dr$ If the force which emanates from the fixed centre was repulsive, it is only necessary to change the sign of this quantity, which will then become $\text{R}dr$, $\text{R}$ being regarded, in all cases, as a positive quantity.

It follows from this, that if the moveable is solicited by any number whatever of forces, such as R, R′, R″, &c., which emanate from fixed centres, and of which the distances from this material point, are $r$, $r'$, $r''$, &c., we shall have

$$\text{X}dx + \text{Y}dy + \text{Z}dz = \mp \text{R}dr \mp \text{R}'dr' \mp \text{R}''dr'' \mp \&c.,$$

the superior signs have place in the case of attractions, and the inferior signs in the case of repulsions. Now, supposing that each of these forces is a given function of the corresponding distance, all the terms of this value of $\text{X}dx + \text{Y}dy + \text{Z}dz$ will be differentials depending on one sole variable, and, consequently, this formula will be an exact differential, which was to be proved.

It appears from this and equation (d), that the increment of the square of the velocity arising from each of the forces R, R′, R″, &c, will be the same as if it solely existed; for example, with respect to the force R, this increment will be expressed by $\mp 2\int \text{R}dr$, the integral being taken in such a manner, that it may vanish for the initial value of $r$.

159. In the case of a heavy material point, which moves on a given curve in a vacuo, and without any friction on this curve, equation (d) will be reduced to

$$v^2 = k^2 + 2g(z - c),$$

$g$ denoting the gravity, and the direction of the axis of the positive $z$s being supposed to be that of this force, so that we may have

$$\text{X} = 0, \quad \text{Y} = 0, \quad \text{Z} = g.$$

Let ADBC (fig. 39) be the given curve, B its lowest and A its most elevated point, which need not exist in the same vertical as B, and D the point from which moveable sets out. If the origin of the vertical coordinates $z$ be placed in this last point, and if the initial velocity $k$ be due to the height $h$, we shall then have

$$c = 0, \quad k^2 = 2gh,$$

and, consequently,

$$v^2 = 2g(h + z).$$

It follows from this, that when the moveable attains the point B, the *maximum* velocity will be the same as if it fell from the height $h$, increased by that of the point D, above the horizontal plane drawn through the point B. In virtue of this acquired velocity, the moveable will ascend along BCA; its velocity will diminish continually, and if $h = 0$, it will be nothing at the point C situated in the same horizontal plane as D. Having reached the point C, the moveable will redescend along CB, and it will thus oscillate from D towards C, and from C towards D. When the constant $h$ does not vanish, the moveable will ascend above the point C. If the elevation of the point A above the horizontal plane, which comprises D and C, is greater than $h$, the moveable will not reach the point A; its velocity will be cypher at a certain point C′, and if through C′ there be drawn a horizontal plane which intersects the curve in another point D′, the moveable will oscillate indefinitely from C′ towards D′, and from D′ toward C′. The oscillations will be all *isochronous*, or of equal duration This is evident with respect to those which are performed in the same direction, and it appears also, that the duration of each oscillation from C′ to D′ is the same as that from D′ to C′, for any element whatever of the curve will be described with the same velocity in the two cases. This common duration of each of the entire oscillations, will depend on the form of the curve and on the magnitude of $h$.

When the elevation of A above the horizontal plane which passes through the point of departure is equal to $h$, the moveable will approach indefinitely to the point A, but will not attain to it until after the lapse of an infinite time. When this elevation is greater than $h$, the moveable will pass beyond the point A, and traverse the entire circumference of the given

curve. When it returns to the point D, its velocity will be the same as at the commencement of the motion, hence it follows, that it will perform an infinite series of revolutions, the duration of each of which will be equal, and dependent on the form of the curve and on the magnitude of $h$.

If the given curve($h$) is comprised in a vertical plane, which is a tangent to a cylinder of any base whatever, and if this plane is enveloped on the cylinder, so that the given curve may become a line of double curvature, the motion of the moveable, whether it be oscillatory, or one of revolution, will not be at all changed, it being always however supposed, that the point of departure and initial velocity remain the same, for then the value of $t$ in a function of $s$, determined as has been already stated (No. 157), will only depend on that of $z$ in a function of $s$, which will not be changed, whatever be the base of the vertical cylinder on which the given curve is traced.

160 In all cases in which equation (d) obtains, and in which the moveable is not constrained to move on a given curve, that which it describes in passing from one given point A to another given point B, possesses the following remarkable property. If the moveable be entirely free, the integral $\int vds$, taken from the point A unto the point B, is less than if it moved along any other curve, terminating at these two points, if it is constrained to move on a given surface, this property of the trajectory obtains only relatively to all curves traced on this surface, and which always terminate at the points A and B. In these two cases, $ds$ is the differential element of any curve whatever, which corresponds to the coordinates $x, y, z$, and $v$ is a function of these three variables and of a constant $k$, which is given by equation (d).

The demonstration of this theorem consists in proving that in virtue of the equations of motion, the variation of $\int vds$ is nothing, (the limits of this integral being supposed to be fixed). In consequence of this, $\int vds$ will be either a *maximum*

or *minimum*, but it is evident that it must be a *minimum* in this case, in which the moveable is entirely free, for the integral $\int vds$ increases indefinitely with the length of the curve, and, consequently, cannot be susceptible of having a maximum value.

Now, by the rules of the calculus of variations, we have

$$\delta . \int vds = \int \delta . vds, \quad \delta . vds = \delta vds + v\delta ds,$$

moreover, $dt$ being the element of the time, we have $ds = vdt$, therefore,

$$\delta vds = \tfrac{1}{2} dt\delta . v^2.$$

If equation (d) be differentiated, and if the variations $\delta x, \delta y, \delta z$, be substituted for the differentials $dx, dy, dz$, we shall have

$$\tfrac{1}{2} \delta . v^2 = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z.$$

By substituting for $\cos\lambda$, $\cos\nu$, $\cos\mu$, their values given in No. 151, and observing that($\iota$)

$$\frac{d\mathrm{L}}{dx}\delta x + \frac{d\mathrm{L}}{dy}\delta y + \frac{d\mathrm{L}}{dz}\delta z = \delta \mathrm{L},$$

equations (3) of same number will give

$$\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z - \mathrm{NV}\delta \mathrm{L}.$$

Now if the moveable is entirely free, the term $\mathrm{NV}\delta\mathrm{L}$ will not occur in this equation, and when it is constrained to move on a surface of which the equation is $\mathrm{L} = 0$, this term is cypher, for as all the curves, which are compared to the trajectory of the moveable, must also be traced on this surface, we have $\delta\mathrm{L} = 0$, therefore, in all cases this term must be suppressed, it follows from this that

$$\delta vds = \tfrac{1}{2} dt\delta . v^2 = \frac{d^2x}{dt}\delta x + \frac{d^2y}{dt}\delta y + \frac{d^2z}{dt}\delta z.$$

With respect to $v\delta ds$ the second term of the variation of $vds$, we have

$$ds^2 = dx^2 + dy^2 + dz^2,$$

and, consequently,

$$\delta ds = \frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta dy + \frac{dz}{ds}\delta dz,$$

hence, because $ds = vdt$, by changing the order of the characteristics $d$ and $\delta$ in the second member, we shall have$(k)$

$$v\delta ds = \frac{dx}{dt} d\delta x + \frac{dy}{dt} d\delta y + \frac{dz}{dt} d\delta z,$$

and by adding together these two parts of the value of $\delta$ $vds$, there results,

$$\delta\ vds = d\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right);$$

consequently we obtain,

$$\int\delta . vds = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z + \text{constant},$$

for the indefinite integral of $\delta$ $vds$ But as the two extreme points A and B are supposed to be fixed, the variations $\delta x$, $\delta y$, $\delta z$, which respect these points, must be nothing. Therefore, the definite integral $\int\delta$ $vds$, taken from the point A to the point B, which is equal to the variation $\delta$ $\int vds$, will be reduced to cypher, which it was required to demonstrate

161. When the moveable, being constrained to move on a curved surface, is not solicited by a given force, its velocity is constant (No. 157), and the integral $\int vds$ is then the product $vs$ Consequently, the arc $s$ described by the moveable is, in general, the shortest line between the points A and B, and from the uniformity of the motion, it follows that, in this case, the moveable goes from one point to another, in a less time than if it was forced to describe on the given surface any other curve than its trajectory However, if this surface is closed on all sides, as a sphere, for example, the points A and B will be the extremities of two arcs of a

great circle, one of which will be less and the other greater than all other arcs of lesser circles of the sphere, terminating at the same points, and the moveable can describe either the one or the other of these two portions of the same great circle, according to the direction of the initial velocity $h$, which is tangential to the sphere.

The differential equation of the trajectory may be presented under a form, in which the property of the shortest line on any surface whatever will be immediately apparent, which consists in this, that its osculating plane in each point is normal to this surface.

The forces X, Y, Z, being supposed to be cypher, equations (3) of No. 151 are reduced to

$$\frac{d^2x}{dt^2} = \text{N}.\cos\lambda, \quad \frac{d^2y}{dt^2} = \text{N}.\cos\mu, \quad \frac{d^2z}{dt^2} = \text{N}.\cos\nu.$$

Because $v$ is constant, and that $vt = s$, we have

$$\frac{d^2x}{dt^2} = v^2\frac{d^2x}{ds^2}, \quad \frac{d^2y}{dt^2} = v^2\frac{d^2y}{ds^2}, \quad \frac{d^2z}{dt^2} = v^2\frac{d^2z}{ds^2},$$

the arc $s$ being assumed as the independent variable, and this being the case, the preceding equations may be replaced by the following:

$$\frac{dxd^2y - dyd^2x}{ds^3} = \frac{\text{N}}{v^2}\left(\frac{dx}{ds}\cos\mu - \frac{dy}{ds}\cos\lambda\right),$$

$$\frac{dzd^2x - dxd^2z}{ds^3} = \frac{\text{N}}{v^2}\left(\frac{dz}{ds}\cos\lambda - \frac{dx}{ds}\cos\nu\right),$$

$$\frac{dyd^2z - dzd^2y}{ds^3} = \frac{\text{N}}{v^2}\left(\frac{dy}{ds}\cos\nu - \frac{dz}{ds}\cos\mu\right),$$

which may be deduced from them without any difficulty. If they be multiplied by $\cos\nu$, $\cos\mu$, $\cos\lambda$, respectively, and then added together, the quantity N disappears, and we have simply

$$\left.\begin{aligned}\frac{dxd^2y - dyd^2x}{ds^3}\cos\nu + \frac{dzd^2x - dxd^2z}{ds^3}\cos\mu + \\ \frac{dyd^2z - dzd^2y}{ds^3}\cos\lambda = 0,\end{aligned}\right\} \qquad (c)$$

hence from the values of $\cos\lambda$, $\cos\mu$, $\cos\nu$, cited in No. 151, we shall have

$$\left.\begin{array}{c}\dfrac{dxd^2y-dyd^2x}{ds^3}\cdot\dfrac{d\text{L}}{dz}+\dfrac{dzd^2x-dxd^2z}{ds^3}\cdot\dfrac{d\text{L}}{dy}+\\ \dfrac{dyd^2z-dzd^2y}{ds^3}\cdot\dfrac{d\text{L}}{dx}=0,\end{array}\right\}\quad(i)$$

for the second differential equation of the trajectory. If in this, there be substituted the value of one of the three coordinates $x$, $y$, $z$, in a function of the two others, deduced from $\text{L}=0$, the equation of the given surface on which the curve is supposed to be traced, and if the equation between the two variables, which results from it, be then integrated, the two constant arbitraries which the integral will contain being determined, by subjecting the curve to pass through the two points A and B of the given surface, the equation which will be obtained in this manner will be independent, as is evident, of the magnitude and direction of the initial velocity $k$, and must be that of the shortest line between these two points Now if $\alpha$, $\beta$, $\gamma$ be the angles which the normal to the osculating plane of any curve whatever, at the point of which the coordinates are $x$, $y$, $z$, makes with their productions in the positive direction, and if, in order to abridge, we make

$$[(dxd^2y-dyd^2x)^2+(dzd^2x-dxd^2z)^2+(dyd^2z-dzd^2y)^2]^{\frac{1}{2}}=h,$$

we shall have by formulæ (3) of No. 19, in which these angles are represented by $\lambda$, $\mu$, $\nu$,

$$\cos\alpha=\frac{1}{h}(dyd^2z-dzd^2y),$$

$$\cos\beta=\frac{1}{h}(dzd^2x-dxd^2z),$$

$$\cos\gamma=\frac{1}{h}(dxd^2y-dyd^2x),$$

consequently in virtue of equation (e), we shall have

$$\cos\lambda\cos\alpha+\cos\beta\cos\mu+\cos\gamma\cos\nu=0,$$

which shews that the normal to the osculating plane of the trajectory, and the normal to the given surface are perpendicular, the one to the other, hence we infer, that equation (f) which belongs(g) to the shortest line, is also that of the curve which has every where its osculating plane normal to the given surface, so that these two lines are one and the same curve traced on this surface, when both the one and the other are constrained to pass through the same extreme points A and B.

It follows from this, that when these two points belong to one of the lines of curvature of the given surface, this line is the shortest from one point to another, for its osculating plane in any point whatever, contains two consecutive normals to the given surface, and is, consequently, normal to this surface.

### III Digression on the Motion of Light

162 The theorem of No 160 is known by the denomination of *the principle of least action*, which was given to it, from the metaphysical point of view in which it was first considered, but this has been since properly abandoned. However it may be useful here, to give one of the first applications which was made of this principle, namely, that which is relative to the reflexion and refraction of light in the system of emission

As long as a ray of light moves in a medium of uniform density, its velocity and direction remain the same; but when it passes from one medium to another, its direction is inflected, and its velocity changes At the instant of the transit, the light describes a curve of inappreciable extent, which consequently we may neglect without any sensible error. Therefore, the trajectory of each luminous particle may be supposed to consist of two lines, each of which is described with a uniform velocity Thus, if $y, y'$ be the lengths of these lines, $u$ the

velocity of light in the first medium, and $n'$ its velocity in the second, $ny$ will be the value of the integral $\int vds$ taken from the point of departure of the particle to its entrance into the second medium, and $n'y'$ the value for the part of this integral relative to the second medium, consequently, this integral taken in the entire extent of the trajectory, will be expressed by $ny + n'y'$, and it is this sum, which, by the principle of least action, should be a *minimum*

Before we proceed further, it is necessary to observe, that if the second medium is a diaphanous and crystallized substance, the velocity of light in this substance will, in general, depend on the *direction* of the luminous ray; so that it will be constant for the same ray, but variable from one ray to another The phenomenon of *double refraction* which is observable in *Iceland spar*, and the greater number of transparent crystals, arises from the difference of velocity of the different luminous rays which traverse them. The velocity $n'$ should then be regarded as a function of the angles which determine the direction of each ray, and the law of refraction depends on the form of this function Laplace succeeded, by means of the form which he assigned to this function, in deducing from the principle of the least action, the law of double refraction, which was originally discovered by Huyghens, and afterwards confirmed by the experiments of Malus, but as this is not the place to enter into a detail of this theory, we shall restrict ourselves to the consideration of the case in which all the rays move with the same velocity, whatever be their directions Therefore, in the following investigation the velocities $n$ and $n'$ will be considered as given quantities for each medium in particular, and independent of the direction of the luminous rays

163. Let now A and B (fig 40) be the two extreme points of the trajectory, and through them let a plane pass, intersecting the surface of separation of the two media, which is supposed to be plane, in the right line CD Let the line ALB, the parts of which are AE and LB, represent the projection of the

trajectory on this plane, and through the points A, B, E, let AF, BG, HEK be drawn perpendicular to the line CD. Since the position of the points A and B is given, the three lines AF, BG, FG are known, but the position of the point E, and the angles AEH, BEK are unknown, and must be determined by the condition of the *minimum*. Therefore, let AF $= a$, BG $= b$, FG $=$ C, AEH $= x$, BEK $= x'$

From the right angled triangles AFE and BGE, we obtain

$$\text{EF} = a \operatorname{tang} x, \quad \text{EG} = b \operatorname{tang} x',$$

consequently, there results

$$a \operatorname{tang} x + b \operatorname{tang} x' = c. \tag{a}$$

The luminous ray traverses the surface which separates the two media, in a point of which E is the projection on the plane of the figure. If $z$ denotes the distance of this unknown point from the point E, $y$ will be the hypothenuse of a right angled triangle, of which $z$ and AE are the two lesser sides, and $y'$ will be the hypothenuse of another triangle, of which $z$ and BE are the two lesser sides, but from the triangles AEF, and BEG, we obtain

$$\text{AE} = \frac{a}{\cos x}, \quad \text{BE} = \frac{b}{\cos x'},$$

hence we shall have

$$y = \sqrt{z^2 + \frac{a^2}{\cos^2 x}}, \quad y' = \sqrt{z^2 + \frac{b^2}{\cos^2 x'}}.$$

If these values be substituted in the expression $ny + n'y'$, there will result a function of $z$, $x$, $x'$ which should be a *minimum* with respect to these three variables, the two last of which are connected together by equation (a) It is, therefore, in the first place necessary, that the differential of this function taken with respect to $z$, should be equal to zero, hence we obtain

$$\frac{ndy}{dz} + \frac{n'dy'}{dz} = \frac{nz}{y} + \frac{n'z}{y'} = 0.$$

But this condition can only be satisfied by assuming $z = 0$, which shews that the luminous ray traverses, at the point E, the surface that separates the two media, and consequently that it does not deviate from the plane drawn through the points A and B perpendicular to this surface. Therefore, making $z = 0$, we shall have simply,

$$ny + ny' = \frac{na}{\cos x} + \frac{nb}{\cos x'};$$

and by putting the complete differential of this quantity equal to cypher, there arises

$$\frac{na \sin x dx}{\cos^2 x} + \frac{n'b \sin x' dx'}{\cos^2 x'} = 0,$$

but by differentiating likewise equation (a), we have at the same time

$$\frac{adx}{\cos^2 x} + \frac{bdx'}{\cos^2 x'} = 0,$$

and if $\frac{dx'}{dx}$ be eliminated between these two equations, there results

$$n \sin x = n' \sin x'. \qquad \text{(b)}$$

This and equation (a) will determine the values of $x$ and $x'$, which answer to the *minimum* of $ny + n'y'$. The value of $x$ being obtained, the point E can be constructed, by taking $\text{EF} = a \operatorname{tang} x$, then the lines AE and BE can be determined, and the line AEB, the parts of which are AE and EB, will be the path of the luminous ray in passing from the point A to the point B.

The angle AEH contained between EH which is normal to the surface of separation of the two media, and the incident ray AE, is what is termed *the angle of incidence*, the angle BEK contained between EK, the production of this normal and the refracted ray BE, is *the angle of refraction*. These two angles have been denoted by $x$ and $x'$ Hence equation (b) will make known the angle of refraction when the angle of

incidence is given, and from this equation it appears, that the sine of the angle of incidence is to the sine of the angle of refraction in a constant ratio.

This is, in fact, the known law of ordinary refraction, which was first discovered by Des Cartes. The ratio of the two sines depends on that of the velocities $n$ and $n'$ relative to the media which are considered; and for this reason, it varies with the different descriptions of transparent media.

164. If the luminous ray, instead of penetrating the second medium, is reflected at the surface of separation, its velocity will be constant for the entire length of the trajectory, which then exists altogether in the same medium. Hence the integral $\int v ds$ will be equal to the entire length of the trajectory, multiplied by this constant velocity, consequently, in virtue of the principle of least action, this length should be a *minimum*.

Let us therefore suppose, that, as in the preceding number, the surface of separation is a plane. Let A and B (fig 41) be the two extreme points of the trajectory, through these points let a plane be drawn perpendicular to this surface, and intersecting it in the line CD each particle of light will, in its passage from the point A to the point B, describe the line AEB, the parts of which are AE and EB, and it will be the shortest of all those that are reflected from the surface of separation. Now it is immediately evident, that this line must exist in the plane perpendicular to this surface, for every other trajectory is necessarily longer than its projection on this plane. Moreover, it is easy to show without the assistance of the calculus, that the shortest line between A and B, which meets the surface of separation, is that which makes equal angles with the line CD, that is to say, if

$$AEC = BED,$$

the line AEB will be shorter than any other line such as AE'B, of which the point E', like the point E, appertains to the line CD.

In fact, from A let the perpendicular AF be let fall on this line, and produce it so that AF′ may be equal to AF, and let the lines A′E, A′E′ be drawn. The two angles AEC, A′EC will be equal, hence the two angles A′EC, BED will be so likewise on account of the preceding equation; consequently A′EB will be a right line, and we shall have

$$A'E + BE \angle A'E' + BE',$$

and as

$$A'E = AE \text{ and } A'E' = AE'$$

there will result

$$AE + BE \angle AE' + BE';$$

which was to be proved.

If at the point E, the perpendicular EH be erected to the line CD, AEH and BEH will be the angles of incidence and reflexion of the luminous ray which issues from the point A to the point B. These angles will be equal, because they are the complements of the equal angles AEC and BED, hence results the known law of the reflexion of light, which consists in this, that the angle of incidence is always equal to the angle of reflexion.

165 If the theory of the emission of light be admitted, the laws of reflexion and refraction can be inferred from the expression of the square of the velocity of a point subjected to the action of attractive forces (No. 168), in a more direct manner, than by making use of the principle of least action As this question presents us with an example of the motion of a material point, which is interesting, as well from a consideration of the nature of the forces which operate, as from its physical application, we will give the solution of it in the ordinary case, in which the two media traversed by the light are not crystallized.

In this theory, each luminous particle is supposed to be subjected to the attraction of all the material points of the medium which it traverses, and this force is regarded as an un-

known function of the distance, for all that is known respecting it is, that it decreases with extreme rapidity when the distance increases, so that when the distance is of a sensible magnitude, it becomes altogether insensible Thus, for example, if $r$ denotes the distance of the attracted from the attracting point, $a$ a line of a finite but insensible magnitude, and $e$ the base of the Naperian system of logarithms, then a force of this

N

nature may be represented($n$) by $A \; e^{\frac{-r}{a}}$, A being the intensity relative to an infinitely small distance $r$; when this distance is of a sensible magnitude, and is, consequently, a considerable multiple of $a$, the value of this function will be no longer sensible.

If the luminous ray moves in a homogeneous medium of a uniform density, the attractions which it experiences mutually destroy each other's effect, and its motion becomes uniform and rectilinear. But if it attains the point M (fig 42) situated at an insensible distance from the surface CD, which separates the two media, and which, for greater clearness, we will suppose to be horizontal let a perpendicular MP be drawn from this point on CD, and then in the upper medium let there be drawn two planes C′D′ and C″D″ parallel to CD, the mutual distance of which may be equal to MP, and let the first pass through the point M, it is evident that the attractions exercised on the luminous rays at the point M, by the two strata of the upper medium, which are comprised, the one between CD and C′D′, the other between C′D′ and C″D″, will be equal and contrary, therefore they will destroy each other's effects, and the moveable will only be solicited by the part of the medium it traverses, which is above C″D″, and by the entire attraction of the inferior medium These two forces will be perpendicular to CD; and they will vary with the distance MP, according to unknown laws, but such, that each of these forces will be insensible when MP is not so, and they will attain their *maxima* when this distance vanishes, in which case the move-

able will have attained to the surface which separates the two media

At the end of the time $t$, let $z$ be the distance MP, and $Z$, $Z'$ the unknown functions of $z$, which express the accelerating forces arising from the attraction of the inferior medium, and from the part of the other medium above C″D″. The entire accelerating force tending to diminish $z$, will be the difference $Z-Z'$, consequently, we shall have, in the superior medium,

$$\frac{d^2z}{dt^2}+Z-Z'=0, \tag{1}$$

for the equation of the vertical motion of a luminous particle

When the moveable, after traversing the surface CD in a point E, has penetrated into the lower medium as far as a point M′, which is such that M′P′, the perpendicular to CD, may be also represented by $z$, it is easy to see that the accelerating force, which will tend to diminish this variable, will be then the difference $Z'-Z$, so that we shall have

$$\frac{d^2z}{dt^2}+Z'-Z=0, \tag{2}$$

for the equation of vertical motion in the inferior medium

As to the horizontal motion or that parallel to CD, it will be uniform, and the horizontal velocity will undergo no change in passing from one medium to the other, for the attractive forces of each medium parallel to CD, will destroy each other's effect so that a luminous ray is not subject to any accelerating force in this direction Thus, if $k$ be the velocity of light at A, a point of the superior medium, situated at a *sensible* distance from CD, and $\alpha$ the acute angle that the direction of this velocity makes with the vertical, $k \sin \alpha$ will be the velocity parallel to CD at any instant whatever If the luminous ray penetrates by a sensible quantity into the inferior medium, and if $k'$ and $\alpha'$ denote what $k$ and $\alpha$ become at A′, any point of this medium, situated at a sensible distance from CD,

we can likewise represent the horizontal velocity of the moveable by $k' \sin a'$, so that we should have

$$k \sin a = k' \sin a'. \tag{3}$$

It appears also, *a priori*, that the trajectory of the moveable will be a plane and vertical curve, therefore it only remains to determine its velocity perpendicular to CD, whether in the superior or inferior medium, at $z$ any distance whatever from this surface CD

166. Let this velocity be denoted by $u$, so that we may have $\frac{dz^2}{dt^2} = u^2$ for the two media By multiplying equation (1) by $2dz$, then integrating and denoting the constant arbitrary by $c$, we shall have in the superior medium

$$u^2 = c + 2\int z' dz - 2\int z dz.$$

N Let these two integrals be supposed to vanish with $z$, and let their values at a sensible distance from CD be $h$ and $h'$, these integrals $h$ and $h'$ can be extended from zero to infinity, for by hypothesis beyond a sensible value, the functions $z$ and $z'$, and consequently, the corresponding parts of $\int z dz$ and $\int z' dz$ vanish, or become insensible.

Therefore, we may write if we please,

$$h = \int_0^\infty z dz, \quad h' = \int_0^\infty z' dz.$$

Moreover, for any sensible value of $z$, we have

$$u^2 = k^2 \cos^2 a,$$

hence in that case we shall have

$$k^2 . \cos^2 a = c + 2h' - 2h;$$

and by eliminating $c$ from the general value of $u^2$, there will result at any point whatever,

$$u^2 = k^2 \cos^2 a + 2h - 2h' + 2\int z' dz - 2\int z dz.$$

Let $k_1$ be the velocity of the moveable at E any point of the

surface CD, and let $a_1$ be the angle which its direction makes with the vertical At this point we shall have $u^2 = k_1^2 \cos^2 a_1$, and as in this case $z = 0$, the two last terms of the preceding formula will vanish, and it will be reduced to

$$k_1^2 \cos^2 a_1 = k^2 \cos^2 a + 2h - 2h'. \tag{4}$$

In order, therefore, that the luminous ray may reach the surface which separates the two media, it is necessary that the second member of this equation should be a positive quantity, and, consequently, that

$$h' \angle h + \tfrac{1}{2} k^2 \cos^2 a$$

If this condition be not satisfied, in which case the attraction of the upper medium should surpass that of the inferior, the vertical velocity will be destroyed before it reaches the plane CD, therefore there will be a point of the trajectory, at which the tangent will be horizontal. Having attained this point, the moveable will retrograde, the two branches of this curve, which terminate in this same point, will be similar, since they are described by the action of equal forces, for the same value of $z$, and when this distance $z$ is of a sensible magnitude, these two branches will be changed into right lines, which make equal angles with the vertical, or, in other words, the angles of incidence and reflexion will be equal. If, on the contrary, the attraction of the inferior medium surpasses that of the superior, and if the preceding condition is satisfied, the luminous ray will penetrate into the inferior medium with a velocity the direction of which will be perpendicular to CD, and which will be determined by equation (4) In this hypothesis we shall have, by equation (2), relative to this medium,

$$u^2 = k_1^2 \cos^2 a_1 + 2\int z dz - 2\int z' dz,$$

the integrals being always supposed to vanish when $z = 0$ At a sensible distance from CD, $u^2 = k'^2 \cos^2 a'$, therefore we shall have

$$k'^2 \cos a^{2\prime} = k_1^2 \cos a_1^2 + 2h - 2h'.$$

and eliminating $k_1^2 \cos a_1^2$ by means of equation (4), there will result

$$k'^2 \cos a^{2\prime} = k^2 . \cos a^2 + 4h - 4h'. \qquad (5)$$

Therefore, in order that the luminous ray, after having traversed the surface CD, may penetrate to a sensible depth into the inferior medium, it will be necessary that

$$h' \angle h + \tfrac{1}{4} k^2 \cos^2 a,$$

and this is the only condition that is necessary.

It may happen that $h'$, although less than $h + \frac{1}{2} k^2 \cos^2 a$, surpasses $h + \frac{1}{4} k^2 \cos^2 a$, in which case the moveable will not penetrate into the inferior medium a sensible distance beyond CD, but it will return into the superior medium, and the two branches of the trajectory which it describes will be similar on each side of the point at which it commenced to retrograde. Consequently, the light will be reflected, as in the preceding case, making the angle of incidence equal to the angle of reflexion, so that there are *two distinct cases of reflexion* in the theory we are considering.

167. Let us now suppose that neither the one nor the other of these two cases obtains, so that the luminous ray must be *refracted*. By equation (3) we have

$$k'^2 \sin^2 a' = k^2 \sin^2 a,$$

and by adding the corresponding members of equation (5), and of this, there will result

$$k'^2 = k^2 + 4h - 4h'; \qquad (6)$$

which shows that the increase of the square of the velocity of the moveable, in passing from the point A of the superior medium, to the point A′ of the inferior medium, is (as it ought to be (No. 157)) independent of the route along which it moves.

Likewise, from equations (3) and (6) there results

$$\frac{\sin a'}{\sin a} = \frac{k}{\sqrt{k^2 + 4(h - h')}}, \qquad (7)$$

from this formula it appears, that the sine of the angle of incidence is to the sine of the angle of refraction in a constant ratio, and it is evident that the value of this ratio is a function of $k$, the velocity of light in one of these two media, and of $h-h'$, the difference of their *refractive powers* $h$ and $h'$

If the inferior medium is terminated by two parallel planes, and if the medium below is the same as that above it, experiment shows that the light, after having undergone two refractions, and traversed the two faces of the intermediate medium, resumes a direction parallel to that which it had in the superior medium This likewise results from equation (7) For if $a''$ be the angle which the luminous ray makes with the vertical after it emerges from the intermediate medium, it is necessary, in order to determine $\sin a''$, to interchange among themselves the quantities, $h$ and $h'$, and to put $k'$, $a'$, $a''$ instead of $k$, $a$, $a'$ Consequently, we shall have

$$\frac{\sin a''}{\sin a'} = \frac{k'}{\sqrt{k'^2 + 4(h'-h)}},$$

or, in consequence of equations (6) and (7),

$$\frac{\sin a''}{\sin a'} = \frac{\sqrt{k^2 + 4(h-h')}}{k} = \frac{\sin a}{\sin a'}$$

which gives in fact

$$a'' = a$$

The phenomenon of *dispersion* which arises from a different value of the angle of refraction $a'$, for the differently coloured rays, of which the same incident ray is composed, may be attributed, agreeably to formula (7), either to an inequality in their velocity $k$, or to a different action in each medium on these different rays, from which results unequal values of $h - h'$.

168 Every thing else being the same, it appears from equation (7) that the ratio of the sine of the angle of incidence to the sine of the angle of refraction, must change with the velocity of

the light. Now, for a star situated in the plane of the ecliptic, there is an epoch in the course of the year, in which the direction of the velocity of the earth is contrary to that of light, and another epoch in which the direction of the first mentioned velocity is the same as that of the second, this renders the velocity of light, relatively to the medium in which it moves along with the earth, sensibly greater in the first case than in the second The ratio in question must consequently be different at these two epochs, but from extremely accurate experiments instituted by M. Arago, it appears that this ratio does not vary in a sensible manner, during the entire course of the year, and moreover, that its magnitude is the same for the sun and for the different stars, from which light on the theory of emission emanates.

Whatever theory of light be adopted, it is a very remarkable fact, that the composition of its velocity with that of the earth, which is indicated in the apparent motion of the stars, termed *aberration*, has, notwithstanding, no appreciable influence on the refraction of the light, which is transmitted to us from them on different days of the year

In a vacuo, the motion of light, whether direct or reflected, is always uniform, and its velocity is independent of the source from whence it emanates This velocity is so great, that light traverses in 493,34 seconds, the mean distance of the sun from the earth from which it follows, that it describes 30950 myriametres in a second.

A luminous ray emitted from the sun or a star, should, like any other projectile, experience a diminution in its velocity, caused by its gravity towards this star, that is to say, by the attraction in the inverse ratio of the square of the distances from its centre, which the mass of the body exerts on each *material* particle of light, but this diminution is a very small fraction of the final velocity of light Thus, for example, as it will be shown hereafter, that the intensity of gravity at the surface of the sun is twenty-seven times and a half greater

than the intensity of the terrestrial gravity, and as the sun's radius is 110 times the radius of the earth, it follows from what has been observed in No 143, that the velocity of light, in order to be 30950 myriametres per second at a considerable distance from the sun, must, when issuing from the surface, be greater by a little less than (*k*) two millionths only.

# CHAPTER IV.

## OF THE CENTRIFUGAL FORCE

169 THE pressure of a material point on a curve which it is constrained to describe, is not the same as when it is in equilibrio on this curve In consequence of the motion, a particular pressure arises, which is termed *the centrifugal force*, because it was first considered in the circle, where it is directed along the production of the radius, and continually tends to increase the distance of the moveable on which it acts from the centre It is this force which we now proceed to consider in any curve whatever

Let $M_1M$ and $MM'$ (fig. 43) be two equal and consecutive elements of the given curve, $H$ and $H'$ their middle points, $MT$ and $M'T'$ their productions Their plane, and the angle $TMT'$, will be the osculating plane and angle of contact of the curve at the point $M$, and if in this plane the line $MO$ be drawn, dividing the angle $M_1MM'$ into two equal parts, its direction will coincide with that of the radius of curvature at this same point $M$, and consequently the centre of curvature will exist in a certain point of this line such as $O$. Let $ds$ denote $M_1M$ the element of the curve, which will be also equal to $HMH'$, moreover, let $\delta$ be the infinitely small angle $TMT'$, and $\rho$ the radius of curvature $MO$, we shall have (No 18)

$$\delta = \frac{ds}{\rho}$$

This being established, let us first abstract from the consideration of the given forces which may act on the moveable, and let us suppose, that at the end of the time $t$, it reaches the

point M, with the velocity $v$ If it was entirely free it would continue to move on the line MT with the same velocity, but it is by hypothesis forced to describe a given curve, this produces a change in the direction of its motion, which thus becomes MT′ Now if there be erected to MT′ the perpendicular MK, such that it may exist in the osculating plane, and fall without the concavity of the curve, we can substitute for the velocity $v$, the direction of which is MT, two other velocities, of which one is equal to $v \cos \delta$, and directed along MT′, and the other equal to $v \sin \delta$, and directed along MK, the effect of the curve will be to destroy the last of these two velocities, so that the first only remains, or in other words, this effect will be the same thing as if there was impressed on the moveable a velocity equal and contrary to $v \sin \delta$ Therefore if the given curve be replaced by an infinitismal polygon, its resistance consists in impressing on the moveable at each summit of this polygon such as M, an infinitely small velocity $v \sin \delta$, in a direction opposite to that of MK

In order that this resistance may be completely assimilated to $f$, a motive force which acts *incessantly* on the moveable, the velocity $v \sin \delta$ may be supposed to be produced while this material point moves from H to H′, and $dt$ may be assumed as the time during which this action continues The change in the direction of this force may be also neglected in this interval, and it may be assumed, for example, parallel to the line MO. Then the measure of the corresponding accelerating force will be, like each of the forces U, U′, U″, &c. of No 147, the velocity $v \sin \delta$, which is produced in the instant $dt$, divided by $dt$, and if the mass of the moveable be denoted by $m$, there will result, for the value of $f$,

$$f = \frac{mv \sin \delta}{dt}.$$

Consequently, if $\delta$ be substituted for $\sin \delta$, and $\frac{ds}{\rho}$ for $\delta$, we shall obtain, since $ds = vdt$,

$$f=\frac{mv^2}{\rho}.$$

The pressure that the curve experiences, and which is solely due to the state of motion of the material point that describes it, or the centrifugal force that acts on this moveable, is equal and contrary to this force $f$. It follows therefore, that, at any point whatever of the given curve, such as M, the centrifugal force exists in the osculating plane, and is directed from the concavity of this curve along MN, the production of its radius of curvature, and that its intensity is in the inverse ratio of this radius, and in the direct ratio of the mass of the moveable and of the square of its velocity.

170. As this velocity along the side $M_1M$ is $v$, and as it becomes $v \cos \delta$ along the following side MM′, it follows that its magnitude is not affected by the curve, for the quantity $v(1-\cos\delta)$ may be neglected, being an infinitely small quantity of the second order, from which there can only result an infinitely small diminution of velocity on a part of the curve of which the magnitude$(a)$ is finite. Hence, then the motion on any curve whatever is finite, when the moveable is not solicited by a given force. This has been already observed in No 157, but moreover, the reason why this is the case, is because the angle of contact is infinitely small, for in a point where two different curves intersect at a finite angle, the moveable will experience a finite loss of velocity in passing from one curve to another, which loss will be equal to the primitive velocity multiplied by the versed sine of this angle.

When the moveable is solicited by one or more given forces, its velocity varies with the components of these forces that are *tangential* to the trajectory, and their *normal* components exert, as in a state of rest, a pressure on this curve, which must be combined with the centrifugal force.

In general, let $m$R be the resultant of the given forces which act on the moveable when it attains the point M. If this motive force be resolved into two others, the one, in the direction of the

tangent, the other normal to the trajectory, and represented respectively by $m$T and $m$Q, the first force is that which causes the velocity to vary, and the second will produce the part of the pressure that is independent of the state of motion of the moveable. If by the rule for the composition of forces, the resultant of $m$Q and of the centrifugal force $f$ or $\frac{mv^2}{\rho}$, be taken, the entire pressure exerted on the point M of the given curve will be obtained, both in magnitude and direction. This force, divided by the mass of the moveable, or the resultant of the accelerating forces Q and $\frac{v^2}{\rho}$, ought to coincide with the force P of No. 152. It is in fact this, which we now proceed to verify.

171. We may substitute for equations (5) of this number, the following, which can be immediately deduced from them:

$$\left.\begin{aligned}
\frac{dxd^2y - dyd^2x}{dsdt^2} &= \text{Y}\frac{dx}{ds} - \text{X}\frac{dy}{ds} - \text{P}\left(\frac{dx}{ds}\cos\omega' - \frac{dy}{ds}\cos\omega\right),\\
\frac{dzd^2x - dxd^2z}{dsdt^2} &= \text{X}\frac{dz}{ds} - \text{Z}\frac{dx}{ds} - \text{P}\left(\frac{dz}{ds}\cos\omega - \frac{dx}{ds}\cos\omega''\right),\\
\frac{dyd^2z - dzd^2y}{dsdt^2} &= \text{Z}\frac{dy}{ds} - \text{Y}\frac{dz}{ds} - \text{P}\left(\frac{dy}{ds}\cos\omega'' - \frac{dz}{ds}\cos\omega'\right),
\end{aligned}\right\} \quad (1)$$

whichever of these is considered as the independent variable, we have

$$\frac{dxd^2y - dyd^2x}{dt^2} = \frac{dx^2}{dt^2}\frac{d\cdot\frac{dy}{dx}}{dt}dt,$$

we have also, at the same time,

$$\frac{dx^2}{dt^2} = \frac{dx^2}{ds^2}\frac{ds^2}{dt^2}, \quad \frac{d\,\frac{dy}{dx}}{dt} = \frac{d\,\frac{dy}{dx}}{ds}\frac{ds}{dt},$$

and because $v = \frac{ds}{dt}$, there will result ($b$)

$$\frac{dxd^2y - dyd^2x}{dsdt^2} = v^2 \frac{dx^2}{ds^2} \frac{d \cdot \frac{dy}{dx}}{ds} = v^2 \frac{(dxd^2y - dyd^2x)}{ds^3}.$$

In the same manner, there may be obtained,

$$\frac{dzd^2x - dxd^2z}{dsdt^2} = v^2 \frac{(dzd^2x - dxd^2z)}{ds^3},$$

$$\frac{dyd^2z - dzd^2y}{dsdt^2} = v^2 \frac{(dyd^2z - dzd^2y)}{ds^3}.$$

If $q$, $q'$, $q''$, denote the angles that the force Q makes with the parallels to the axes of $x$, $y$, $z$, we shall likewise have, X, Y, Z being the components in the direction of these parallels, of Q and of the tangential force T,

$$X = T\frac{dx}{ds} + Q\cos q, \quad Y = T\frac{dy}{ds} + Q\cos q', \quad Z = T\frac{dz}{ds} + Q\cos q'';$$

and by means of these, and of the preceding values, equations (1) will become

$$\left.\begin{aligned}
v^2\left(\frac{dxd^2y - dyd^2x}{ds^3}\right) &= Q\left(\frac{dx}{ds}\cos q' - \frac{dy}{ds}\cos q\right) - \\
&\quad P\left(\frac{dx}{ds}\cos\omega' - \frac{dy}{ds}\cos\omega\right), \\
v^2\left(\frac{dzd^2x - dxd^2z}{ds^3}\right) &= Q\left(\frac{dz}{ds}\cos q - \frac{dx}{ds}\cos q''\right) - \\
&\quad P\left(\frac{dz}{ds}\cos\omega - \frac{dx}{ds}\cos\omega''\right), \\
v^2\left(\frac{dyd^2z - dzd^2y}{ds^3}\right) &= Q\left(\frac{dy}{ds}\cos q'' - \frac{dz}{ds}\cos q'\right) - \\
&\quad P\left(\frac{dy}{ds}\cos\omega'' - \frac{dz}{ds}\cos\omega'\right)
\end{aligned}\right\} \quad (2)$$

Now, if $\gamma$, $\gamma'$, $\gamma''$ be the angles which the direction of the centrifugal force, that is to say MN, the production of the radius of curvature MO, makes with the parallels to the axes of

$x, y, z$, drawn through the point M, and if $x', y', z'$, be the coordinates of the centre of curvature O, we shall have

$$x - x' = \rho \cos \gamma, \quad y - y' = \rho \cos \gamma', \quad z - z' = \rho \cos \gamma'',$$

and by combining equations (2) with the formulæ of No. 20, we may without difficulty deduce from them (c)

$$\frac{v^2}{\rho} \cos \gamma = \text{Q}\left[\frac{dy}{ds}\left(\frac{dx}{ds}\cos q' - \frac{dy}{ds}\cos q\right) - \frac{dz}{ds}\left(\frac{dz}{ds}\cos q - \frac{dx}{ds}\cos q''\right)\right]$$

$$- \text{P}\left[\frac{dy}{ds}\left(\frac{dx}{ds}\cos \omega' - \frac{dy}{ds}\cos \omega\right) - \frac{dz}{ds}\left(\frac{dz}{ds}\cos \omega - \frac{dx}{ds}\cos \omega''\right)\right],$$

$$\frac{v^2}{\rho} \cos \gamma' = \text{Q}\left[\frac{dz}{ds}\left(\frac{dy}{ds}\cos q'' - \frac{dz}{ds}\cos q'\right) - \frac{dx}{ds}\left(\frac{dx}{ds}\cos q' - \frac{dy}{ds}\cos q\right)\right]$$

$$- \text{P}\left[\frac{dz}{ds}\left(\frac{dy}{ds}\cos \omega'' - \frac{dz}{ds}\cos \omega'\right) - \frac{dx}{ds}\left(\frac{dx}{ds}\cos \omega' - \frac{dy}{ds}\cos \omega\right)\right],$$

$$\frac{v^2}{\rho} \cos \gamma'' = \text{Q}\left[\frac{dx}{ds}\left(\frac{dz}{ds}\cos q - \frac{dx}{ds}\cos q''\right) - \frac{dy}{ds}\left(\frac{dy}{ds}\cos q'' - \frac{dz}{ds}\cos q'\right)\right]$$

$$- \text{P}\left[\frac{dx}{ds}\left(\frac{dz}{ds}\cos \omega - \frac{dx}{ds}\cos \omega''\right) - \frac{dy}{ds}\left(\frac{dy}{ds}\cos \omega'' - \frac{dz}{ds}\cos \omega'\right)\right]$$

But because the forces P and Q are perpendicular to the tangent of the trajectory, we have

$$\frac{dx}{ds}\cos q + \frac{dy}{ds}\cos q' + \frac{dz}{ds}\cos q'' = 0,$$

$$\frac{dx}{ds}\cos \omega + \frac{dy}{ds}\cos \omega' + \frac{dz}{ds}\cos \omega'' = 0;$$

this reduces the coefficients of Q(d) in the three preceding equations to $-\cos q$, $-\cos q'$, $-\cos q''$, and those of $-$P to $-\cos \omega$, $-\cos \omega'$, and $-\cos \omega''$; therefore we shall have

$$\frac{v^2}{\rho}\cos \gamma + \text{Q}\cos q = \text{P}\cos \omega,$$

$$\frac{v^2}{\rho}\cos \gamma' + \text{Q}\cos q' = \text{P}\cos \omega',$$

$$\frac{v^2}{\rho}\cos \gamma'' + \text{Q}\cos q'' = \text{P}\cos \omega''.$$

from which it appears, that the force P is the resultant in magnitude and direction of the two forces $\frac{v^2}{\rho}$ and Q, as may be easily verified.

172 When the moveable is merely constrained to move on a given surface, it is necessary that the resultant of the motive forces $m$Q and $\frac{mv^2}{\rho}$, which we know already is perpendicular to its *trajectory*, should be moreover normal to this *surface* Therefore, denoting this resultant by $m$N, and the angles, whether acute or obtuse, that its two components make with a determinate part of the normal to the surface, at the point where the moveable exists, by $\phi$ and $\psi$, we shall have

$$\text{N} = \pm\left(\text{Q}\cos\phi + \frac{v^2}{\rho}\cos\psi\right).$$

The force N will act in the direction of this part of the normal, or in the direction of its production, according as the quantity comprised between the crotchets is positive or negative, and in order that N may be always a positive quantity, we should take the superior sign in the first case, and the inferior sign in the second. This accelerating force N should be equal and contrary to that which occurs in equations (3) of No. 151, in fact, these last differ from equations (5) of No 152 only in this, that they contain N, $\lambda$, $\mu$, $\nu$, in place of $-$ P, $\omega$, $\omega'$, $\omega''$, and by the preceding analysis we can deduce from them the components of the force N, which will be equal and contrary to those that have been found for the force P

In this same case of a given surface, if $\phi'$, $\psi'$ denote the angles that the forces $m$Q and $\frac{mv^2}{\rho}$ make with an axis, drawn through the point where the moveable exists on the curve, tangential to this surface, and perpendicular to the trajectory, so that we may have

$$\cos^2\phi + \cos^2\phi' = 1, \quad \cos^2\psi + \cos^2\psi' = 1,$$

it is necessary that the sum of the components of these two forces acting in the direction of this tangential axis, should be equal to cypher, because their resultant is normal to the same point of the surface, consequently we shall have

$$\text{Q} \cos \phi' + \frac{v^2}{\rho} \cos \psi' = 0,$$

by means of which equation, $\psi'$ the inclination of the osculating plane of the trajectory on the plane which touches the given surface, can be determined.

When the moveable is not subjected to the action of any given force, or, more generally, when it is only subjected to the action of a force tangential to its trajectory, we shall have $\text{Q} = 0$, hence there results $\cos \psi' = 0$ and $\psi' = 90°$, so that the osculating plane of this curve will be constantly perpendicular to the given surface. As this is in general the property of the shortest line between two given points on this surface, it is this line that the moveable will describe, as has been already stated (No 161), but we see now besides, that a force which is tangential to a surface, such as friction against the given surface, or the resistance of a medium, will not cause the moveable to deviate from the shortest line between the two points, from the one to the other of which it passes

173 Finally, if the moveable be entirely free, it is necessary that the component of the motive force $m\text{R}$, which is normal to the trajectory, should be in equilibrio with its centrifugal force $\frac{mv^2}{\rho}$, since in this case, there is nothing in the curve, or given surface, which can destroy the normal resultant of these two forces It is necessary, therefore, in the first place, that the osculating plane of the trajectory should be that which passes through the tangent and the given direction of the force $m\text{R}$, naming $\theta$ the angle which this direction, at any point whatever, makes with the radius of curvature MO, this angle should be acute, in order that the normal component of the force $m\text{R}$ may act in a direction opposite to the

centrifugal force which is directed along MN, and this being so, we should have

$$R \cos \theta = \frac{v^2}{\rho}. \qquad (a)$$

When the accelerating force R, to the action of which the moveable is subjected, is a central force directed to a known point, and when the curve described about this fixed centre is known from observation, the radius of curvature $\rho$ and the angle $\theta$ that it makes with the direction of the force R may be deduced from the equation of this curve, there may likewise be obtained from this equation and the proportionality of the areas to the times (No 155), the expression of the velocity $v$ at any point whatever of the trajectory, consequently, equation (a) will determine the value of R, or the law of the central force, by the action of which the moveable is made to describe the given curve($f$). It is in this manner that Newton discovered the law of the force directed towards the centre of the sun, which causes each planet to describe an ellipse of which this point occupies one of the foci; but it will be shown in the sequel, that proceeding from the same data, this determination can be effected by a much simpler process.

174. Huyghens, to whom we are indebted for the measure of the centrifugal force, deduced it from the consideration of circular motion, and although this method is less direct than the preceding, it may, notwithstanding, be useful briefly to explain it here

Let M (fig. 44) be a material point attached to a fixed point C, by an inextensible thread CM, if by means of a percussion a velocity $a$ be impressed on it in a direction perpendicular to the length of the thread, and if in order to simplify the question, no other given motive force be supposed to act on the moveable; this material point then describes a circle AMB, of which the centre and radius are the fixed point and the length of the thread During this motion, the thread which retains the moveable will experience, in the direction

of its length, a certain *tension* which is, in fact, the centrifugal force By applying to the moveable a force equal to this tension, and constantly directed towards the fixed centre, we may abstract from the consideration of the thread, and consider the material point as entirely free It is therefore, in virtue of this central force, the magnitude of which is unknown, combined with the velocity $a$, that the circle will be described.

It follows immediately, that the circular sectors traced by the radius of the moveable, will be proportional to the times (No 155), this requires that the arcs of the circle which are described should be so also Hence the circular motion will be uniform, and if $s$ denotes the arc described in the time $t$, we shall have $s = at$. Let $m$ be the mass of the moveable, $m\alpha$ the central force, and consequently, $\alpha$ the accelerating force which is to be determined. Whatever the nature of this force may be, we can consider it as constant in magnitude and direction for an infinitely short interval of time Thus, while the moveable describes MM′, an infinitely small arc of the circle, the force $\alpha$ may be supposed to be constant and parallel to CM, the radius which is drawn to the origin of this arc, hence it follows, that if the moveable was not actuated by the velocity $a$, the central force would cause it to describe, in an infinitely short interval of time, the versed sine MN, or the projection of the arc MM′ which it actually describes, on CM. Now, the measure of every accelerating force is twice the infinitely small space, which it is capable of making the moveable describe in an infinitely short time, divided by the square of the time (No 118), therefore if $\varepsilon$ denotes the versed sine MN, and $\tau$ the time in which the arc MM′ is described, we shall have

$$\alpha = \frac{2\varepsilon}{\tau^2},$$

but if we denote this arc by $\sigma$, and the radius CM by $r$, we have

$$\varepsilon = \frac{\sigma^2}{2r},$$

the arc being taken for the chord, hence, because $\sigma = a\tau$ we shall have

$$\alpha = \frac{a^2}{r}.$$

This value of $\alpha$ is therefore that of the centrifugal force referred to the unit of mass, in a circle described with a uniform motion. It follows at once, that this force in any curve whatever, will be equal to the square of the velocity divided by the radius of curvature; for since in the trajectory any two consecutive elements are common with its osculating circle, we can suppose that, during an infinitely small portion of time, the moveable will move circularly about the centre of curvature, and that consequently it has the centrifugal force which corresponds to this motion. If this accelerating force be multiplied by $m$, we shall obtain the same value as for the force denoted by $f$ in No. 169.

175. In order to compare the centrifugal force in the circle with the gravity, let $a$ be the velocity acquired in falling through the height $h$, so that we may have $a^2 = 2gh$ (No. 130), $g$ denoting the gravity, then there will result

$$\frac{\alpha}{g} = \frac{2h}{r},$$

which shews that the centrifugal force is to the gravity, as twice the height through which the moveable should fall to acquire the velocity $a$, to the radius.

If the moveable is a body of which the dimensions are very small with respect to its distance from the point C, the value of $\alpha$ may be considered for its entire extent, as nearly constant, and consequently we may assume $\frac{\alpha}{g}$ as the expression of the ratio of the centrifugal force arising from circular motion, to the weight of the body on which it acts.

When the motion is not performed in a horizontal plane, the velocity of the moveable, the centrifugal force and tension of the thread attached to the point C, will be variable. If the motion takes place in a vertical plane, and if $2gh$ denotes the square of the velocity when the moveable is in the horizontal plane passing through the point C, let $z$ be its distance from this plane, at any instant whatever, which is considered as positive when the moveable is situated below, and as negative when it lies above this plane, $2g\,(h+z)$ will at this instant be the expression for the square of the velocity (No 159), and $2mg\dfrac{(h+z)}{r}$ for the centrifugal force In order to obtain the entire tension, there should be added to this force the component of the weight acting in the direction of the production of its radius, which component it is easy to shew is equal to $\dfrac{mgz}{r}$[g] Therefore, naming $\theta$ the total tension of the thread at any instant, we shall have

$$\theta = \frac{mg(2h+3z)}{r}$$

This force expresses likewise the pressure which the point C experiences at any instant in the direction of the radius terminating at the moveable. It will attain its *maximum*, when the moveable is at the lowest point of the circle, i e when $z = r$, and its *minimum*, when it is at the highest point, i. e, when $z = -r$ If $h$ is less than $\dfrac{3r}{2}$ the tension will become negative, or will be converted into a contraction[h] during a part of the motion. It is therefore then necessary, that the thread should be inflexible in order that the circular motion may have place. In this discussion, the weight and centrifugal force of the thread are neglected, because its mass is considered as extremely small with respect to that of the moveable. It will be shewn hereafter, how this ought to be taken into account if it should be necessary to do so.

176. Let us now revert to the consideration of the uniform motion performed in a circle, and let T denote the time in which the moveable describes the entire circumference; we shall then have

$$a = \frac{2\pi r}{T},$$

and, consequently,

$$a = \frac{4\pi^2 r}{T^2},$$

from which it appears, that the centrifugal force is in the direct ratio of the radius of the circle, and in the inverse ratio of the square of the time of an entire revolution. When a solid body revolves about a fixed axis, all its points describe, in the same time, circles, the planes of which are perpendicular to the axis; their centres lie in this axis, and their radii are the perpendiculars let fall from each point on this same axis; consequently, the centrifugal forces of their several points are to each other as these perpendiculars. Thus, for example, the centrifugal force of bodies at the surface of the earth, and which revolve with it about its axis, is proportional to the radii of the parallels which they describe; and moreover, this force at each place acts in the direction of the production of the radius of the parallel drawn to this point.

177. The force which causes bodies to descend to the surface of the earth, and which is termed *weight*, is due principally to the attraction of the terrestrial spheroid on these bodies. But whatever be the cause of it, there can be no doubt, that the centrifugal force diminishes this tendency of heavy bodies; so that except at the pole, where the centrifugal force is nothing, the weight is in every other place less than if the earth had no rotatory motion. At the equator, the centrifugal force and weight act in opposite directions the one to the other; therefore, the weight is equal to the excess of the attraction of the earth over the centrifugal force; consequently we have

$$g = \text{G} - \frac{4\pi^2 r}{\text{T}^2},$$

$g$ being this weight, G the terrestrial attraction, or the weight which would have place if the earth was immoveable, $r$ the radius of the equator, and T the time of the earth's rotation

As the second term of the second member of this expression is very small relatively to the first, we have, very nearly,

$$g = \text{G}\left(1 - \frac{4\pi^2 r}{g\text{T}^2}\right)$$

In order to find the numerical value of $\frac{4\pi^2 r}{g\text{T}^2}$, the radius of the meridian may be assumed in place of $r$ the radius of the equator, from which it differs but little, we shall then have

$$2\pi r = 40000000^m$$

Assuming the second for the unit of time, and neglecting, in this computation, the small variation of gravity at the surface of the earth, we have also (No 115)

$$g = 9^m,80896$$

Moreover, by (No 111)

$$\text{T} = 86164,$$

hence, we obtain very nearly,

$$\frac{4\pi^2 r}{g\text{T}^2} = \frac{1}{289}.$$

Hence, at the equator, the weight is diminished by $\frac{1}{289}$, in consequence of the motion of rotation of the earth about its axis. If this motion should become more rapid, T would diminish, and the centrifugal force would differ less from gravity Since 289 is the square of 17, it is evident that if the rotation was performed in the seventeenth part of a day, the centrifugal force would be equal to that of gravity, in this case the weight would be equal to cypher, and bodies remitted to themselves would remain in equilibrio

In this computation, we have only considered the centrifugal force arising from the motion of heavy bodies about the axis of the earth, and in fact, it is easy to conceive, that the motion of translation about the sun, which is common to the earth, to its axis, and to all these bodies, cannot influence their tendency to deviate from this moveable line. For if we suppose, for example, a thread parallel to the equator attached to this axis, and terminating in a body situated at the surface, it is evident that its tension will not be in any respect changed by the effect of a motion which carries along, at the same time, the axis, the thread, and the body, without changing their relative positions.

178 The centrifugal force diminishes the weight in all points of the earth's surface, but by a greater quantity at the equator than at any other point, both because the centrifugal force decreases in passing from the equator to the poles, and also because the angle which it makes with the vertical increases Naming $r$ the radius of the equator, $\mu$ the latitude of any place on the surface of the earth, and $u$ the radius of the corresponding parallel, we shall have

$$u = r \cos \mu ,$$

the non-sphericity of the earth being neglected, the angle $\mu$ will be that which the production of $u$, or the direction of the centrifugal force makes with the vertical, therefore the vertical component of the centrifugal force will be obtained by multiplying its intensity $\frac{4\pi^2 u}{T^2}$ by $\cos \mu$, this gives

$$\frac{4\pi^2 r \cos^2 \mu}{T^2}$$

for the diminution of the centrifugal force arising from the rotation of the earth, and, by what precedes, the actual value of this quantity will be

$$\frac{\cos^2 \mu}{289}$$

This would be all the diminution that the weight would experience, if the earth was a homogeneous sphere, and as it is proportional to the square of the cosine of latitude, the entire diminution from the pole where $\mu = 90°$, to the equator where $\mu = 0$, amounts to $\frac{1}{289}$. But the earth is a spheroid flattened at its poles, and for this reason, the attraction which it exerts on bodies situated on its surface, diminishes from the poles to the equator, this diminution in each point of the surface is also proportional to the square of the cosine of latitude, it should be added to that which is produced by the centrifugal force, and by this addition the coefficient $\frac{1}{289}$ becomes $\frac{1}{200}(e)$ very nearly Therefore it is this fraction $\frac{1}{200}$ which will express, as has been observed (No 117), the total increment of the weight of a body transferred from the equator to the pole.

# CHAPTER V.

## EXAMPLES OF THE MOTION OF A MATERIAL POINT ON A CURVE OR ON A GIVEN SURFACE

### I. *Oscillation of the simple Pendulum*

179. A *pendulum* is, in general, a solid heavy body, which oscillates about a fixed and horizontal axis But, in order to facilitate the comparison of the durations of the oscillations of different pendulums, and the corresponding intensities of gravity, geometers have devised an ideal pendulum, that is termed the *simple pendulum*, and which consists of a heavy material point, attached to a fixed point, by means of an inflexible inextensible string, that is supposed to be void of gravity and of uniform density, the length of this string is that of the pendulum.

In a subsequent chapter it will be shewn, that there always exists a *simple* pendulum of which the oscillations coincide, both as to their durations and amplitudes, with those of any other pendulum whatever It will also be shewn there, how, when the form and dimensions of the second description of pendulum are given, the length of the first can be determined, and in the discussion on this subject it will appear, that if this agreement obtains between the motions of two such pendulums in a vacuo, it will also subsist in a resisting medium, whatever be the function of the velocity which expresses the resistance Thus, it will be sufficient to consider the motion of the simple pendulum either in a vacuo, or in a resisting medium, which we propose to do in this first section.

Let C (fig. 45) be the point of suspension, CB the vertical

passing through this fixed point, and CA the initial position of the pendulum If the material point at the extremity of CA, moves from the point A in a direction perpendicular to CA, and in the plane of the lines CA and CB, with a velocity represented by $k$, it is evident that it will not deviate from this vertical plane, and that it will describe arcs of a circle, of which the centre is A and radius CA.

After any time such as $t$, let M be the position of the moveable, from M and A, let the perpendiculars MP and AD be let fall on the vertical CB, and let us make

$$\text{CP} = z, \quad \text{CD} = c$$

Then, if $g$ denote the gravity, and $v$ the velocity of the moveable at the point M, we shall have, when the motion is performed in a vacuum, (No, 159),

$$v^2 = k^2 + 2g(z - c),$$

and if $s$ denotes the arc described by the moveable, we have $\frac{ds}{dt} = v$, consequently there results($a$)

$$dt = \frac{ds}{\sqrt{k^2 + 2g(z - c)}}$$

If $\theta$ denotes the angle MCB, which will be positive when the pendulum exists to the left of CB, and negative when the pendulum lies to the right of the vertical, likewise if $\alpha$ be the angle ACB, or the initial value of $\theta$, we shall have

$$s = a(\alpha - \theta), \quad v = \frac{ds}{dt} = -a\frac{d\theta}{dt},$$

$a$ representing CM or CA the length of the pendulum. We shall have also

$$z = a \, . \cos\theta, \quad c = a \quad \cos\alpha,$$

by means of these values, that of $dt$ will become

$$dt = \frac{-a d\theta}{\sqrt{k^2 + 2ga(\cos\theta - \cos\alpha)}}. \qquad (1)$$

This is the expression which it is proposed to integrate either exactly or by approximation.

181. There is only one case in which the integration can be effected in a finite form, and that is when($b$)

$$k^2 = 2ga\,(1 + \cos\alpha)\,;$$

this equation obtains, when the moveable departs from the point A with a velocity acquired in falling through a height equal to ED, E being the most elevated point of the circle in which the pendulum moves Making $\theta = 2\psi$, and observing that

$$1 + \cos 2\psi = 2\cos^2\psi,$$

we then have($c$)

$$dt = -\sqrt{\frac{a}{g}}\,\frac{d\psi}{\cos\psi}.$$

If this expression be integrated, and the constant arbitrary determined, so that $\psi = \frac{1}{2}\alpha$ when $t = 0$, it becomes, by substituting $\frac{1}{2}\theta$ in place of $\psi$,

$$t = \tfrac{1}{2}\,.\sqrt{\frac{a}{g}}\log\frac{(1 - \sin\frac{1}{2}\theta)(1 + \sin\frac{1}{2}\alpha)}{(1 + \sin\frac{1}{2}\theta)(1 - \sin\frac{1}{2}\alpha)}.$$

If the point A coincides with the point E, we shall have $\alpha = \pi$, and this will render the preceding value of $t$ infinite, whatever may be the magnitude of the angle $\theta$ This indicates that the moveable does not leave the point E, in fact, in this case, its initial velocity will be nothing, and as the tangent at the point E is horizontal, it will remain at rest.

At the point B, $\theta = 0$, therefore in every other case,

$$\tfrac{1}{2}\sqrt{\frac{a}{g}}.\log\frac{1 + \sin\frac{1}{2}\alpha}{1 - \sin\frac{1}{2}\alpha},$$

expresses the time that the moveable takes to describe the arc AB. It will ascend on the semi-circumference BA′E with the velocity acquired at this point, but it is evident from what has been stated in No. 159, that it would require an infinite time to reach the point E; and this is in fact evident from the pre-

ceding expression, in which when $\theta = -\pi$, $t = \infty$. Whatever be the initial velocity $k$ and the angle $a$, formula (1) may be integrated by elliptic functions; so that the times of oscillations, or of revolutions of a pendulum, may be always determined by means of the numerical tables of these functions, but as in practice, it is only necessary to know the durations of very small oscillations, we shall here restrict ourselves to the consideration of such

182 In order that the pendulum may make only small oscillations on each side of the vertical CB, it is necessary that the angle $a$ and the velocity $k$ should be inconsiderable, as this velocity may be rendered entirely evanescent ($d$), by making the moveable depart from a point a little more elevated than A, that is to say, by a suitable increase of the angle $a$, the generality of the question will not therefore be affected by supposing $k = 0$, this supposition reduces equation (1) to

$$dt = -\sqrt{\frac{a}{g}} \cdot \frac{d\theta}{\sqrt{2\cos\theta - 2\cos a}}. \qquad (2)$$

By known formulæ, we have

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{1.2.3.4} - \&c,$$

$$\cos a = 1 - \frac{a^2}{2} + \frac{a^4}{1.2.3.4} - \&c.$$

And since by hypothesis, the angles $a$ and $\theta$ are very small, their fourth powers may be neglected, therefore we shall have

$$dt = -\sqrt{\frac{a}{g}} \, \frac{d\theta}{\sqrt{a^2 - \theta^2}}.$$

By integrating and observing that $t = 0$ when $\theta = a$, there results

$$t = \sqrt{\frac{a}{g}} \operatorname{arc}\left(\cos = \frac{\theta}{a}\right),$$

hence we can deduce

$$\theta = a \cos t \sqrt{\frac{g}{a}}, \quad \frac{d\theta}{dt} = -a\sqrt{\frac{g}{a}} \cdot \sin t \sqrt{\frac{g}{a}}.$$

These formulæ indicate, agreeably to what has been already pointed out (No 159), that the pendulum will make an indefinite series of equal and isochronous oscillations on each side of the vertical CB; the velocity vanishes, and the moveable will return to the point A, where $\theta = a$, whenever $t\sqrt{\frac{g}{a}}$ will be a multiple of $2\pi$, and to the point A′ equally elevated as A, and where $\theta = -a$, as often as $\theta$ will be an odd multiple of $\pi$ If T denotes the time employed in passing from one of these extreme points to the other, that is to say, the time of an entire oscillation, we have($e$)

$$T = \pi\sqrt{\frac{a}{g}}.$$

The durations of the two semi-oscillations, the one descending, and the other ascending, will be respectively equal to each other and expressed by $\frac{T}{2}$.

In general, at two instants separated by an interval of time equal to T, the pendulum will exist on opposite sides of the vertical CB, in points which are equally distant from this line, and will be actuated by equal and opposite velocities, for if in the values of $\theta$ and $\frac{d\theta}{dt}$, $t + T$ be substituted in place of $t$, it is evident that the only change which they undergo is a change of sign($f$)

The pendulum coincides with the vertical when $\theta = 0$, in which case $t$ is an odd multiple of $\frac{T}{2}$, hence it follows, that

$$\frac{d\theta}{dt} = \pm a\sqrt{\frac{g}{a}},$$

and, consequently,

$$v = \pm a\sqrt{ga},$$

expresses the velocity of the moveable at the point B. Naming $b$ the height DB of its point of departure above B, we shall have

$$b = a\ (1 - \cos \alpha) = \tfrac{1}{2} a\alpha^2,$$

because the fourth and higher powers of $\alpha$ are neglected. Consequently, abstracting from the consideration of the sign, the velocity acquired at the lowest point will be($g$)

$$v = \sqrt{2gb},$$

which expresses, as it ought to do, the velocity acquired in falling through the height $b$

183 The value of T is, as we have seen, independent of the angle $\alpha$, it will even subsist, and be rigorously exact, when this amplitude $\alpha$ is infinitely small Therefore, if the pendulum deviates by an infinitely small quantity from the vertical, it will take a finite time equal to $\frac{1}{2}\pi\sqrt{\frac{a}{g}}$ to return to it In this movement, the moveable will describe an infinitely small space in a finite time, this arises from the circumstance of the intensity of its accelerating force being infinitely small. In fact, this force is that of gravity resolved in the direction of a tangent to the trajectory, now in the extent of the infinitely small arc that is terminated at the lowest point of this curve, the tangent makes with the vertical an angle which differs from a right angle, by an infinitely small quantity, consequently the cosine of this angle, by which the gravity must be multiplied in order to obtain its component, is infinitely small, therefore, this component must be also infinitely small

This result may be extended to the oscillations of a heavy material point on any curve whatever, of which the osculating plane at the lowest point is vertical, for within an *infinitely* small extent, the curve coincides with its osculating circle, and in an extent which is only *very* small, it deviates from it very little, hence it follows, that if C be the centre of this

circle, the duration of the very small oscillations on the curve on each side of its point B, is the same as for a simple pendulum whose point of suspension is C, and of which the length will be CB, the radius of curvature corresponding to this point B. Therefore, the duration of very small oscillations on all vertical curves, which have the same curvature at their lowest point, is the same and independent of their amplitude. When the osculating plane in this point is not vertical, we must substitute in the expression for T, in place of the gravity $g$, its component in this plane, which is equal to $g \sin \iota$, $\iota$ denoting the inclination of the given plane on the horizontal plane.

184 When the angle $a$ is of a finite magnitude, but very small, then the preceding value of T is only an approximate one In fact, if the fourth powers of $a$ and $\theta$ are retained in the values of $\cos a$, $\cos \theta$, and if they are substituted in formula (2), we shall have($h$)

$$dt = -\sqrt{\frac{a}{g}} \frac{d\theta}{\sqrt{a^2 - \theta^2}\sqrt{1 - \frac{1}{12}(a^2 + \theta^2)}}$$

At this degree of approximation, we must assume

$$\left(1 - \frac{1}{12}(a^2 + \theta^2)\right)^{-\frac{1}{2}} = 1 + \frac{1}{24}(a^2 + \theta^2),$$

therefore we shall have

$$dt = -\sqrt{\frac{a}{g}}\left(\frac{d\theta}{\sqrt{a^2 - \theta^2}} + \frac{(a^2 + \theta^2)\,d\theta}{24\sqrt{a^2 - \theta^2}}\right),$$

which formula may be integrated by the known rules. By integrating from $\theta = a$ to $\theta = -a$, in order to obtain T the duration of an entire($\iota$) oscillation, we find

$$T = \pi\sqrt{\frac{a}{g}}\left(1 + \frac{a^2}{16}\right),$$

from which it appears, that this duration is a little increased by the magnitude of the amplitude.

It follows from this expression, that if $n$ denotes the num-

ber of *infinitely small* oscillations of any pendulum in a given time, and $n'$ the number of oscillations of the same pendulum and in the same time, when their amplitude $a$ is only *very* small, we shall have(*k*)

$$n = n'\left(1 + \frac{a^2}{16}\right),$$

for this number $n'$ must diminish in the same ratio as the duration of each oscillation is increased by the magnitude of this amplitude

185. Although, in the different applications of the pendulum, philosophers always take precautions that the amplitude of the oscillations should be very small, by which means the correction relative to the magnitude of $a$, which has been determined above, will be always sufficiently accurate, it may, nevertheless, be useful to know the converging series, by means of which the duration of an oscillation may be expressed, whatever be its amplitude

For this purpose, let $x$ and $\beta$ be the versed sines of the angles $\theta$ and $a$, so that

$$1 - \cos\theta = x, \quad 1 - \cos a = \beta,$$

and also, at same time,

$$d\theta = \frac{dx}{\sqrt{2x - x^2}}$$

The formula (2) will become

$$dt = -\tfrac{1}{2}\sqrt{\frac{a}{g}}\,\frac{dx}{\sqrt{\beta x - x^2}\sqrt{1 - \frac{1}{2}x}},$$

and, in order to deduce from these the duration of $\frac{1}{2}$T, a semi-oscillation, we must integrate from $x = \beta$, (which answers to $\theta = a$,) to $x = 0$, (which answers to $\theta = 0$ )(*l*)

Now, developing by the formula for expanding a binomial, we obtain

$$(1 - \tfrac{1}{2}x)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\frac{x}{2} + \frac{1\,3}{2\,4}\cdot\frac{x^2}{4} + \frac{1\,3\,5}{2\,4.6}\cdot\frac{x^3}{8} + \&c.,$$

in this series, the general term is

$$\frac{1\ 3\ 5\quad .\ 2n-1}{2\ 4\ 6\ldots\ .\quad 2n}\left(\frac{x}{2}\right)^n,$$

and it always converges, because $x$ is constantly less than 2. If, therefore, the order of integration be reversed, which we are permitted to do, by changing at the same time the sign of $dt$, and if we make

$$\int_0^\beta \frac{x^n dx}{\sqrt{\beta x - x^2}} = A_n,$$

($n$ being either cypher or any number whatever) there will result

$$T = \sqrt{\frac{a}{g}}\cdot\left(A_0 + \tfrac{1}{2}\cdot\tfrac{1}{2}A_1 + \frac{1\ 3}{2.4}\cdot\tfrac{1}{4}A_2 + \frac{1\ 3.5}{2.4.6}\cdot\tfrac{1}{8}A_3 + \&c\right),$$

T being the duration of an entire oscillation.

The values of the definite integrals $A_0$, $A_1$, $A_2$, $A_3$, are connected together in such a manner, that one of them being known, it is easy to deduce successively all the others In fact, we have($m$)

$$\int\frac{x^n dx}{\sqrt{\beta x - x^2}} = \int\frac{(x-\frac{1}{2}\beta)\,x^{n-1}dx}{\sqrt{\beta x - x^2}} + \frac{\beta}{2}\int\frac{x^{n-1}\,dx}{\sqrt{\beta x - x^2}},$$

$$\int\frac{(x-\frac{1}{2}\beta)x^{n-1}dx}{\sqrt{\beta x - x^2}} = -x^{n-1}\sqrt{\beta x - x^2} + (n-1)\int x^{n-2}\sqrt{\beta x - x^2}\,dx,$$

$$\int x^{n-2}\sqrt{\beta x - x^2}\,dx = \beta\cdot\int\frac{x^{n-1}dx}{\sqrt{\beta x - x^2}} - \int\frac{x^n\,dx}{\sqrt{\beta x - x^2}},$$

hence we infer

$$\int\frac{x^n dx}{\sqrt{\beta x - x^2}} = -\,x^{n-1}\sqrt{\beta x - x^2} - (n-1)\int\frac{x^n dx}{\sqrt{\beta x - x^2}}$$
$$+\frac{(2n-1)}{2}\beta\int\frac{x^{n-1}dx}{\sqrt{\beta x - x^2}},$$

and, consequently,

$$\int\frac{x^n dx}{\sqrt{\beta x - x^2}} = \frac{-x^{n-1}}{n}\sqrt{\beta x - x^2} + \frac{(2n-1)}{2n}\beta\int\frac{x^{n-1}dx}{\sqrt{\beta x - x^2}}.$$

At the two limits $x=0$ and $x=\beta$, we have $\sqrt{\beta x - x^2}=0$, and, therefore, by taking the definite integrals we shall have, by means of this last equation,

$$A_n = \frac{(2n-1)\beta}{2n} A_{n-1}.$$

If in this formula we make $n=1$, $n=2$, $n=3$, &c., successively, we obtain from it

$$\begin{aligned} A_1 &= \tfrac{1}{2}\beta A_0, \\ A_2 &= \tfrac{3}{4}\beta A_1 = \frac{1\ 3}{2.4}\beta^2 A_0, \\ A_3 &= \tfrac{5}{6}\beta A_2 = \frac{1.3.5}{2\ 4.6}\beta^3 A_0, \\ &\text{\&c.}, \end{aligned}$$

consequently, we shall have, generally,

$$A_n = \frac{1.3\ 5\ \ldots\ .\ 2n-1}{2\ 4.6\ .\ ..\ \ 2n}\beta^n A_0;$$

and as to the value of $A_0$, it is, when taken between $x=o$ and $x=\beta$,

$$A_0 = \int_0^\beta \frac{dx}{\sqrt{\beta x - x^2}} = \pi.$$

By substituting these values of $A_0$, $A_1$, $A_2$, &c., in that of $T$, there will result

$$T = \pi\sqrt{\frac{a}{g}}\left[1 + \left(\frac{1}{2}\right)^2\left(\frac{\beta}{2}\right) + \left(\frac{1\ 3}{2.4}\right)^2\left(\frac{\beta}{2}\right)^2 + \left(\frac{1\ 3.5}{2.4.6}\right)^2\left(\frac{\beta}{2}\right)^3 + \text{\&c.}\right],$$

for the required value of $T$, and which necessarily converges, since $\frac{1}{2}\beta$ is always less than unity.

If the fourth power of $a$ be neglected, we shall obtain $\beta = \frac{1}{2}a^2$, hence this series will be reduced to its two first terms, and the value of $T$ will coincide with that of the preceding number.

186. Let us now proceed to consider the motion of a simple pendulum in a resisting medium. If the preceding notations be retained, the force of gravity resolved in the direction of the tangent MT will be $g \, . \sin \theta$, because the angle which this line makes with the vertical MN is the complement of the angle MCB or $\theta$. As the accelerating force arising from the resistance which we denote by V, acts in a direction contrary to $g \, . \sin \theta$, the equation of the motion will be (No. 152)

$$\frac{d^2 s}{dt^2} = g \sin \theta - \mathrm{V}. \qquad (3)$$

(in which $s$ denotes the arc AM). We may make different hypotheses as to the value of V, considered as a function of the velocity of the moveable; the simplest is that in which it is supposed to be proportional to this velocity, in which case, we have

$$\mathrm{V} = \frac{g}{k} \cdot \frac{ds}{dt},$$

$k$ denoting a given constant velocity. We have also

$$s = a(\alpha - \theta), \quad \sin \theta = \theta - \frac{\theta^3}{1 \, 2.3} + \&c.,$$

if, therefore, $\theta$ be considered, as heretofore, a very small angle, and if its third power be neglected, equation (3) will become

$$\frac{d^2 \theta}{dt^2} + \frac{g}{k} \cdot \frac{d\theta}{dt} + \frac{g}{a} . \theta = 0,$$

of which the complete integral is($n$)

$$\theta = \left( c \cos t\gamma \sqrt{\frac{g}{a}} + c' \sin t\gamma \sqrt{\frac{g}{a}} \right) e^{\frac{-gt}{2k}},$$

in which $c, c'$, denote two constant arbitraries, $e$ the base of the Naperian system of logarithms, and also, for the sake of abridging,

$$\sqrt{1 - \frac{ga}{4k^2}} = \gamma,$$

$c$ and $c'$ are determined by the conditions $\theta = a$ and $\frac{d\theta}{dt} = 0$, when $t = 0$, which gives

$$c = a, \quad c' = \frac{a\sqrt{ga}}{2\gamma k},$$

consequently, we shall have

$$\theta = a\left(\cos t\gamma\sqrt{\frac{g}{a}} + \frac{\sqrt{ga}}{2\gamma k}\sin t\gamma\sqrt{\frac{g}{a}}\right)e^{\frac{-gt}{2k}},$$

and, by differentiating,(o)

$$\frac{d\theta}{dt} = -\frac{a}{\gamma}\sqrt{\frac{g}{a}}\left(\sin t\gamma\sqrt{\frac{g}{a}}\right)e^{\frac{-gt}{2k}},$$

for the expressions, by means of which we can determine at any instant whatever, the position of the pendulum and its angular velocity

At the end of each oscillation, we have $\frac{d\theta}{dt} = 0$; this is the case as often as $t\gamma\sqrt{\frac{g}{a}}$ is a multiple of $\pi$. It follows, therefore, that the oscillations are isochronous, as in a vacuum, and that we have(p)

$$\mathrm{T} = \frac{\pi}{\gamma}\sqrt{\frac{a}{g}},$$

for the duration of an entire oscillation, so that it is increased by the resistance of the medium, in the ratio of unity to the fraction $\gamma$. As to the amplitudes of the oscillations, it is evident from the form of the exponential $e^{\frac{-gt}{2k}}$, that they continually diminish. Naming $a_n$ the amplitude of the $n^{th}$ oscillation, that is to say, supposing that $\theta = (-1)^n a_n$ when $t = n\mathrm{T}$, there will result(q)

$$a_n = ae^{\frac{-n\pi\sqrt{ga}}{2\gamma k}},$$

which shews, that the successive amplitudes constitute a

decreasing geometrical progression, of which the ratio is $e^{\frac{-\pi\sqrt{ga}}{2\gamma k}}$.

This oscillatory motion supposes that T is a real quantity; and, in fact, this is the case in all experiments of the pendulum, in which the length is never very considerable, its density being at the same time always very great, compared with that of the air in which it moves, and as the velocity $k$ is proportional to the ratio of the first density to the second, it is very considerable compared with $\frac{1}{2}\sqrt{ga}$, and, consequently, $\gamma$ is a real quantity, that differs very little from unity. However, if $2k$ was less than $\sqrt{ga}$, $\gamma$ would be imaginary, and of the form $\beta\sqrt{-1}$, $\beta$ denoting a real quantity, by the known formulæ, the sines and cosines which occur in the expression for $\theta$, will then be transformed into exponentials, and when this transformation is effected, it is evident from inspection, that the time required to lapse, before $\theta$ vanishes, is infinite(?), so that the pendulum will approach indefinitely near the vertical CB, without ever passing or even attaining to it.

187 It appears from experiment, that according as the amplitudes of the oscillations performed in the air diminish, they tend more and more to decrease in a geometric progression, for example, they deviate little from this progression, when the angle $\alpha$ is the third of a degree, or somewhat less. Experiment likewise proves, that this decrease is very slow, thus, in the experiments of Borda, in which the amplitudes of the oscillations constitute, as to sense, a geometric progression, the amplitude was reduced to about two-thirds, after 1800 oscillations. Therefore, if to this example, the expression for $a_n$, given above, be applied, we shall obtain

$$e^{\frac{-1800\pi\sqrt{ga}}{2\gamma k}} = \tfrac{2}{3},$$

and, consequently,

$$\frac{1800\pi\sqrt{ga}}{2k} = \gamma \log \tfrac{3}{2} = \gamma\,(0{,}40546),$$

but we have

$$\frac{ga}{4k^2} = 1 - \gamma^2,$$

hence there results

$$(1800)^2\pi^2(1 - \gamma^2) = \gamma'^2 (0,40546)^2,$$

from which we obtain

$$\gamma = 1,00000000257 \ldots\ldots,$$

or $\gamma = 1$, *q p.*, we are therefore permitted to neglect the consideration of the resistance of the air in computing the value of T. It follows from this, that when the oscillations are very small, the resistance of the air may be assumed to be proportional to the velocity, as has been supposed, and also that this resistance does not sensibly influence their duration. But when the amplitudes are of some magnitude, it appears from observation, that they do not decrease in a geometric progression, so that it becomes necessary to make some other hypothesis on the law of resistance

188. Let it be assumed that this force is proportional to the square of the velocity, and let

$$\text{v} = \frac{g}{k^2}\frac{ds^2}{dt^2},$$

$k$ being a given constant velocity, which will always be very great, so that if we make

$$\frac{2ga}{k^2} = \mu,$$

$\mu$ will be a very small fraction. Since $ds = -a d\theta$, equation (3) will become

$$\frac{d^2\theta}{dt^2} + \frac{g}{a}\sin\theta = \tfrac{1}{2}\mu\frac{d\theta^2}{dt^2}, \qquad (4)$$

multiplying this equation by $2d\theta$, integrating and making

$$\int\frac{d\theta^2}{dt^2}\,d\theta = y, \quad \frac{d\theta^2}{dt^2} = \frac{dy}{d\theta},$$

we shall have

$$\frac{dy}{d\theta} - \frac{2g}{a}\cos\theta - \mu y = 0,$$

this being a linear equation of the first order, its complete integral is(*s*)

$$y = ce^{\mu\theta} + 2g\frac{(\sin\theta - \mu\cos\theta)}{(1+\mu^2)\,a},$$

*c* being a constant arbitrary, and *e* the base of the Naperian system; differentiating with respect to $\theta$, and substituting $\frac{d\theta^2}{dt^2}$ in place of $\frac{dy}{d\theta}$, there results

$$\frac{d\theta^2}{dt^2} = \mu ce^{\mu\theta} + 2g\frac{(\cos\theta + \mu.\sin\theta)}{(1+\mu^2)\,a},$$

which is a first integral of equation (4) given in a finite form. In order to determine *c*, let us assume, as has been done above, that $\frac{d\theta}{dt} = 0$, when $\theta = a$; then there will result

$$\mu c = -2g\frac{(\cos a + \mu\sin a)}{(1+\mu^2)\,a}e^{-\mu a},$$

consequently, we shall have, at any instant whatever,

$$\frac{d\theta^2}{dt^2} = \frac{2g}{(1+\mu^2)\,a}[\cos\theta + \mu\sin\theta - (\cos a + \mu\sin a)\,e^{-\mu(a-\theta)}]. \quad (5)$$

Therefore, at the lowest point, where $\theta = 0$, we shall have

$$\frac{a^2 d\theta^2}{dt^2} = \frac{2ga}{1+\mu^2}[1 - (\cos a + \mu\sin a)\,e^{-\mu a}],$$

for the expression for the square of the acquired velocity, which is evidently less than in a vacuum. In virtue of this velocity, the moveable will ascend on the arc BA′ to a point $A_1$ less elevated than A′, and for which we shall have $\frac{d\theta}{dt} = 0$.

If the corresponding value of $\theta$ be denoted by $-a_1$, there will result

$$(\cos a_1 - \mu \sin a_1)\, e^{\mu a_1} = (\cos a + \mu \sin a)\, e^{-\mu a},$$

and if the exponentials be developed according to the powers of $\mu$, the square of $\mu$, which is a very small fraction, being neglected, we shall obtain($t$)

$$\cos a_1 - \mu (\sin a_1 - a_1 . \cos a_1) = \cos a + \mu (\sin a - a \cos a).$$

The value of $a_1$ which can be deduced from this equation, differs very little from $a$, therefore, by making $a_1 = a - \delta$, and neglecting the square of $\delta$ and the product $\mu\delta$, there results

$$\delta \sin a = 2\mu (\sin a - a \cos a),$$

so that we shall have($u$)

$$a_1 = a - \frac{2\mu}{\sin a} (\sin a - a \cos a),$$

as an expression, abstracting from the sign, for the magnitude of $\theta$ at the end of the first oscillation

This result does not imply that the oscillations are very small, however, if they are so small, that the fourth power of $a$ may be neglected in the expression for $a_1$ it will become($v$)

$$a_1 = a - \frac{2\mu a^2}{3}.$$

After the moveable attains the point $A_1$, it will descend again, and will thus continue to oscillate from one side to the other of the point B, until the amplitudes of these oscillations to all appearance vanish. If $a_2$ denotes the amplitude of the second ascending demi-oscillation, it is evident that it can be deduced from $a_1$, as $a_1$ has been deduced from $a$, so that we shall have

$$a_2 = a_1 - \frac{2\mu a_1^2}{3}.$$

and in like manner, if $a_3$, $a_4$, &c., be the successive amplitudes of the other ascending demi-oscillations, we shall have

$$a_3 = a_2 - \frac{2\mu a_2^2}{3}, \quad a_4 = a_3 - \frac{2\mu a_3^2}{3}, \text{ \&c.},$$

which shews that they no longer decrease in geometric progression, as in the case of a resistance proportional to the velocity.

189 In order to determine the time in which the angle $\theta$ is described, the value of $dt$ deduced from equation (5) must be integrated, and this can be always effected by the method of quadratures, when the numerical values of $a$, $\mu$, $\theta$, are known. But when the oscillations are small, the value of $\theta$ considered as a function of $t$, may be obtained in a converging series, and *vice versa.*

The initial velocity of the moveable is always assumed equal to cypher; and the value of $\theta$ at any instant whatever will be a function of $a$ and $t$, which should become cypher when $a = 0$; therefore it can be represented by

$$\theta = a\theta_1 + a^2\theta_2 + a^3\theta_3 + \text{\&c},$$

$\theta_1, \theta_2, \theta_3$, &c, being coefficients independent of $a$. By substituting this series in equation (4), and then developing the two members according to the powers of $a$, we shall obtain by putting the coefficients of the same powers respectively equal to each other, a series of differential equations of the second order, by means of which the unknown quantities $\theta_1, \theta_2, \theta_3$, &c., can be determined. Moreover, in order that we should have $\theta = a$ and $\frac{d\theta}{dt} = 0$, when $t = 0$, whatever may be the magnitude of $a$, it is necessary that the initial values of $\theta_2, \theta_3$, &c, $\frac{d\theta_2}{dt}, \frac{d\theta_3}{dt}$, &c., should all vanish, and that those of $\theta_1$ and $\frac{d\theta_1}{dt}$ should be unity and cypher; it is by means of these conditions that the constant arbitraries which are contained in the complete integrals of these series of equations, can be determined. In this manner, we may compute as many terms as we

please, of the preceding series. If the approximation be limited to the square of $a$, in which case the cube and higher powers of this quantity should be neglected, we obtain

$$\frac{d^2\theta}{dt^2} = \frac{a d^2\theta_1}{dt^2} + a^2 \frac{d^2\theta_2}{dt^2},$$

$$\sin\theta = a\theta_1 + a^2\theta_2$$

$$\frac{d\theta^2}{dt^2} = a^2 \frac{d\theta_1{}^2}{dt^2},$$

and if these values be substituted in equation (4), there results, by equalling the coefficients of $a$ and of $a^2$ in its two members,

$$\frac{d^2\theta_1}{dt^2} + \frac{g}{a}\theta_1 = 0,$$

$$\frac{d^2\theta_2}{dt^2} + \frac{g}{a}\theta_2 = \tfrac{1}{2}\mu \frac{d\theta_1{}^2}{dt^2}.$$

If the first of these two equations be integrated, and if the two constant arbitraries be determined in such a manner that $\theta_1 = 0$ and $\frac{d\theta_1}{dt} = 0$, when $t = 0$, we shall have

$$\theta_1 = \cos t \sqrt{\frac{g}{a}}.$$

Hence there results

$$\frac{d\theta_1{}^2}{dt^2} = \frac{g}{a}\sin^2 t \sqrt{\frac{g}{a}} = \frac{1}{2}\frac{g}{a}\left(1 - \cos 2t \sqrt{\frac{g}{a}}\right),$$

consequently the second equation will become(*t*)

$$\frac{d^2\theta_2}{dt^2} + \frac{g}{a}\theta_2 = \frac{g\mu}{4a}\left(1 - \cos 2t \sqrt{\frac{g}{a}}\right);$$

and then

$$\theta_2 = -\frac{\mu}{3}\cos t \sqrt{\frac{g}{a}} + \tfrac{1}{4}\mu + \tfrac{1}{12}\mu \cos 2t \sqrt{\frac{g}{a}},$$

will be its complete integral subject to the conditions $\theta_2 = 0$ and $\frac{d\theta_2}{dt} = 0$, when $t = 0$.

By means of these expressions for $\theta_1$ and $\theta_2$, that for $\theta$ becomes

$$\theta = \left(a - \frac{a^2\mu}{3}\right)\cos t\sqrt{\frac{g}{a}} + \frac{a^2\mu}{4} + \frac{a^2\mu}{12}\cos 2t\sqrt{\frac{g}{a}},$$

and because $v = -\frac{ad\theta}{dt}$, we shall, at the same time, have

$$v = \left(a - \frac{a^2\mu}{3}\right)\sqrt{ga}\sin t\sqrt{\frac{g}{a}} + \frac{a^2\mu\sqrt{ga}}{6}\sin 2t\sqrt{\frac{g}{a}},$$

these formulæ make known the position and velocity of the moveable at any instant whatever.

190. If in the last equation, we substitute in place of $\sin 2t\sqrt{\frac{g}{a}}$, its value $2\sin t\sqrt{\frac{g}{a}}\cos t\sqrt{\frac{g}{a}}$, the equation $v=0$, which obtains at the end of each oscillation, will assume the form

$$\left(1 - \frac{a\mu}{3} + \frac{a\mu}{3}\cos t\sqrt{\frac{g}{a}}\right)\sin t\sqrt{\frac{g}{a}} = 0.$$

As the angle $a$ is very small, the first factor cannot vanish, the second is cypher as often as $t\sqrt{\frac{g}{a}}$ is a multiple of $\pi$. It follows, therefore, that the interval of time which lapses between two consecutive velocities, respectively equal to cypher, or T the duration of an entire oscillation, is($u$)

$$T = \pi\sqrt{\frac{a}{g}},$$

hence it appears, that when the resistance of the air is proportional to the square of the velocity, the duration of the oscillation is not affected by it. However, it increases the time which the moveable takes to attain the point B. In fact, if we denote it by $t'$, and make $\theta = 0$, we have

$$\left(1 - \frac{a\mu}{3}\right)\cos t'\sqrt{\frac{g}{a}} + \frac{a\mu}{4} + \frac{a\mu}{12}\cos 2t'\sqrt{\frac{g}{a}} = 0$$

The least value of $t'\sqrt{\frac{g}{a}}$, which satisfies this equation, differs little from $\frac{1}{2}\pi$, let therefore,

$$t'\sqrt{\frac{g}{a}}=\tfrac{1}{2}\pi+\delta,$$

neglecting the square of $\delta$ and the product $\alpha\delta$, we obtain(v)

$$\delta=\tfrac{1}{6}\alpha\mu,$$

and, consequently,

$$t'=\tfrac{1}{2}\pi\sqrt{\frac{a}{g}}\left(1+\frac{\alpha\mu}{3\pi}\right),$$

hence it appears that the resistance of the air increases the duration of the first descending semi-oscillation in the ratio of $1+\frac{\alpha\mu}{3\pi}$ to unity, and since it does not influence the duration of an entire oscillation, it must diminish in the same ratio, the duration of the ascending semi-oscillation.

By substituting this value of $t'$ in that of $v$, and neglecting the cube of $\alpha$, we obtain(x)

$$v=\left(1-\frac{\alpha\mu}{3}\right)\alpha\sqrt{ga}$$

hence we infer, that the velocity acquired at the lowest point is diminished by the resistance of the air, in the ratio of $1-\frac{\alpha\mu}{3}$ to unity

If $-\alpha_1$ denotes the value of $\theta$ at the end of the first entire oscillation, which answers to that of $t\sqrt{\frac{g}{a}}=\pi$, we shall obtain as before(y)

$$\alpha_1=\alpha-\frac{2\mu\alpha^2}{3}.$$

These different results are independent of the magnitude of $\mu$, the coefficient of the resistance, and only suppose that the angle $\alpha$ is very small, they apply equally to the motion of a

pendulum in an aeriform fluid and in a liquid, provided that the coefficient $\mu$ be determined for each medium in particular. When $a$ is very small, it is unnecessary to consider the case of a resistance proportional to the cube or any higher power of the velocity, because in the values of $\theta$ and $v$, there can only result terms depending on powers of $a$ higher than the square, which we have shewn in the preceding calculations ought to be neglected. Therefore, if what has been stated in No. 187 be considered, it will be manifest, that the resistance of the air does not influence the duration of very small oscillations of the pendulum, for which the correction relative to the magnitude of the amplitude is neglected (No 184) When this correction, however, is taken into account, the resistance has a small influence, because it causes the amplitudes to vary during the continuance of the motion

191 It does not follow from this, that the duration of the oscillations of a heavy body, however small they may be supposed to be, is the same in the air and in a vacuum, for this fluid, by the pressure which it exercises on the moveable, increases this duration, by diminishing the gravity. It can be shewn by experiment, and it will be hereafter demonstrated when we come to treat of *Hydrostatics*, that a body at rest, when plunged in a fluid, loses a part of its weight equal to the weight of that portion of the fluid which it displaces Thus, P being the weight of this body in a vacuum, P′ its weight in air, Π the weight of a volume of air equal to that of the body, we have

$$\mathrm{P}' = \mathrm{P} - \Pi$$

Naming $\rho$ the ratio of the density of the air to that of the body, $g$ the gravity in a vacuum, $g'$ what this force becomes in air, and $m$ the mass of the body, we have

$$\Pi = \mathrm{P}\rho, \quad \mathrm{P} = mg, \quad \mathrm{P}' = mg',$$

consequently there results

$$g' = g(1 - \rho).$$

Now, if $T, T'$ represent the durations of the small oscillations of the same pendulum, when the accelerating forces are respectively $g$ and $g'$, we shall have $T = \pi\sqrt{\frac{a}{g}}$, $T' = \pi\sqrt{\frac{a}{g'}}$, and, consequently,

$$T' = \frac{T}{\sqrt{1-\rho}}.$$

Likewise if $a'$ be the length of the pendulum which, actuated by the gravity $g'$, makes oscillations isochronous with the pendulum actuated by the force of gravity $g$, and of which the length is $a$, we must have

$$\sqrt{\frac{a}{g}} = \sqrt{\frac{a'}{g'}},$$

and, consequently,

$$a' = a(1-\rho).$$

Therefore, if the loss of weight which a body sustains in a state of repose be solely taken into account, the durations of oscillations made in the air will be increased in the ratio of unity to $\sqrt{1-\rho}$ when the length of the pendulum is supposed to be the same, and if the duration be the same in the air and in the vacuum, the length will be diminished in the ratio of $1-\rho$ to unity. Moreover, M. Bessel has shewn by experiment, that the loss of weight which any body sustains in the air, is not the same when the body is at rest, and when it has an oscillatory motion The loss is greater in the second case, it is therefore necessary, in the preceding formulæ, to multiply $\rho$ by a factor $f$ greater than unity, and depending on the form of the moveable The same result was also obtained by M Poisson in a memoir, published in the eleventh volume of the Transactions of the Academy, *On the simultaneous Motions of a Pendulum and of the ambient Air*, by his analysis he found $f = \frac{3}{2}$ when the pendulum consists, like that of Borda, of a sphere attached to the extremity of a very slender thread, its length being very considerable compared with the diameter

of this sphere, so that the correction relative to the density of the air, which, previously to the observation of M Bessel, was applied to the duration of small oscillations and length of the simple pendulum, must be increased by one-half. In all cases, the coefficient $f$ is independent of the density of the pendulum, and also of the density and nature of the fluid in which it vibrates, so that we may always determine it by experiment, by comparing the durations of the oscillations of two pendulums of the same form and of different densities, in the same fluid, or even of the same pendulum in two different fluids, such, for instance, as air and water.

192 Now let $n$ be the number of infinitely small oscillations that any pendulum makes in a vacuum during a given time $\tau$. In order to deduce this number, by the rule of No 184, from that of very small oscillations, which is given by observation, and at the same time to take into account the variation of the amplitudes during this time $\tau$, we must assume for the angle $\alpha$ the mean of the extreme amplitudes which are also furnished by observation This being premised, T the duration of an infinitely small oscillation of this pendulum will be

$$\mathrm{T} = \frac{\tau}{n},$$

and the error into which we may fall, in measuring the time $\tau$, will have so much less influence on the value of T, as the number $n$ will be more considerable. If the form and dimensions of the vibrating body be known, we can determine, by means of a formula that will be given in a subsequent chapter, the length of the simple pendulum, of which the motion is the same as that of this body, and then this length can be reduced to what it would be in a vacuum, in the way just now explained. Hence, if $a$ denotes what it becomes after this reduction, we shall have

$$\frac{\tau}{n} = \pi\sqrt{\frac{a}{g}},$$

(where $g$ denotes the force of gravity in a vacuum) from which we obtain

$$g = \frac{\pi^2 n^2 a}{\tau^2}. \tag{a}$$

It is by means of this formula, that the measure of the gravity, or $g$, the velocity which heavy bodies acquire in falling vertically in a vacuum, during a unit of time, is determined. From experiments made by Borda at the Observatory of Paris, with a pendulum, the length of which was about two metres, there results

$$a = 0^m,993855,$$

(the unit of time being assumed equal to a second,) hence we deduce,

$$g = 9^m,80896,$$

in the latitude of Paris, which is equal to 48° 50′ 14″. M. Bessel having made bodies of every species of matter to vibrate, such as metals, ivory, marble, meteoric stones, &c., invariably found that the values of $g$ were sensibly equal, the greatest differences, on one side or the other of the mean value, scarcely amounting to the hundredth thousandth part of this value, and even this may be attributed to the inevitable errors of observation *There can therefore exist no doubt as to the perfect equality of the attraction exercised by the earth on all bodies, whatever may be their nature, which are situated at the same place of its surface, for this equality results from that of the values of the gravity g, since this force is the excess of the terrestrial attraction over that part of the centrifugal force common to all bodies, which is resolved in the direction of the vertical*

193 It is demonstrated in the *Mechanique Celeste*, that if the surface of the earth is the same as that of the level surface of the sea when at rest, the variation, at this surface, of the length of the simple pendulum, which vibrates in a unit of time, is proportional to the cosine of double the latitude,

so that if $\lambda$ denote this length, in a place of which the latitude is $\psi$, we should have(z)

$$\lambda = l(1 - \omega \cos 2\psi), \tag{b}$$

$l$ and $\omega$ being constant quantities which must be determined by observation It is likewise shewn, that the coefficient $\omega$ is connected with the compression of the terrestrial spheroid by the equation

$$2\omega + \delta = \tfrac{5}{2} r,$$

in which $\delta$ is this compression, (so that the radius of the equator and that of the pole are to each other in the ratio of $1 + \delta$ to unity,) and $r$ denotes the ratio of the centrifugal force to the force of gravity at the equator, the numerical value of this last is (No. 177)

$$r = \tfrac{1}{289}$$

The formula (b) is, in fact, confirmed by experiment, when we abstract from the consideration of all those local circumstances, which, as we shall see in the sequel, may influence the attraction of the earth and the length of the pendulum From a comparison of a great number of observations, made at different latitudes, we obtain

$$\omega = 0{,}002588\ ,$$

from this it follows, that $\delta$ is very nearly equal to $r$. The constant $l$ is the value of $\lambda$ corresponding to $\psi = 45°$, it differs but little from that value of $\lambda$, which corresponds to the latitude of Paris; and by this last, we have

$$0{,}993855 = l\,[1 + 0{,}002588 \,.\, \sin(7° 40' 28'')]\,,$$

hence we obtain

$$l = 0^m{,}993512$$

If in formula (a) we make $n = 1$ and $\tau = 1$, and then substitute successively $l$ and $\lambda$ in place of $a$, we shall have, if $p$ and $p'$ denote the corresponding values of $g$,

$$p = \pi^2 l, \quad p' = \pi^2 \lambda,$$

consequently, we shall have

$$p = 9^m,80557,$$

and for any latitude whatever,

$$p' = p\,(1 - 0,002588 \cos 2\psi)$$

Since

$$\cos 2\psi = 2 \cos^2\psi - 1,$$

it is evident that the diminution of gravity, as we go from the pole to the equator, is proportional to the square of the cosine of latitude, conformably to what is stated in No 178

If the same pendulum be transferred to different places on the earth, it is evident from equation (a), that $n$ the number of its oscillations, in the same time $\tau$, will vary proportionably to the square root of the gravity Thus, for example, a clock regulated at Paris, by the diurnal motion of the earth, and then transferred to the equator, will lose

Naming $n$ and $n'$ the number of oscillations which this pendulum performs in a sidereal day, in these two places respectively, we shall have

$$n = 86164, \quad n' = n\sqrt{\frac{1 - 0,002588}{1 + 0,002588 \sin(7^\circ 40' 28'')}},$$

and, consequently,

$$n' = 86037,$$

so that a clock, when transferred to the equator, loses about 127 seconds in 24 hours. It was from observing that a clock lost when carried to the equator, that philosophers first established the variation of gravity on the surface of the earth.

## II *Motion on a Cycloid.*

194 Let ABC (fig 46) be the trajectory of a heavy material point, the plane of which is vertical. Let us suppose that this moveable departs from any point D without initial velo-

city, and at the end of the time $t$, let it be at M, from D and M let fall the perpendiculars DE and MP on the vertical passing through the point B, which is the lowest point on the curve, then by making EP $= z$, and denoting the velocity acquired at the point M by $v$, and the gravity by $g$, we shall have (No 159)

$$v = \sqrt{2gz},$$

provided that the gravity is the sole force which acts on the moveable. Likewise let $s$ denote the arc BM, since it decreases as the time increases, we shall have

$$v = -\frac{ds}{dt},$$

and if we make

$$\text{EB} = h, \quad \text{PB} = x = h - z,$$

there will result

$$\sqrt{2g}\,dt = -\frac{ds}{\sqrt{h-x}}, \qquad (1)$$

whatever be the nature of the given curve. This curve being by hypothesis a cycloid, we shall have (No. 73)

$$s^2 = 4ax,$$

$a$ denoting the diameter BF of its generating circle. Hence we obtain

$$\sqrt{\frac{2g}{a}}\,dt = -\frac{dx}{\sqrt{hx - x^2}},$$

and, by integrating,

$$t\sqrt{\frac{2g}{a}} = \text{arc}\left(\cos = \frac{2x-h}{h}\right)$$

a constant arbitrary is not added, because we have $t = 0$, at the origin of the motion, or when $x = h$.

If $t'$ denotes the time that the moveable takes to reach the point B which corresponds to $x = 0$, we shall have

$$t'\sqrt{\frac{2g}{a}} = \text{arc}\,(\cos = -1) = \pi,$$

and, consequently,

$$t' = \pi \sqrt{\frac{a}{2g}}$$

Hence it appears, that this time is independent of $h$ the height of the point D, (from whence the moveable commenced its motion,) above the lowest point B. So that this property that obtains only approximately, for all curves of which the height $h$ is very small, is rigorously true for the cycloid, whatever this height, within the limit $a$ or BF, may be Hence it follows, that all moveables which commence to move simultaneously from different points of the cycloid, will arrive at the lowest point in the same time

It appears, therefore, that $\pi \sqrt{\frac{2a}{g}}$ expresses the time of an entire oscillation from one side to another of the point B, now, it is evident, that this is the time in which very small oscillations of a pendulum, whose length $2a$ is the radius of curvature of the cycloid at this point, are performed (No. 72). And it thus accords with the result of No. 185, relative to the duration of very small oscillations on any curve whatever, which duration, in the case of the cycloid, is the same as those of oscillations of any amplitude whatever.

195 The time of describing the arc DB of the cycloid is likewise independent of the length of this arc, when the motion is performed in a resisting medium, provided that the resistance is proportional to the first power of the velocity

In fact, if we denote this force by $\frac{gv}{k}$ as in No. 186, the force of gravity resolved in the direction of the tangent MT is $g\frac{dx}{ds}$, because $\frac{dx}{ds}$ is the cosine of the angle TMN that this line makes with the vertical MN, therefore the force which acts at the point M, and which tends to diminish the arc MN or $s$, is the difference $g\frac{dx}{ds} - g\frac{v}{k}$, consequently, we shall have for the equation of the motion,

$$\frac{d^2s}{dt^2} = -g\left(\frac{dx}{ds} - \frac{v}{k}\right),$$

or, what is the same thing,

$$\frac{d^2s}{dt^2} + \frac{g}{k}\frac{ds}{dt} + \frac{gs}{2a} = 0,$$

because

$$v = -\frac{ds}{dt}, \quad \frac{dx}{ds} = \frac{s}{2a}$$

If at the commencement of the motion, i e , when $t = 0$, the velocity is nothing, and if then $s = a$, by determining the two constant arbitraries by means of these conditions; and making, for the sake of abridging,

$$\sqrt{1 - \frac{ga}{2k^2}} = \gamma,$$

the integral of the preceding equation will become (No. 186)

$$s = a.\left(\cos t\gamma\sqrt{\frac{g}{2a}} + \frac{\sqrt{2ga}}{2\gamma k}\sin t\gamma\sqrt{\frac{g}{2a}}\right)e^{-\frac{gt}{2k}}.$$

Therefore, if $t'$ denotes the time when the point B, at which $s = 0$, is attained, we shall have

$$\cos t'\gamma\sqrt{\frac{g}{2a}} + \frac{\sqrt{2ga}}{2\gamma k}\sin t'\gamma\sqrt{\frac{g}{2a}} = 0,$$

by means of which equation we can deduce a value of $t'$ independent of $a$, which was required to be done. If the resistance be very small, or the velocity $k$ very great, we shall have $\gamma = 1$, very nearly, and the preceding equation will give($a'$)

$$t'\sqrt{\frac{g}{2a}} = \tfrac{1}{2}\pi + \frac{\sqrt{2ga}}{2k},$$

which shews that the time $t'$ is somewhat increased by this resistance.

196. If the line BF be continued to O, so that OF may be equal to BF, this point O will be the centre of the circle which osculates the cycloid at the point B, and if two equal semi-cycloids OA and OC be described, touching the lines OB and AC, and having the diameter of their generating circle equal to OF, OA will be the evolute of AB, and OC that of BC (No 72), consequently, if a thread of a constant length, equal to OB, or $2a$ be attached to the point O, and be then made to envelope the two curves OA and OC successively, its other extremity will trace the cycloid ABC

This suggests a means of constructing a cycloidal pendulum. For this purpose, let the curves OA and OC be traced in relief, and let OB be an inextensible and perfectly flexible thread attached to the fixed point O, if we also attach a heavy body to its other extremity B, and cause this thread to deviate from the vertical position in such a manner, that it may envelope entirely, or in part, either of the curves OA or OC, and that the part which is not enveloped may be a tangent to this curve, then when the moveable is remitted to itself, the inferior extremity of the thread will describe the curve ABC, and from No. 194 it follows, that the duration of the oscillations of this pendulum, in a vacuum, will be rigorously independent of their amplitude But in practice this method is not susceptible of much precision, and besides, when the body oscillates in the air, the durations of the oscillations will not be equal when the amplitudes are considerable, for the resistance of this fluid is not then proportional to the first power of the velocity

197 Those curves are termed *tautochronous*, on which if a material point moves, it will always arrive at the lowest point at the same time, from whatever point of the curve it may have commenced to move. Thus in a vacuum, the cycloid is a tautochronous curve, and moreover, it is the only curve to which this property belongs in a vacuum, as we now proceed to demonstrate. If $t'$ denotes the time, which the moveable takes

to go, without any initial velocity, from the point D to the lowest point B, on ADB any curve whatever, the value of $t'\sqrt{2g}$ will be given by the integral of formula (1), taken from $x=h$, to $x=0$, or, which is the same thing, from $x=0$ to $x=h$, the sign of this formula being changed, consequently, we shall have

$$t'\sqrt{2g}=\int_o^h \frac{ds}{\sqrt{h-x}},$$

and in order that the curve should be tautochronous, it is necessary to determine $s$ as a function of $x$, such that this expression for $t'\sqrt{2g}$ may be independent of $h$. Now, if we suppose that this unknown function is developed according to the ascending powers of $x$, so that we may have

$$s=\text{A}x^\alpha+\text{B}x^\beta+\text{C}x^\gamma+\&\text{c},$$

A, B, C, &c, $\alpha$, $\beta$, $\gamma$, &c., being indeterminate coefficients and exponents, then as the abcissa $x$ and the arc $s$ have their origin at the same point B, we must have $x=0$ and $s=0$ at the same time, it is therefore necessary that all the exponents $\alpha$, $\beta$, $\gamma$, &c., should be positive, and that none of them should be cypher. It is likewise evident, *à priori*, that the least of them should be less than unity, for as, by hypothesis, the point of B is the lowest point of the given curve, the tangent there is horizontal or perpendicular to the axis of $x$, this requires that we should have $\frac{ds}{dx}=\infty$ when $x=0(b')$. If the differential of this series be taken, and then substituted for $ds$ in the preceding formula, there results

$$t'\sqrt{2g}=\text{A}\alpha\int_o^h\frac{x^{\alpha-1}\,dx}{\sqrt{h-x}}+\text{B}\beta\int_o^h\frac{x^{\beta-1}\,dx}{\sqrt{h-x}}+\text{C}\gamma\int_o^h\frac{x^{\gamma-1}\,dx}{\sqrt{h-x}}+\&\text{c}.$$

And if we assume $x=hx'$ and $dx=hdx'$, the limits of the integrals relative to this new variable $x'$ will be zero and unity, for example, we shall have$(c')$

$$\int_0^h \frac{x^{a-1}\,dx}{\sqrt{h-x}} = h^{a-\frac{1}{2}} \int_0^1 \frac{x'^{a-1}\,dx'}{\sqrt{1-x'}};$$

and if, for the sake of abridging, we make,

$$\int_0^1 \frac{x'^{a-1}\,dx'}{\sqrt{1-x'}} = \text{A}', \quad \int_0^1 \frac{x'^{\beta-1}\,dx'}{\sqrt{1-x'}} = \text{B}', \text{ \&c.},$$

there will result

$$t'\sqrt{2g} = a\text{AA}'h^{a-\frac{1}{2}} + \beta\text{BB}'h^{\beta-\frac{1}{2}} + \gamma\text{CC}'h^{\gamma-\frac{1}{2}} + \text{\&c.},$$

it is important to observe, that none of these integrals A′, B′, C′, &c. can become equal to cypher, for the values of the differentials, of which they constitute the sums (No 13), do not change their signs between the limits of the integrations, therefore, as these values are all positive, the values of the integrals will be so likewise. Now, it is evident, that the value of $t'$ cannot be independent of $h$, unless all the terms of the preceding series vanish, with the exception of that in which the exponent of $h$ is zero, or which corresponds to that one of the exponents $a$, $\beta$, $\gamma$, &c, which is equal to $\frac{1}{2}$. Let us suppose, that this term is the first, and that we have $a = \frac{1}{2}$. In order that the second term may disappear, it is necessary that the product $\beta$BB′ should be cypher, this implies that B should be zero, because neither $\beta$ nor B′ do vanish. In like manner it may be shewn, that the other coefficients C, D, &c., are also equal to zero, so that the equation of the tautochronous curve is reduced to the following,

$$s = \text{A}x^{\frac{1}{2}}, \text{ or } s^2 = \text{A}^2 x,$$

this indicates that it is a cycloid, whose base is horizontal, and whose summit is at the point B which the moveable always attains in the same time.

If $a$ denotes the diameter of the generating circle, we shall have $\text{A}^2 = 4a$, and, consequently,

$$t'\sqrt{2g} = \text{A}'\sqrt{a}$$

Besides, as $a = \frac{1}{2}$, $(d')$ we have,

$$A' = \int_0^1 \frac{dx'}{\sqrt{x' - x'^2}} = \pi,$$

hence we obtain

$$t' = \pi \sqrt{\frac{a}{2g}},$$

as in No. 194.

198. It is also the equation of the cycloid that we arrive at when we wish to determine the *brachystochrone* curve in a vacuum, that is to say, the curve AMB (fig 47), which a heavy material point must describe, in order to pass in the *least* time, and without any initial velocity, from a given point A to another given point B

In order to determine this curve, let $x, y, z$, be the three rectangular coordinates of the point M, where the moveable exists at the end of the time $t$, likewise let $s$ denote the arc AM which it has traversed If the axis of $x$ be vertical, and in the direction of gravity, and if $a$ denote the value of $x$ at the point A, the velocity $\frac{ds}{dt}$ acquired at the point M, will be equal to the velocity acquired in falling through the height $x - a$, therefore, if $g$ represents the gravity, we shall have

$$\frac{ds}{dt} = \sqrt{2g(x-a)},$$

and if in order to abridge, we make

$$\sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}} = u,$$

in which case $ds = udx$, there will result

$$\sqrt{2g}\, dt = \frac{udx}{\sqrt{x-a}}.$$

Therefore, if $\beta$ denotes the value of $x$ at the point B, and $t'$ the time which the moveable takes to pass from the point A to the point B, we shall have

$$t'\sqrt{2g}=\int_a^\beta \frac{udx}{\sqrt{x-a}}.$$

Consequently, the question is reduced to the determination of the curve, for which this integral is a minimum, but, for greater generality, let the integral

$$\mathrm{U}=\int_a^\beta \mathrm{X}u dx$$

be considered, in which X is a given function of $x$, for this will be useful in the sequel, in the resolution of another problem of the same kind, in the one we are now concerned with, we shall assume $(x-a)^{-\frac{1}{2}}$ for X.

199 Let $\iota$ denote a constant quantity infinitely small, and let $\delta y$ and $\delta z$ denote two arbitrary functions of $x$, subject solely to the condition of vanishing when $x=a$ and $x=\beta$, and of not becoming infinite for the intermediate values of $x$. Let U' and $u'$ be what U and $u$ become when $y+\iota\delta y$ and $z+\iota\delta z$ are substituted in place of $y$ and $z$, so that we shall have

$$\mathrm{U}'=\int_a^\beta \mathrm{X}u'dx,$$

an integral which will correspond to another curve AM'B, passing, in like manner as the required curve AMB, through the points A and B, and deviating almost insensibly from this last. We shall have by this means

$$\mathrm{U}'-\mathrm{U}=\int_a^\beta \mathrm{X}(u'-u)\,dx,$$

and, from the property which the curve AMB is supposed to possess, this difference U' − U must be positive, whatever may be the values of $\delta y$ and of $\delta z$, and whatever may be the sign of $\iota$ Now, if the difference $u'-u$ be developed according to the powers of $\iota$, and if $\iota\delta u$ be the first term of its development, the first term of that of U' − U will be $\iota\int_a^\beta \mathrm{X}\delta u\,dx$, hence it follows, that we must have

$$\int_a^\beta \text{x}\delta u dx = 0, \qquad \text{(a)}$$

otherwise the difference $\text{U}' - \text{U}$ would change its sign at the same time as $i$

This condition must be satisfied, both when U is a *maximum* and a *minimum* When it is fulfilled, the difference $\text{U}' - \text{U}$ will be, in general, an infinitesimal of the second order, and it will have the same sign as the coefficient of $i^2$ in its development, consequently, it will be a *maximum* or a *minimum* according as this coefficient is negative or positive. But, it is evident from the nature of the question, that the time $t'$ is not susceptible of having a *maximum* value, consequently this coefficient will be positive in the problem of the *brachystochrone*, and it will be only necessary that the condition expressed by equation (a) be satisfied

The quantity $i\delta u$ is in fact the differential of $u$, taken with respect to $y$ and to $z$, and in which the increments of these quantities are represented by $i\delta y$ and $i\delta z$ If the factor $i$, which is common to $i\delta u$ and its value, be suppressed, we shall have

$$\delta u = \frac{1}{u}\frac{dy}{dx}\frac{d\delta y}{dx} + \frac{1}{u}\frac{dz}{dx}\frac{d\delta z}{dx};$$

so that equation (a) will become

$$\int_a^\beta \frac{\text{x}}{u}\frac{dy}{dx}\frac{d\delta y}{dx}dx + \int_a^\beta \frac{\text{x}}{u}\frac{dz}{dx}\frac{d\delta z}{dx}.dx = 0.$$

If we integrate this equation by parts, we shall obtain, because by hypothesis $\delta z$ and $\delta y$ are cypher at the two limits $x = a$, $x = \beta(e')$

$$\int_a^\beta \frac{\text{x}}{u}\frac{dy}{dx}\frac{d\delta y}{dx}\ dx = -\int_a^\beta \frac{d\left(\frac{\text{x}}{u}\frac{dy}{dx}\right)}{dx}\delta y\, dx,$$

$$\int_a^\beta \frac{\text{x}}{u}\frac{dz}{dx}\ \frac{d\delta z}{dx}dx = -\int_a^\beta \frac{d\left(\frac{\text{x}}{u}\frac{dz}{dx}\right)}{dx}\delta z\, dx,$$

hence the preceding equation will be changed into

$$\int_{a}^{\beta}\left[\frac{d\left(\frac{\lambda}{u}\frac{dy}{dx}\right)}{dx}\delta y+\frac{d\left(\frac{\lambda}{u}\frac{dz}{dx}\right)}{dx}\delta z\right]dx=0$$

But, as $\delta z$ and $\delta y$ are arbitrary functions of $x$, this integral cannot be equal to cypher, unless that the quantity contained under $\int$ the sign of the integration, be of itself equal to cypher, we shall consequently have

$$\frac{d\left(\frac{\lambda}{u}\frac{dy}{dx}\right)}{dx}\delta y+\frac{d\left(\frac{\lambda}{u}\frac{dz}{dx}\right)}{dz}\delta x=0, \qquad \text{(b)}$$

200 If the sought curve AMB, and any other curve AM′B, were required to be traced on a given surface, the equation of which was $L=0$, then the values of $y$ and $z$ considered as functions of $x$, which it is required to determine, and these values respectively increased by $i\delta y$ and $i\delta z$, must successively satisfy this equation, hence we infer

$$\frac{dL}{dy}\delta y+\frac{dL}{dz}\delta z=0,$$

by means of which, one of the two quantities $\delta y$ and $\delta z$ may be eliminated from equation (b), and as the other will disappear at the same time, we shall have

$$\frac{dL}{dz}\cdot\frac{d\left(\frac{\lambda}{u}\frac{dy}{dx}\right)}{dx}-\frac{dL}{dy}\cdot\frac{d\left(\frac{\lambda}{u}\frac{dz}{dx}\right)}{dx}=0$$

In this case the two equations of the required curve will be $L=0$ and this last equation, by means of which we can determine the curve of quickest descent on a given surface If, on the contrary, the question was to determine the *minimum* of U among all the curves which are terminated at the points A and B, without restricting it to exist on any particular surface, then the quantities $\delta y$ and $\delta z$ will be arbitrary and independent of each other. Their coefficients must consequently be separately equal to cypher in the preceding equation (b), and thus it will be decomposed into two others, namely,

$$\frac{d.\left(\frac{\text{x}}{u}\frac{dy}{dx}\right)}{dx}=0, \quad \frac{d\left(\frac{\text{x}}{u}\frac{dz}{dx}\right)}{dx}=0,$$

we shall restrict ourselves to the consideration of this last case.

If we integrate and denote by $a$ and $a'$ the two constant arbitraries, which are introduced by the integration, we obtain

$$\frac{\text{x}}{u}\frac{dy}{dx}=a, \quad \frac{\text{x}}{u}\frac{dz}{dx}=a', \tag{c}$$

and, consequently,

$$a'\frac{dy}{dx}-a\frac{dz}{dx}=0,$$

integrating again, and denoting by $\gamma$ a third arbitrary quantity, there results

$$a'y-az=\gamma,$$

which shews that the required curve is one of single curvature, and comprised in a plane perpendicular to that of the axes of $y$ and $z$. For greater simplicity, let the plane of this curve be assumed to be that of the axes of $x$ and $y$, we shall have then

$$u=\sqrt{1+\frac{dy^2}{dx^2}},$$

and it will be only necessary to consider the first equation (c), which will become($g'$)

$$\text{x}dy=a\sqrt{dx^2+dy^2},$$

hence we obtain

$$dy=\frac{adx}{\sqrt{\text{x}^2-a^2}}. \tag{d}$$

It only now remains for us to integrate this formula, which will depend on the form of the function x, and then to determine $a$ and the new constant arbitrary introduced by this integration, from the condition that the required curve passes through the given points A and B.

201. Before we proceed farther, let us suppose that $c$ is

any constant whatever, and that $x+c$ is substituted in place of $x$ in the preceding formulæ. The integral $u$ will become

$$u=\int_a^\beta x\sqrt{1+\frac{dy^2}{dx^2}}\,dx+c.\int_a^\beta\sqrt{1+\frac{dy^2}{dx^2}}.dx,$$

and the value of $y$, which renders this function a *minimum*, will be furnished by the equation

$$dy=\frac{adx}{\sqrt{(x+c)^2-a^2}}. \qquad \text{(e)}$$

Now, as this sum of the integrals that $u$ represents, is a *minimum*, for *all* the curves which are terminated at the points A and B, it is evident that the first integral

$$\int_a^\beta x\sqrt{1+\frac{dy^2}{dx^2}}.dx$$

will be a *minimum*, if we only consider, among all these curves, those which answer to the same value of the second integral.

This simple remark enables us to extend immediately to problems of relative *maxima* and *minima*, the solutions of problems of absolute *maxima* and *minima*; in the sequel, another instance of the application of this principle will occur. As in the present case, the second integral contained in $u$ is the length of the required curve, it follows that by means of equation (e) we can determine, among all isoperimetrical curves, that for which the first integral is a *maximum* or a *minimum*. If $l$ denotes the given length, common to all these curves, we shall have

$$\int_a^\beta\sqrt{1+\frac{dy^2}{dx^2}}.dx=l,$$

which condition will be satisfied by means of the indeterminate constant $c$, that has been introduced into the formula (e).

202. Let us now apply formula (d) to the curve of quickest descent.

Since

$$X = \frac{1}{\sqrt{x - \alpha}},$$

we shall have

$$dy = \frac{(x - \alpha)\,dx}{\sqrt{a(x - \alpha) - (x - \alpha)^2}},$$

$\frac{1}{\sqrt{a}}$ being substituted in place of $a$.($g'$) Now, this differential equation is that of a cycloid (No 72) passing through A, (the point from which the moveable commences its motion,) the base being horizontal and the diameter of the generating curve being equal to $a$, which establishes what was proposed to be proved in No 198.

By integrating, we obtain($h'$)

$$y - \alpha' = \tfrac{1}{2} a \operatorname{arc}\left(\cos = \frac{a - 2x + 2\alpha}{a}\right) - \sqrt{a(x - \alpha) - (x - \alpha)^2},$$

$\alpha'$ being what $y$ becomes when $x = \alpha$ If $\beta'$ denotes the value of $y$ when $x = \beta$, we shall have

$$\beta' - \alpha' = \tfrac{1}{2} a \operatorname{arc}\left(\cos = \frac{a - 2\beta + 2\alpha}{a}\right) - \sqrt{a(\beta - \alpha) - (\beta - \alpha)^2}$$

As the coordinates $\alpha$ and $\alpha'$, $\beta$ and $\beta'$, of the points A and B, are given, the constant $a$ can be determined by this last equation, and then the preceding value of $y$ will not contain any unknown quantity

By means of the value of $dy$, we obtain($i'$)

$$u = \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{\sqrt{a}}{\sqrt{a - x + \alpha}},$$

therefore we shall have (No. 198)

$$t'\sqrt{2g} = \int_\alpha^\beta \frac{\sqrt{a}\,dx}{\sqrt{a(x - \alpha) - (x - \alpha)^2}},$$

and, consequently, ($k'$)

$$t' = \sqrt{\frac{a}{2g}}.\text{arc}\left(\cos = \frac{a - 2\beta + 2\alpha}{a}\right),$$

expresses the shortest time in which the moveable can pass from the point A to the point B

If these two points exist in the same vertical, we shall have $\beta' = \alpha'$; a condition which can be satisfied by assuming $a = \infty$, for we have ($l'$)

$$\text{arc}\left(\cos = \frac{a - 2\beta + 2\alpha}{a}\right) = \text{arc}\left(\sin = 2\frac{\sqrt{a(\beta-\alpha)-(\beta-\alpha)^2}}{a}\right),$$

and, when $a = \infty$, we can substitute this arc for its sine, which reduces the preceding value of $\beta' - \alpha'$ to cypher, and as at the same time, the value of $y$ becomes equal to $\alpha'$, it follows that the moveable will not deviate from the vertical direction The value of $t'$ will likewise be in this case

$$t' = \sqrt{\frac{a}{2g}}\ \frac{2\sqrt{a(\beta - \alpha) - (\beta - \alpha)^2}}{a} = \frac{\sqrt{2(\beta - \alpha)}}{\sqrt{g}},$$

which is in fact the time that the moveable takes to traverse the height $\beta - \alpha$, in descending from the point A situated above the point B

As the determination of the line of quickest descent is a problem of pure curiosity, we have restricted ourselves to the consideration of the simplest case, namely, that in which the motion is performed in a vacuum, the extreme points being given If these points A and B are not fixed and given, but only subjected to exist on the given curves DAE and FBG, or on surfaces that are given, the brachystochrone, when the motion is performed in a vacuum, will still be a cycloid, and, by means of the rules of the calculus of variations, we can determine, in all these cases, the coordinates of these two points In a resisting medium, this line will be a different curve, the differential equation of which may be obtained by the rules of

this calculus, and as this equation depends on the law of the resistance with respect to the velocity of the moveable, it follows that the curve must be different for each law.

### III. *Motion on a given Surface*

203. The simple pendulum treated of in No. 179, will furnish us with an example of the motion of a material point on a given surface, provided we suppose, that after it is drawn out of the vertical CB, and transferred to CA (fig. 45), there is impressed on it a velocity, the direction of which does not exist in the vertical plane ACD. The pendulum will then deviate from this plane, and the material point by which it is terminated will move on the surface of a sphere described from the point C as a centre, and with a radius equal to $a$ the length of this pendulum. The percussion that the moveable will experience at the commencement of the motion, may be resolved into two forces, the one acting in the direction of AC, or of its production, and which will be destroyed by the resistance of the fixed point C, the other will be perpendicular to AC, and will produce the initial velocity of the pendulum, which we shall denote by $k$. As the motion is supposed to be performed in a vacuum, the gravity is the only given accelerating force that acts on the moveable.

This being established, let CM be the position of the pendulum at the end of the time $t$, and let $x, y, z$, denote the rectangular coordinates of the point M. Likewise, let $m$ denote the mass of the moveable, and $m$N the unknown tension of the thread CM, acting in the direction of its production. If the origin of the coordinates $x, y, z$, be at C, the components of the accelerating force N in the direction of their productions, will be

$$\frac{x}{a}\text{N}, \quad \frac{y}{a}\text{N}, \quad \frac{z}{a}\text{N}$$

Now, if to the moveable a force equal and contrary to N,

be applied, we may abstract altogether from the consideration of the thread CM, and consider the moveable as entirely free; therefore, if we suppose that the axis of the positive $zs$ is vertical, and its direction to be that of gravity, the three equations of the motion will be

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{x}{a}\text{N} = 0, \\ \frac{d^2y}{dt^2} + \frac{y}{a}\text{N} = 0, \\ \frac{d^2z}{dt^2} + \frac{z}{a} - g = 0 \end{aligned}\right\} \quad (1)$$

which accord with equations (3) of No. 151 By eliminating the unknown N, they will be reduced to two, and these combined with the equation of the sphere, namely,

$$x^2 + y^2 + z^2 = a^2,$$

will furnish us with three equations, by means of which $x, y, z$, may be determined in functions of $t$.

204 Multiplying equations (1) by $x, y, z$, respectively, and then adding them together, there results

$$\frac{xd^2x + yd^2y + zd^2z}{dt^2} + \text{N}a - gz = 0$$

The first differential of the equation of the sphere will give us

$$xdx + ydy + zdz = 0, \quad (2)$$

and the second

$$xd^2x + yd^2y + zd^2z + dx^2 + dy^2 + dz^2 = 0$$

Hence, if $v$ denotes the velocity of the moveable, at the end of the time $t$, so that we may have

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = v^2,$$

there will result($m'$)

$$\text{N} = \frac{v^2}{a} + \frac{gz}{a},$$

and, in fact, the tension $m\text{N}$ must be the sum of the centrifugal force $\frac{mv^2}{a}$ and of $\frac{mgz}{a}$ the component of the weight which acts in the direction of the radius CM

Likewise, if equations (1) be multiplied by $dx$, $dy$, $dz$, respectively, and then added together, the unknown quantity N will disappear, in virtue of equation (2), and we shall have

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = g\,dz.$$

If this be integrated, we obtain

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = 2gz + b, \tag{3}$$

$b$ denoting a constant arbitrary As the initial value of the first member of this equation is $k^2$, if $\gamma$ denotes that of $z$, we shall have

$$k^2 - 2g\gamma = b,$$

and, at any instant whatever,

$$v^2 = k^2 + 2g\,(z - \gamma),$$

as we know already

Finally, if the first equation (1) be multiplied by $y$, and then subtracted from the second multiplied by $x$, we obtain

$$\frac{xd^2y}{dt^2} - \frac{yd^2x}{dt^2} = 0,$$

hence, by integrating, there results

$$xdy - ydx = c\,dt, \tag{4}$$

$c$ denoting a constant arbitrary.

In this manner, the solution of the problem will only depend on the three differential equations (2), (3), (4), which are all of the first order, and with respect to the first of these, we know already, that its integral is the equation of the sphere

We can separate the variables and reduce the question to quadratures by means of the following calculus

205 From equation (2) we obtain

$$xdx + ydy = -zdz,$$

if the two members of this equation, and also those of equation (4), be raised to the square, we obtain, by adding the resulting equations together,

$$(x^2 + y^2)(dx^2 + dy^2) = z^2dz^2 + c^2dt^2$$

If $a^2 - z^2$ be substituted instead of $x^2 + y^2$, and if $dx^2 + dy^2$ be eliminated by means of equation (3), we shall have

$$(a^2 - z^2)\left[(2gz + b)\,dt^2 - dz^2\right] = z^2dz^2 + c^2dt^2,$$

hence we obtain

$$dt = \frac{adz}{\sqrt{(a^2 - z^2)(2gz + b) - c^2}} \qquad (5)$$

Let $r$ denote the radius vector of the projection of the moveable on the horizontal plane of the axes of $x$ and $y$, and $\psi$ the angle that this radius makes with the axis of $x$, we shall have

$$x = r\cos\psi, \quad y = r\sin\psi, \quad xdy - ydx = r^2d\psi,$$

and, because $r^2 = a^2 - z^2$, equation (4) will become

$$(a^2 - z^2)\,d\psi = cdt,$$

by substituting for $dt$ its preceding value, we deduce

$$d\psi = \frac{cadz}{(a^2 - z^2)\sqrt{(a^2 - z^2)(2gz + b) - c^2}} \qquad (6)$$

The integrals of these expressions of $dt$ and $d\psi$ will make known the values of $t$ and $\psi$ in functions of $z$, they may be always reduced to elliptic functions, and cannot be obtained under a finite form, unless the quantity of the third degree with respect to $z$, contained under the radical, has a double factor. The value of $\psi$, and the equation of the sphere, will determine the *trajectory* of the moveable, and the value of $t$ as a function of $z$, or of $z$ as a function of $t$, will then make known the *position* of the moveable on this curve, at each instant.

The constant $b$ may be known from the given quantities $k$ and $\gamma$. The constant arbitraries which are introduced by the integrations of $dt$ and $d\psi$, are determined by the conditions $t = 0$ and $\psi = 0$, when $z = \gamma$, the second of these conditions, supposes that the axis of $x$ exists in the vertical plane ACB, from which the pendulum commences its motion. There remains, therefore, only the constant quantity $c$ to determine. Now, as the direction of $u$, the velocity of the moveable, is perpendicular to CM, the radius of the sphere on which the body moves, if it be resolved into two, one perpendicular to the vertical plane MCB, and the other comprised in this plane, the first component will be the velocity of the horizontal projection of the moveable, perpendicular to its radius vector $r$, consequently, if it be represented by $u$, we shall have (No. 156)

$$u = \frac{r\,d\psi}{dt},$$

we have also in virtue of equation (4)($n'$)

$$u = \frac{c}{r} = \frac{c}{\sqrt{a^2 - z^2}},$$

therefore, if $\varepsilon$ denotes the angle that the initial velocity $k$ makes with the perpendicular to the plane ACB, so that we may have $u = k \cos \varepsilon$ at the commencement of the motion, there will result

$$c = k\sqrt{a^2 - \gamma^2} \cos \varepsilon$$

When the velocity $k$ is cypher, we shall have $c = 0$, $b = -2g\gamma$, and, consequently,

$$dt = \frac{a\,dz}{\sqrt{2g\sqrt{(a^2 - z^2)}(z - \gamma)}},$$

this expression coincides with the value of $dt$ given in No. 185, because $a - z$ and $a - \gamma$ are what have been denominated $a\iota$ and $a\beta$ in that value.

206. Let us consider the case in which the pendulum has

received a very small initial velocity, and, therefore, deviates very little from the vertical CB If the direction of this velocity be horizontal, it will be perpendicular to the plane ACB, and, consequently, we shall have $\varepsilon = 0$ Let $\beta$ denote a very small fraction, and let

$$k = \beta \sqrt{ga}$$

Likewise let $a$ and $\theta$ represent the angles ACB and MCB, if their fourth powers be neglected, we shall have

$$\gamma = a - \tfrac{1}{2} a a^2, \quad z = a - \tfrac{1}{2} a \theta^2,$$
$$b = -2ga + ga(a^2 + \beta^2), \quad c^2 = ga^3 a^2 \beta^2,$$

and formulæ (5) and (6) will become(o′)

$$\left.\begin{aligned} dt &= -\sqrt{\frac{a}{g}} \frac{\theta d\theta}{\sqrt{(a^2-\theta^2)(\theta^2-\beta^2)}}, \\ d\psi &= -\frac{a\beta d\theta}{\theta\sqrt{(a^2-\theta^2)(\theta^2-\beta^2)}} \end{aligned}\right\} \quad (a)$$

The angle $\psi$ will make known the position of the vertical plane MCB, in which the pendulum exists at each instant, it evidently may increase indefinitely The angle $\theta$ will determine also at each instant, the position of the pendulum in this variable plane, it is considered as a positive quantity, and the positions of the pendulum, which are equidistant from the vertical CB on opposite sides, correspond to the same angle $\theta$, and to values of $\psi$ which differ from each other by 180°.

From the value of $\frac{d\theta}{dt}$, deduced from the first equation $(a)$, it appears that the angle $\theta$ will be always comprised between the limits $a$ and $\beta$. If $\beta = a$, we shall have $\theta$ always $= a$, if equations (a) be divided the one by the other, there results, in all cases,$(p')$

$$d\psi = \sqrt{\frac{g}{a}} \frac{a\beta}{\theta^2} dt, \quad (b)$$

therefore, when $\theta = a = \beta$, we shall have

$$\psi = t\sqrt{\frac{g}{a}},$$

consequently, the pendulum will then describe uniformly the surface of a right cone with a circular base, and the time of an entire revolution will be $2\pi\sqrt{\frac{a}{g}}$, $(q')$ that is to say, the same as the time of a double oscillation in the vertical plane ACB. Thus, two pendulums having the same length $a$, and which depart together from the same line CA, the one without any initial velocity, the other with a velocity perpendicular to the plane ACB and equal to $a\sqrt{ga}$, will return after the lapse of the same time, to this line CA.

207 The value of $dt$ may be expressed under the following form,

$$dt = -2\sqrt{\frac{a}{g}}\frac{\theta d\theta}{\sqrt{(a^2-\beta^2)^2-(2\theta^2-a^2-\beta^2)^2}},$$

and if, for greater simplicity, we make

$$2\theta^2 - a^2 - \beta^2 = (a^2-\beta^2)x, \quad 4\theta d\theta = (a^2-\beta^2)dx,$$

the radical becomes $\pm(a^2-\beta^2)\sqrt{1-x^2}$, and there results

$$dt = \mp\tfrac{1}{2}\sqrt{\frac{a}{g}}\frac{dx}{\sqrt{1-x^2}}.$$

Since $\theta = a$, and $x = 1$, when $t = 0$, we obtain from this

$$t = \pm\tfrac{1}{2}\sqrt{\frac{a}{g}}\operatorname{arc}(\cos = x),$$

and conversely,

$$x = \cos 2t\sqrt{\frac{g}{a}}.$$

Therefore, at any instant whatever, we shall have

$$\theta^2 = \tfrac{1}{2}(a^2+\beta^2) + \tfrac{1}{2}(a^2-\beta^2)\cos 2t\sqrt{\frac{g}{a}},$$

which shews that the oscillations of the pendulum, of which the two extremities correspond to $\theta = a$ and $\theta = \beta$, in the variable plane MCB, will be isochronous, and their duration will be $\frac{1}{2}\pi\sqrt{\frac{a}{g}}$, which is equal to half the time of an oscillation in the fixed($r'$) plane ACB. Substituting this value of $\theta^2$ in equation (b), and observing that

$$\cos 2t\sqrt{\frac{g}{a}} = \cos^2 t\sqrt{\frac{g}{a}} - \sin^2 t\sqrt{\frac{g}{a}},$$

there results

$$d\psi = \sqrt{\frac{g}{a}} \frac{a\beta dt}{a^2\cos^2 t\sqrt{\frac{g}{a}} + \beta^2\sin^2 t\sqrt{\frac{g}{a}}},$$

and, because $\psi = 0$ when $t = 0$, we infer($t'$)

$$a \text{ tang} . \psi = \beta \text{ tang}\, t\sqrt{\frac{g}{a}}$$

This being the case, the motion of the plane MCB will be no longer uniform as when $a = \beta$, but it is evident that this plane will perform successively the four quarters of its entire revolution, in times respectively equal to each other and to $\frac{1}{2}\pi\sqrt{\frac{a}{g}}$, which is half the time in which an oscillation is performed in this variable plane

From this last equation we deduce

$$\cos^2\psi = \frac{a^2\cos^2 t\sqrt{\frac{g}{a}}}{a^2\cos^2 t\sqrt{\frac{g}{a}} + \beta^2\sin^2 t\sqrt{\frac{g}{a}}},$$

$$\sin^2\psi = \frac{\beta^2\sin^2 t\sqrt{\frac{g}{a}}}{a^2\cos^2 t\sqrt{\frac{g}{a}} + \beta^2\sin^2 t\sqrt{\frac{g}{a}}},$$

we have also

$$x^2 = (a^2 - z^2)\cos^2\psi = a^2\theta^2\cos^2\psi,$$
$$y^2 = (a^2 - z^2)\sin^2\psi = a^2\theta^2\sin^2\psi,$$

by substituting for $z$ its approximate value; therefore, since

$$\theta^2 = \alpha^2\cos^2 t\sqrt{\frac{g}{a}} + \beta^2\sin^2 t\sqrt{\frac{g}{a}},$$

we obtain

$$x^2 = a^2\alpha^2\cos^2\psi, \quad y^2 = a^2\beta^2\sin^2\psi,$$

and, consequently, $(v)$

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = a^2,$$

hence it appears, that the trajectory of the projection of the moveable on the horizontal plane passing through the point C, is an ellipse which has its centre in this point, and one of its axes in the plane ACB, from which the pendulum commences its motion with a velocity, the direction of which is perpendicular to this plane

# CHAPTER VI

## EXAMPLES OF THE MOTION OF A BODY ALTOGETHER FREE.

### 1. *Motion of Projectiles*

208 THE theory of projectiles, which are discharged by *artillery* with great velocity, and are at the same time subjected to the action of gravity, and also to the resistance of the air, will constitute the subject matter of this section.

In the first place, let us abstract from the consideration of this resistance, and let us consider the motion of a heavy material point which moves from the point O (fig 48), with a velocity $a$ in the direction of the line OA It is evident, that the moveable will not deviate from the vertical plane passing through($a$) this line Let OMD be its trajectory in this plane, to which OA will be a tangent If in this same plane, two axes be drawn, the one horizontal and the other vertical and directed from the horizon, these axes may be taken for those of the coordinates, and at the end of any time, such as $t$, let M be the position of the moveable, $x$ its abscissa OP, and $y$ its ordinate PM, then if $\alpha$ denotes the acute angle AO$x$, which the direction of the initial velocity $a$ makes with the axis O$x$, its components will be, $a\cos\alpha$ in the direction of this axis, and $a\sin\alpha$ in the direction of the axis O$y$, the angle $\alpha$ will be negative, if the right line OA falls below O$x$.

From what has been established in No 148, it is evident that the motions of the projections of the moveable on the two axes O$x$ and O$y$ will be independent of each other, the motion of its horizontal projection will be therefore uniform, and

equal to the velocity $a \cos \alpha$, and as that of its vertical projection arises from the initial velocity $a \sin \alpha$, and from $g$ the constant force of gravity acting in an opposite direction to this velocity, we shall consequently have

$$x = ta . \cos \alpha, \quad y = ta \sin \alpha - \tfrac{1}{2} gt^2,$$

eliminating $t$ between these equations, and substituting $\sqrt{2gh}$ for $a$, ($h$ being the height through which the moveable must fall to acquire the velocity $a$) there will result

$$y = x \operatorname{tang} \alpha - \frac{x^2}{4h \cos^2 \alpha},$$

for the equation of the trajectory

Therefore, this curve is a parabola, of which the axis is vertical, its summit determined by the equation $\frac{dy}{dx} = 0$, has for coordinates($b$)

$$x = 2h \cos \alpha \sin \alpha, \quad y = h \sin^2 \alpha,$$

and it meets the axis O$x$ in a second point B, which is obtained by making $y = 0$ in the equation of the curve, hence, if $b$ denotes the distance OB, we obtain

$$b = 4h \sin \alpha \cos \alpha = 2h \sin 2\alpha.$$

This distance $b$ is called the *amplitude of the projection* If the motion is performed in a vacuum, it is a *maximum*, when $\alpha = 45°$, in this case it is equal to $2h$, or to twice the height through which the body must fall to acquire the initial velocity

Naming $v$ the velocity of the moveable at the end of the time $t$, we shall obtain, by substituting the differentials of the preceding values of $x$ and $y$ in the equation($c$)

$$v^2 = \frac{dx^2 + dy^2}{dt^2},$$

$$v^2 = a^2 - 2agt \sin \alpha + g^2 t^2.$$

As $t$, the time which the moveable employs to reach the point B, while describing the curve OCB, is the same as it takes to describe the line OB with the velocity $a \cos \alpha$, we must consequently have

$$t = \frac{b}{a \cos \alpha} = \frac{4h \sin \alpha}{a},$$

and since $h = \frac{a^2}{2g}$, there results $gt = 2a \sin \alpha$, hence we have $v^2 = a^2$, therefore the velocity of the moveable in this point B, is the the same as at the point O, its direction is along the tangent BE, and EBX, the angle which the direction of the moveable makes in its descent with OB, is the same as AOX, the angle of projection If the moveable, instead of being a material point, be a solid body of any form and dimensions whatever, it will be shewn in a subsequent chapter, that these equations of parabolic motion ought to be referred to its centre of gravity

209 The velocity $a$ being given, if it were required to determine the angle $\alpha$ such, that the moveable may reach a determinate point, of which the coordinates are $x = \beta$, $y = \gamma$, these values should be substituted in the equation of the trajectory, and we shall have by this means,

$$\gamma = \beta \operatorname{tang} \alpha - \frac{\beta^2}{4h \cos^2 \alpha},$$

to determine $\alpha$. By making,

$$\operatorname{tang} \alpha = z, \quad \cos^2 \alpha = \frac{1}{1+z^2},$$

this equation becomes

$$4h\gamma + \beta^2 - 4h\beta z + \beta^2 z^2 = 0,$$

hence we obtain

$$z = \frac{2h}{\beta} \pm \frac{1}{\beta} \sqrt{4h^2 - 4h\gamma - \beta^2}.$$

As there are two values of $z$ or tang $\alpha$ furnished by this equation, it follows that when $4h^2$ is greater than $4h\gamma + \beta^2$,

the proposed object will be struck, if the elevation of the projectile be either of the two angles which are deduced from this equation, if the radical part of this equation vanishes, the two values of $z$ are equal, and these two directions coincide; and when the radical part is impossible, in which case $4h^2$ is less than $4h\gamma+\beta^2$, the mark cannot be reached under any direction whatever

Hence, if in the vertical plane which passes through the initial direction of the moveable($d$), a parabola be described of which the equation is

$$4h\gamma+\beta^2=4h^2,$$

this curve will divide the plane into two parts such, that no mark without this curve and in the vertical plane can be struck by a body projected with the given velocity, while for all points which lie within this curve, there are two different directions, along either of which, if the body be projected, it will reach the mark, and if the object to be struck exist on this parabola itself, there is only one elevation at which the projectile can be discharged to reach the mark.

210 It appears from what has been stated in the two preceding numbers, that the theory of the motion of projectiles would be extremely simple, if the resistance which the air opposes to their motion could be neglected, but when the velocity is very great, as in the cases which we are particularly concerned with, this force is too considerable not to be taken into account, in fact, as we shall presently see, it changes altogether both the form of the trajectory, and the law of its motion on this curve.

It will be proved in a subsequent chapter, that whatever be the form and dimensions of the projectile, its centre of gravity will have the same motion, as a heavy material point whose mass is equal to that of the moveable, the direction and magnitude of the initial velocity being the same in both, and to which besides are applied, parallel to the directions in which

they act, the forces, which, arising from the friction and the resistance of the air, act on the surface of this solid body. It will likewise be shewn, that the motive force which results from these resistances, when transferred to the centre of gravity, may sometimes cause this point to deviate from the vertical plane passing through the direction of the initial velocity; but here we suppose that this is not the case, and that the motive force in question acts always in the direction of a tangent to the trajectory described by the centre of gravity.

This being established, in order to obtain the equations of its motion, let the preceding notations be retained, and let them be supposed to refer to figure 49, in which the trajectory OMD is no longer to be considered a parabola. Let $s$ be the arc OM described by the moveable at the end of the time $t$, and let R denote the motive force arising from the resistance of the air, and which acts in the direction of the part MT of the tangent at M. The cosines of the angles which this line MT makes with the axis drawn through the point M in the directions of the positive $x$s and $y$s, will be $-\frac{dx}{ds}$, $-\frac{dy}{ds}$; therefore, if $m$ denotes the mass of the projectile and $g$ the gravity,

$$\frac{d^2x}{dt^2} = -\frac{R}{m}\frac{dx}{ds}, \quad \frac{d^2y}{dt^2} = -g - \frac{R}{m}\frac{dy}{ds},$$

will be the equations of the motion of its centre of gravity.

Let this projectile be an homogeneous sphere, or one composed of concentrical strata each of which will be homogeneous, then if its mean density be denoted by D, and its radius by $r$, we shall have

$$m = \frac{4\pi D r^3}{3}.$$

Likewise if we suppose, agreeably to the hypotheses which are generally admitted, that the force R is proportional to the square of the velocity of the centre of gravity, to the surface of the projectile, and to the density of the air, there results($e$)

$$\frac{\text{R}}{m} = \frac{n\rho}{\text{D}r} \times \frac{ds^2}{dt^2},$$

$\rho$ being this density, and $n$ a numerical factor to be determined by experiment. This expression satisfies the condition of the homogeneity of the quantities, for $\frac{\text{R}}{m}$ and the ratio of $\frac{ds^2}{dt^2}$ to $r$, are two quantities of the same nature as $g$ the gravity, and the factors $n$ and $\frac{\rho}{\text{D}}$ are abstract numbers. For greater convenience, let

$$\frac{n\rho}{\text{D}r} = c,$$

so that $\frac{1}{c}$ may be a line of a given length, which, when we do not take into account the change of density of the mass of air, traversed by the projectile, may be considered as constant.

211. By substituting in place of $\frac{\text{R}}{m}$, its value $c\frac{ds^2}{dt^2}$, the two equations of motion become(*f*)

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + c.\frac{ds}{dt}.\frac{dx}{dt} = 0, \\ &\frac{d^2y}{dt^2} + c.\frac{ds}{dt}.\frac{dy}{dt} + g = 0 \end{aligned}\right\} \quad (1)$$

The integral of the first is

$$\frac{dx}{dt} = a\cos\alpha e^{-cs},$$

for $\frac{dx}{dt} = a\cos\alpha$, at the point o where $s = 0$, and $e$ is the base of the Naperian system of logarithms. As the form of the second equation differs from the first only in its last term, we may assume, in order to integrate(*g*),

$$\frac{dy}{dt} = p\frac{dx}{dt},$$

$p$ being a new unknown. If we substitute this value of

$\frac{dy}{dt}$ in the second equation (1), we obtain, by taking into account the first ($h$),

$$\frac{dx}{dt}\,\frac{dp}{dt} = -\,g.$$

If the members of this equation be respectively divided by the square of $\frac{dx}{dt}$, and its value given above, there will result

$$\frac{dp}{dt} \div \frac{dx}{dt} = -\,\frac{g}{a^2 \cos^2 \alpha}\, e^{2cs}$$

If $y$ and $p$ be regarded as functions of $x$, we shall have

$$p = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx}, \quad \frac{dp}{dt} \div \frac{dx}{dt} = \frac{dp}{dx},$$

therefore, if we substitute $2gh$ for $a^2$, the preceding equation will become

$$\frac{dp}{dx} = -\,\frac{1}{2h \cos^2 \alpha}\, e^{2cs}, \tag{2}$$

and this will be the differential equation of the trajectory

Since

$$\sqrt{1 + p^2}\;\; dx = ds,$$

if each member of the preceding equation be multiplied by these quantities respectively, we shall obtain

$$\sqrt{1+p^2}\;\; dp = -\,\frac{ds}{2h \cos^2 \alpha}\, e^{2cs},$$

hence, if we integrate and denote the constant arbitrary by $\gamma$, there results

$$p\,\sqrt{1+p^2} + \log\left(p + \sqrt{1+p^2}\right) = \gamma - \frac{1}{2ch \cos^2 \alpha}\, e^{2cs} \tag{3}$$

In order to determine $\gamma$, let $s = 0$ and $p = \text{tang}\, \alpha$ at the same time, this gives

$$\gamma = \frac{1}{2ch\cos^2 a} + \text{tang}\, a\sqrt{1+\text{tang}^2 a} + \log(\text{tang}\, a + \sqrt{1+\text{tang}^2 a})$$

however it is better to retain $\gamma$ in place of this value, as it is a more concise expression.

From the preceding equations, we obtain($i$)

$$dx = -2h\cos^2 a\, e^{-2cs}\, dp, \quad dy = p\, dx, \quad gdt^2 = -dx\, dp$$

Hence, if the exponential be eliminated by means of equation (3), there will result

$$\left.\begin{aligned} cdx &= \frac{dp}{p\sqrt{1+p^2}+\log(p+\sqrt{1+p^2})-\gamma}, \\ cdy &= \frac{pdp}{p\sqrt{1+p^2}+\log(p+\sqrt{1+p^2})-\gamma}, \\ \sqrt{cg}\,.\,dt &= \frac{-dp}{[\gamma - p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})]^{\frac{1}{2}}}, \end{aligned}\right\} \quad (4)$$

formulæ which cannot be integrated under a finite form, in the last, the radical should be considered as a positive quantity, because the angle of which $p$ is the tangent, diminishes when the time increases.

212 Naming $\omega$ this angle, that is to say, the inclination MT$x$ of the tangent to the trajectory, on the horizontal axis O$x$, we shall have

$$p = \text{tang}\,\omega, \quad dp = \frac{d\omega}{\cos^2\omega}$$

The values of $x, y, t$, deduced from equations (4), will be of the form $\int\Omega d\omega$; the integral being taken, so that it may vanish at the point O where $\omega = a$, and $\Omega$ denoting a given function of $\omega$ These three values should be calculated for each point M, by the method of quadratures (No 15) In this manner, the trajectory can be constructed by points, and $t$ the time that the moveable takes to describe each arc OM, of which the length $s$ is given by equation (3), will be known

To determine the velocity of the moveable at the point M, we have($k$)

$$v^2 = (1+p^2)\frac{dx^2}{dt^2} = g^2(1+p^2)\frac{dt^2}{dp^2},$$

and, consequently,

$$cv^2 = \frac{g(1+p^2)}{\gamma - p\sqrt{1+p^2} - \log(p+\sqrt{1+p^2})} \qquad (5)$$

If these integrals be extended to $\omega = 0$, the abscissa and ordinate of C, the most elevated point of the trajectory, can be obtained Then by assigning to $\omega$ negative values, the points of the descending branch CBD of the trajectory can be determined When we arrive at a value, such as $-\alpha'$, of $\omega$, for which $y$ the ordinate of the trajectory vanishes, the corresponding value of $x$ will be equal to OB the amplitude of the projection, which will be no longer, as in the case of a vacuum, double of the ordinate of the point C when it is a maximum, in fact $\alpha$ is then equal to an angle less than 45°, and depending on the magnitude of the initial velocity The angle $\alpha'$, or EB$x$, and the velocity at the point B will likewise differ from $a$ and $\alpha$

Thus it appears, that all the circumstances of the motion will be known, and the problem completely resolved, when the values of the three constant quantities $h$, $a$, $c$, contained in the preceding formulæ, are given, however the numerical calculations which must be performed in each particular case, are extremely tedious

213 The motion of the projectile on the descending branch of the trajectory, continually tends to become vertical and uniform, in fact, if we transfer the origin of the coordinates to C the summit of the curve (fig 50) by making

$$x = x_1 + x', \quad y = y_1 - y', \quad t = t_1 + t',$$

then $x'$, $y'$, are the abscissa and ordinate of any point M′ of the descending branch, referred to the horizontal axis C$x'$, and to the axis C$y'$, which is drawn in the direction of gravity, and $t'$ denotes the time employed in describing the arc CM′ Likewise if $p'$ represents the tangent of the angle M′T′$x'$, which the

tangent to the curve at M′, makes with the axis C$x'$, we shall have ($l$)

$$p' = \frac{dy'}{dx'} = -p,$$

and because

$$\log(\sqrt{1+p'^2} - p') = -\log(p' + \sqrt{1+p'^2}),$$

the first equation (4) will become

$$cdx' = \frac{dp'}{\mathrm{P}'},$$

in which P′ denotes, for conciseness,

$$\gamma + p'\sqrt{1+p'^2} + \log(p' + \sqrt{1+p'^2}$$

As the acute angle M′T′$x'$ continually approximates to a right angle, the variable $p'$ will increase indefinitely, but this will not be the case with respect to $x'$. When $p'$ is very great, we can substitute $p'$ in place of $\sqrt{1+p'^2}$, and then if $\gamma + \log 2$ be neglected with respect to $p'^2$, we shall have ($m$)

$$\mathrm{P}' = p'^2 + \tfrac{1}{2}\log p'^2,$$

or simply $\mathrm{P}' = p'^2$, since the logarithm of a very great quantity $p'^2$, and, *à fortiori*, $\frac{1}{2}\log p'^2$, is very small with respect to this quantity, therefore, for these values of $p'$, we shall have

$$dx' = \frac{dp'}{cp'^2}.$$

If this equation be integrated there will result

$$x' = \mathrm{C} - \frac{1}{cp'},$$

C being an arbitrary constant, and from this integral it appears, that the values of $x'$ do not increase indefinitely with those of $p'$. This being the case, if

$$q = \frac{1}{c}\int_0^{\infty} \frac{dp'}{\mathrm{P}'},$$

$q$ will be a line of finite magnitude, which can be calculated by the method of quadratures, and if we take on $cx'$ a part CA equal to this line, the vertical AB drawn through this point will be an asymptote of the part CD of the trajectory, so that the motion of the projectile on this descending branch approaches indefinitely the vertical direction It may also be observed, that when the values of $p'$ are very great, the two last equations (4) become

$$cdy' = \frac{dp'}{p'}, \quad \sqrt{cg}\, dt' = \frac{dp'}{p'},$$

consequently, there results

$$\frac{dy'}{dt'} = \sqrt{\frac{g}{c}},$$

hence it appears, that when the final and vertical motion of the projectile becomes uniform, the velocity of this motion will be that which a heavy body acquires in falling through a height equal to $\frac{1}{2c}$, in a vacuum, this is also evident from formula (5), by substituting $-p'$ in place of $p$, and then considering $p'$ as a very great quantity ($n$)

If in the first equation (4) we make

$$p = \operatorname{tang} \omega, \quad dp = \frac{d\omega}{\cos^2\omega},$$

and, for conciseness,

$$[\gamma - \tan\omega\sqrt{1+\operatorname{tang}^2\omega} - \log(\operatorname{tang}\omega + \sqrt{1+\tan^2\omega})]\cos^2\omega = \frac{1}{\Omega},$$

we shall obtain

$$x_1 = \frac{1}{c}\int_0^a \Omega d\omega,$$

for the abscissa of the point C If, therefore, we take on $ox$ (fig 49) a point I, such that

$$\text{OI} = x_1 + q,$$

the vertical IG, drawn through this point I, will be the asymptote of the descending branch of the trajectory

214. Let ON be the production of the trajectory OCD, O being the point from which the projectile commences to move, the motion of the projectile will not take place in this part of the curve, but, notwithstanding, we may wish to know its form. Now, if it be constructed by points by means of the two first formulæ (4), taking care that the values of $p$ are positive and greater than tang $\alpha$, it is easy to be assured that it also has an asymptote, which however is not vertical, as in the case of the descending branch.

For this purpose, we may observe that it appears from the value of $\gamma$ of No. 211, there is always given an acute angle $\beta$ greater than $\alpha$, which is such, that $p = \text{tang}\,\beta$ renders the common denominator of these two formulæ cypher, that is to say, an angle $\beta$ which satisfies the equation

$$\gamma - \text{tang}\,\beta\sqrt{1+\text{tang}^2\beta} - \log(\text{tang}\,\beta + \sqrt{1+\text{tang}^2\beta}) = 0 \quad (6)$$

This being so, it appears from the value of $dp$ deduced from either of the two first equations (4), that though the abscissa $x$ and ordinate $y$ increase indefinitely (abstracting from the sign), in this part ON of the curve, the quantity $p$ ceases to increase, when the difference between it and tang $\beta$ is infinitely small, so that $p$ can never pass, nor even rigorously attain to the value $p = \text{tang}\,\beta$, this indicates, that the branch of the curve ON has an asymptote, which intersects the axis O$x$ when produced, in an angle equal to $\beta$, and its distance from the point O may be determined in the following manner; let an axis be drawn through the point O, which may make with the production of O$x$, an angle equal to the complement of $\beta$, and which is, consequently, perpendicular to the asymptote of ON. Let $u$ denote the abscissa of any point of the curve, reckoning from the point O on this axis, as the coordinates of this point with respect to the axes O$x$ and O$y$ are always $x$ and $y$, we shall have

$$u = y\cos\beta - x\sin\beta.$$

By taking the differential of this expression, and by sub-

stituting for $dx$ and $dy$ their values furnished by the two equations (4), we shall obtain

$$c\,du = \frac{(\text{tang}\,\beta - p)\cos\beta\,.\,dp}{(\gamma - p\sqrt{1+p^2}) - \log(p+\sqrt{1+p^2})},$$

in this formula, values should be assigned to $p$ greater or less than tang $\alpha$, according as the point in question is on the part ON, or on the part OM of the curve. If from its denominator the first member of equation (6) be taken, and if, moreover, we make

$$p = \text{tang}\,\omega, \quad dp = \frac{d\omega}{\cos^2\omega},$$

and, for conciseness,

$$\text{tang}\,\beta\sqrt{1+\text{tang}^2\beta} + \log(\text{tang}\,\beta + \sqrt{1+\text{tang}^2\beta}$$
$$-\,\text{tang}\,\omega\sqrt{1+\text{tang}^2\omega} - \log(\text{tang}\,\omega + \sqrt{1+\text{tang}^2\omega}) = \text{U},$$

there will result

$$du = \frac{(\text{tang}\,\beta - \text{tang}\,\omega)\cos\beta\,d\omega}{c\,\text{U}\cos^2\omega}.$$

Now, if we assume

$$r = \frac{\cos\beta}{c}\int_{\alpha}^{\beta}\frac{(\text{tang}\,\beta - \text{tang}\,\omega)\,d\omega}{\text{U}\cos^2\omega},$$

$r$ will be a line of a finite magnitude, which can be calculated by the method of quadratures, and it will express the value of $u$ with respect to the asymptote of ON, that is to say, the length of the perpendicular let fall from the point O on this line, which was required to be determined. The equation of this asymptotic line will be

$$u\cos\beta - x\sin\beta = r,$$

so that if on the production of O$x$ a point H be so taken, that we may have

$$\text{OH} = \frac{r}{\sin\beta},$$

the asymptote of the branch ON will be the line HK, drawn through the point H, and making with O$x$ produced, an angle KHO the supplement of $\beta$. The two asymptotes FG and HK,

produced above the axis $ox$, will meet in a point L, so that the entire curve will be comprised within the angle KLG, the complement of $\beta$, the angle that was determined by equation (6)

215. When O A $x$ or $a$ the angle of projection is very small (fig. 51), the height of the projectile above the horizontal axis $ox$, drawn through the point from whence it was discharged, is inconsiderable. Now, in this case, we can obtain by an approximation that is sufficiently accurate, the equation in $x$ and $y$ of the part OCB of the trajectory, that is situated above $ox$, and we can even extend this equation to a point D, the distance of which from this axis is not very great In fact, in all this part OCB, or even OCD of the trajectory, the tangent to this curve will be very nearly horizontal, and the quantity $p$ will be very small, therefore, if the square of $p$ be neglected, we shall have

$$ds = dx, \quad s = x,$$

and equation (2) will become

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = -\frac{1}{2h\cos^2 a}e^{2cx}$$

By integrating twice successively, and determining the arbitrary constants, so that we may have $\frac{dy}{dx} = \text{tang } a$ and $y = 0$, when $x = 0$, we obtain ($p$)

$$y = x \text{ tang} a - \frac{1}{8c^2h\cos^2 a}(e^{2cx} - 2cx - 1),$$

for the approximate equation of the trajectory, which was proposed to be determined If the exponential that it contains be developed, this equation becomes by reducing and then making $c = 0$, the exact equation of this curve in a vacuum

From the equation $gdt^2 = -dxdp$ of No 212, compared with the preceding value of $\frac{dp}{dx}$, we can infer ($q$)

$$dt = \frac{1}{\sqrt{2gh}\cos a} e^{cx} dx,$$

and, consequently,

$$t = \frac{1}{c\sqrt{2gh}\cos a}\,(e^{ca}-1),$$

by means of which, the time $t$ that the moveable takes to traverse OM, any portion of the curve OCD, can be determined

216 If we suppose that the projectile falls on the ground at a point D, and if $\lambda$ denotes the depression of this point below the horizontal plane drawn through the point O, or DQ the perpendicular to the axis O$x$, also, if $l$ be the distance OQ, and $\tau$ the time employed to go from the point O to the point D, we shall have, at the same time,

$$x = l, \quad y = -\lambda, \quad t = \tau,$$

and, for greater simplicity, by substituting unity for $\cos^2 a$, in the preceding formulæ, there will result

$$\left.\begin{aligned} 8c^2h(\lambda + l\operatorname{tang} a) &= e^{2cl}-2cl-1, \\ \tau c\sqrt{2gh} &= e^{cl}-1 \end{aligned}\right\} \qquad (a)$$

When, therefore, the two constants $h$ and $c$ are given, and the angle $a$, and $\lambda$ the elevation of the point O above the ground, are measured, these equations will make known the horizontal range $l$, and $\tau$ the time in which the projectile describes this range Conversely, when $a$, $\lambda$, $l$, $\tau$, are known by direct measurement, these equations will enable us to determine $c$ the coefficient of the resistance, and $h$ the height due to the initial velocity. If $h$ be eliminated, we obtain

$$4(\lambda + l\operatorname{tang} a)(e^{cl}-1)^2 = g\tau^2(e^{2cl}-2cl-1),$$

an equation from which the value of $c$ may be deduced, and then one of the two preceding will immediately give the value of $h$

There still exists some uncertainty respecting the magnitudes of the ranges and of the initial velocities According to Lombard, for a twenty-four pounder, of which the charge is a third of the weight of the bullet, the initial velocity is about

463 metres for each second, and for a twelve-pounder, of which the charge is also a third of the weight of the projectile, this velocity amounts to 494 metres According to the same author, the corresponding ranges relative to $\lambda = 0$, are 700 metres in the first case, $a$ being supposed $= 1° 15' 6''$, and 660 metres in the second case, in which we suppose that $a = 1° 5' 36''$

In place of the time $\tau$, we may employ for the determination of $h$ and $c$, a second range of the same cannon at a different elevation above the point where the projectile reaches the ground Thus, if we suppose that the weight of the projectile, that of the charge, and the angle $a$ remain the same, the quantities $h$ and $c$ will likewise remain unchanged, and if $\lambda$ and $l$ become $\lambda'$ and $l'$, we shall have

$$8 c^2 h (\lambda' + l' \operatorname{tang} a) = e^{2cl'} - 2cl - 1 ,$$

hence there results

$$\left.\begin{aligned} &(\lambda + l \operatorname{tang} a)\,(e^{2cl'} - 2cl' - 1) \\ &= (\lambda' + l' \operatorname{tang} a)\,(e^{2cl} - 2cl - 1), \end{aligned}\right\} \quad (b)$$

$h$ being eliminated by means of the first equation (a)

Those authors who have treated of the Ballistic pendulum, are by no means agreed as to the magnitude of the number $n$ which occurs in the expression of the coefficient $c$, namely (No. 210),

$$c = \frac{n\rho}{\mathrm{D}\prime}$$

It has been inferred from a very imperfect theory of the resistance of fluids, that this number $n$ is $\frac{3}{8}$, but the result of all experiments that have been made, assign a less number than this, and Lombard made it equal to $\frac{9}{10}$ Equation (b) would furnish a means, the most susceptible of precision of any, for determining $c$, if $a$ the angle of projection was supposed to be known and invariable

## II The Motion of the Planets

217. The laws of the motion of the planets about the sun, are generally known by the denomination of *the laws of Kepler*, because they were discovered by that astronomer, who deduced them from observation They are three in number, and are expressed as follows

1. The planets move in plane curves, and their *radii vectores* describe areas proportional to the times, about the centre of the sun.

2 The orbits or trajectories of the planets are ellipses, of which the sun occupies one of the foci.

3 The squares of the times of the revolutions of the planets, are to each other as the cubes of the greater axes of their orbits

Kepler does not appear to have been aware of the great importance of these laws, it was Newton who applied them to determine the force which retains each planet in its orbit, that is to say, the direction of this force, and the variations of its intensity, from one position to another of the same planet, and also from one planet to another. In fact, it will be proved in this section, that each of these three circumstances is a necessary consequence of the three laws of planetary motion which have been just stated.

As these laws refer to the motion of the centre of gravity of each planet, it is the motion of this point which we proceed to consider, and in every question respecting the position or velocity of a planet, it is the position and velocity of this centre that is always referred to.

218 Let AMBD (fig. 52) be the ellipse described by a planet, AB its greater axis, C its centre, O and O′ its two foci, O that which the centre of the sun occupies, B the *perihelion*, or the point of the orbit nearest to the sun, A the *aphelion*, or the point which is most remote from the sun. After the lapse of

time $t$ which is reckoned from the moment that the planet passes through the perihelion, let M be the position of the moveable. Let $r$ denote its radius vector, $\theta$ the angle MOB, which is termed in astronomy the *true anomaly*. The sector described by the radius during the instant $dt$ will be $\frac{1}{2} r^2 d\theta$ (No. 156), therefore, in consequence of Kepler's first law, we have

$$r^2 d\theta = c dt, \tag{1}$$

$c$ being a constant arbitrary equal to twice the area described in the unit of time, and $ct$ twice the area MOB described in $t$ any time whatever Likewise, let O'M $= r'$, CB $=$ CA $= a$, CO $=$ CO' $= ae$. By a property of the ellipse we have

$$r + r' = 2a,$$

we have likewise, in the triangle O'MO,

$$r'^2 = r^2 + 4 r ae \cos\theta + 4a^2e^2,$$

and if $r'$ be eliminated between these two equations, there results

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}, \tag{2}$$

for the equation of the trajectory.

For all the planets which were known previously to the present century, the *excentricity* $e$ is a very small fraction, with the exception of Mercury, the excentricity of which is more than a fifth, in the case of the orbit of the earth

$$e = 0{,}01685318,$$

or, very nearly, a sixtieth The greatest was that of Mars, which exceeds nine hundredths, it was consequently for this planet that the elliptic motion ought to differ most from the motion of a planet in an excentric($a$) circle, in which the planets were supposed to move previously to the time of Kepler; and in fact, it was from the observations of Ticho-Brahe on this planet, that Kepler first recognized the difference between these two motions. If the values $r$ and $\theta$ be developed

into series arranged according to the powers of $e$, by means of the equation of areas proportional to the times, combined with that of the elliptic trajectory, or with that of the trajectory of the excentric circle, we shall find that when the time $t$ is the same, the corresponding expansions of these two curves differ only in terms, that depend on the square or higher powers of $e$, from which it is evident, that in Kepler's time it was extremely difficult, in consequence of the imperfection of the observations, to discover the difference between these two motions

219 If T denotes the time of the revolution of a planet, and if we assume

$$n = \frac{2\pi}{T},$$

this constant quantity $n$ will be the mean angular velocity, and *$nt$ the mean motion* of the planet

If we conceive a fictitious star whose motion is uniform, to set out from perihelion, and to complete its revolution in the same time as this planet, then its radius vector will describe $nt$, while that of the planet describes the angle $\theta$, and the angle $\theta - nt$ contained at any epoch between these radii, is what astronomers term *the equation of the centre*, it is positive, and the planet precedes the fictitious star in going from perihelion to aphelion(*b*), the contrary takes place in returning from the second point to the first The *maximum* of the equation of the centre depends on the magnitude of the excentricity If we assume the mean day for the unit of time, we shall obtain in the case of the earth's orbit, by substituting 360 for $2\pi$,

$$T = 365^d, 256374, \quad n = 0{,}59', 8''.$$

This value of T is the duration of the *sidereal* year, or the interval of time which lapses between two consecutive returns of the sun to the same fixed star, in its apparent motion about the earth. The interval of time between two consecutive returns to the same *equinox*, is shorter, because the equi-

noctial points have a retrograde motion on *the ecliptic*, that is, a motion the direction of which is contrary to that of the sun. If we suppose that the annual *precession* in 1800 was 50″23427, then since the radius vector of the sun describes this small angle in $0^d,014158$, there results $365^d,242216$, for the length of the *equinoctial* year at the commencement of the present century. The sidereal year is constant, but as the precession of the equinoxes varies a little, the equinoctial year must participate in this inequality, its length diminishes very nearly half a second in a century

220 The value of the constant quantity $c$ will be equal to twice the area of the ellipse divided by T, therefore, as the semi-axis minor is $a\sqrt{1-e^2}$, and the area of the ellipse $\pi a^2\sqrt{1-e^2}$, we shall have

$$c=\frac{2\pi a^2\sqrt{1-e^2}}{\text{T}}.$$

By means of this value and of that of $n$, equation (1) becomes

$$r^2d\theta=na^2\sqrt{1-e^2}\,dt$$

equation (2) gives(c)

$$\theta=\text{arc}\left(\cos=\frac{a(1-e^2)-r}{er}\right),$$

$$d\theta=\frac{a\sqrt{1-e^2}.dr}{r\sqrt{a^2e^2-(r-a)^2}},$$

therefore, we shall have

$$nadt=\frac{rdr}{\sqrt{a^2e^2-(r-a)^2}}.$$

In order to integrate these formulæ, let

$$r=a(1-e\cos u); \qquad \text{(a)}$$

and, consequently,

$$dr=ae\sin u\,du, \quad ndt=(1-e\cos u)\,du,$$

because at the point B, $r = a(1-e)$, it is necessary that $u$ should be equal to cypher at this point, where we have also $t = 0$, therefore, by integrating we obtain

$$nt = u - e\sin u. \tag{b}$$

By substituting in the expression of $d\theta$, for $r$ its value, and observing that $\cos u = \cos^2 \frac{1}{2}u - \sin^2 \frac{1}{2}u$, there results

$$d\theta = \frac{\sqrt{1-e^2}\,du}{1 - e\cos^2 \frac{1}{2}u + e\sin^2 \frac{1}{2}u},$$

and if we assume

$$\tang \tfrac{1}{2}u = z, \quad \frac{du}{\cos^2 \frac{1}{2}u} = 2dz,$$

this value becomes

$$d\theta = \frac{2\sqrt{1-e^2}\,dz}{1-e+(1+e)z^2}$$

If we integrate and observe also that $\theta$ and $u$ vanish at the same time, that is, at the point B, we shall have

$$\tfrac{1}{2}\theta = \text{arc}\left(\tang = z\sqrt{\frac{1+e}{1-e}}\right),$$

and hence we can deduce, by substituting for $z$ its value,

$$\tang \tfrac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}}\tang \tfrac{1}{2}u. \tag{c}$$

These three equations (a) (b) (c) express in a finite form, the values of $r$, $nt$, $\theta$, by means of the variable auxiliary $u$, which is called the *excentric anomaly*. By eliminating $u$ between them, we shall obtain the polar coordinates $r$ and $\theta$ of the planet in functions of the time, under the form of series arranged according to the powers of the excentricity, which will, consequently, converge very rapidly in the case of the planets known previously to the present century.

In the series which will thus arise, the powers of $\cos nt$, which occur in the development of $r$, and those of $\sin nt$,

which are contained in the development of $\theta - nt$, can be replaced by the cosines and sines of multiples of $nt$. If we suppose that these developments of the radius vector and of the equation of the centre, are then arranged according to the sines or cosines of the increasing multiples of $nt$, the values of the coefficients of these two series in functions of the excentricity, may be determined directly by the following analysis.

221. Let

$$r = A_0 + A_1 \cos nt + A_2 \cos 2nt + \ldots\ldots + A_i \cos int + \&c.,$$

$$\theta - nt = B_1 \sin nt + B_2 \sin 2nt + \ldots\ldots B_i \sin int + \&c.;$$

$A_0$, $A_1$, $A_2$, &c., $B_1$, $B_2$, &c. and generally, $A_i$, $B_i$, being the coefficients which it is required to determine.

If $i$ and $i'$ be any two positive integral numbers different from each other, we shall have, by performing the integrations(*d*)

$$\int_0^\pi \cos int \cos i'nt\, d\ nt = 0,$$

$$\int_0^\pi \sin int \sin i'nt\, d\ nt = 0,$$

and when $i' = i$, we shall find,

$$\int_0^\pi \cos^2 int\, d\ nt = \tfrac{1}{2}\pi,$$

$$\int_0^\pi \sin^2 int\, d\ nt = \tfrac{1}{2}\pi.$$

These last formulæ cannot be applied when $i = 0$, in this case, the first integral is equal to $\pi$, and the second to zero. This being agreed on, if the development of $r$ be multiplied by $\cos int\, d\ nt$ and that of $\theta - nt$ by $\sin int\, d\ nt$, and if the results be respectively integrated from $nt = 0$ to $nt = \pi$, then all the terms vanish, except those of which the coefficients are $A_i$ and $B_i$ respectively, hence we infer,

$$A_i = \frac{2}{\pi}\int_0^\pi r \cos int\, d\ nt,$$

$$B_i = \frac{2}{\pi}\int_0^{\pi} (\theta - nt) \sin int\, d\ nt$$

In the case of $i = 0$, we shall have, in this particular case,

$$A_0 = \frac{1}{\pi}\int_0^{\pi} r\, d\,.\,nt,$$

that is to say, the general value of $A_i$ is reduced to the half If we integrate by parts, and at the same time observe, that $\theta - nt = 0$, at the two limits $nt = 0$, and $nt = \pi$, the integral of $B_i$ may be replaced by the following expression($e$)

$$B_i = \frac{2}{i\pi}\int_0^{\pi} \cos int\, d\,(\theta - nt).$$

Since, from what precedes, it is evident that

$$\frac{d\theta}{du} = \frac{\sqrt{1-e^2}}{1-e\ \cos u}, \quad \frac{d\ nt}{du} = 1 - e.\cos u,$$

it follows, that if in place of $r$, $nt$, $\theta$, their values in functions of $u$, deduced from the equations (a), (b), (c), be substituted, we shall have($f$)

$$A_i = \frac{2a}{\pi}\int_0^{\pi} (1 - e\cos u)^2 \cos(iu - ie\sin u)\, du,$$

$$B_i = \frac{2}{i\pi}\int_0^{\pi} [\sqrt{1-e^2} - (1 - e\ \cos u)^2]\,\frac{\cos(iu - ie\sin u)}{1 - e\ \cos u}\, du,$$

because $u = 0$ and $u = \pi$ correspond to $nt = 0$ and $nt = \pi$, these formulæ will make known the numerical values of the coefficients $A_i$ and $B_i$, either by the method of quadratures, or by reducing them into series In order to effect this reduction, we shall have by Taylor's theorem

$$\cos(iu - ie.\sin u) = (1 - \frac{i^2e^2}{2}\sin^2 u + \frac{i^4e^4}{2\ 3\ 4}\sin^4 u - \&c.)\cos iu$$

$$+ (ie\sin u - \frac{i^3e^3}{2\ 3}\sin^3 u + \&c)\sin iu,$$

and hence there will result for $A_i$ and $B_i$ series which are integrable with respect to $u$, the exact values of which can be obtained either immediately, or by means of known formulæ, so that we may continue these developments of $A_i$ and $B_i$ as far as we please. We can even obtain their general terms in functions of $i$ and $e$, but we will not insist longer on this subject here, as it belongs especially to astronomy.

In the case of $i = 0$, we have(h)

$$A_0 = \frac{a}{\pi}\int_0^{\pi}(1-e.\cos u)^2\,du = a\,(1+\tfrac{1}{2}e^2),$$

for as was observed above, only half of the general value of $A_i$ is taken, when $i = 0$, this is the only one of the coefficients $A_0$, $A_1$, $A_2$, &c., $B_0$, $B_1$, $B_2$, &c., the exact value of which can be obtained.

222. If $v$ denotes the velocity of the planet at the end of the time $t$, and $\delta$ the angle which its direction makes with the radius vector $r$ or OM produced, by No. 156 we shall have

$$v^2 = \frac{dr^2 + r^2 d\theta^2}{dt^2}, \qquad v\sin\delta = r\,\frac{d\theta}{dt}.$$

Eliminating $dt$ by means of equation (1), there results

$$v^2 = c^2\left(\frac{d\,\frac{1}{r}}{d\theta}\right)^2 + \frac{c^2}{r^2}, \qquad v\sin\delta = \frac{c}{r}.$$

From equation (2), we likewise obtain

$$\frac{1}{r} = \frac{1+e\cos\theta}{a\,(1-e^2)}, \qquad \frac{d\,\frac{1}{r}}{d\theta} = -\frac{e\sin\theta}{a\,(1-e^2)},$$

hence we have(i)

$$a^2(1-e^2)^2v^2 = (1+2e\cos\theta+e^2)\,c^2,$$

and, consequently,

$$v^2 = \frac{c^2}{a^2(1-e^2)}\left(\frac{2a}{r} - 1\right), \qquad \sin\delta = \frac{a\sqrt{1-e^2}}{r\sqrt{\frac{2a}{r} - 1}}, \quad \text{(d)}$$

hence it appears that in case of elliptic motion, the velocity and direction of the moveable at each point may be determined by means of its radius vector Substituting the value of $c$ of No 220, that of $v^2$ may be written

$$v^2 = \frac{4\pi^2 a^2}{T^2}\left(\frac{2a}{r} - 1\right)$$

These formulæ, when combined with those of the preceding number, will completely determine the motion of a planet in the plane of its orbit, but when the motions of two or more planets are considered at the same time, then it is necessary to refer the position of each of them to another plane, which is generally the *plane of the ecliptic*, or of the earth's orbit

223 Let NON′ (fig 53) be the intersection of the plane of the orbit of a planet with a plane passing through O the centre of the sun, OE a right line drawn in this second plane, OM′ the projection of OM, the radius vector of the planet, on this same plane Let us denote by $\gamma$ the inclination of the two planes, by $\alpha$ the angle NOE, by $\omega$ the angle BON which the radius vector OB drawn to the perihelion of the planet makes with the line ON These three angles $\alpha$, $\omega$, $\gamma$, being given, they determine the plane of the orbit, and the position of the ellipse in this plane Also, let $\phi$ and $\psi$ represent the variable angles MOM′ and M′OE, which the radius vector OM makes with its projection OM′, and this projection with the line OE, these angles will determine, at each instant, the direction of OM the radius drawn to the planet

This being established, in the solid angle, whose vertex is at the point O, and which is contained by the three sides OM, OM′, ON, as the true anomaly, or the angle MOB is represented always by $\theta$, the three faces of this solid angle will be

$$\text{MON} = \text{MOB} + \text{BON} = \theta + \omega,$$
$$\text{M}'\text{ON} = \text{M}'\text{OE} - \text{NOE} = \psi - \alpha,$$
$$\text{MOM}' = \phi,$$

the first is opposite to a right angle, and the third to the angle $\gamma$. By the rules of spherical trigonometry we have

$$\sin\phi = \sin\gamma \sin(\theta + \omega),$$

$$\tang(\psi - \alpha) = \cos\gamma \tang(\theta + \omega),$$

and, as by what precedes the angle $\theta$ is known in a function of $t$, each of the angles $\phi$ and $\psi$ will be also given by means of these formulæ.

When the given plane, on which the angle $\psi$ is reckoned, is the ecliptic, then if the line OE, from which this angle is measured in the direction of the earth's motion, is the line drawn from the sun to the vernal equinox, the angles $\psi$ and $\phi$ are called the *longitude* and *latitude* of the planet in question. The line NON′ is the line of the *nodes* of its orbit; if it passes into the *northern* hemisphere, when it traverses the plane of the ecliptic at the point N, this point is the *ascending* node, and N′ is the *descending* node. According as the planet exists in the northern or southern hemisphere, the latitude $\phi$ is said to be positive or negative, and the angle MON, or $\theta + \omega$ is less or greater than 180 (*k*). The angle $\phi$ extends from $-90°$ to $90°$, and the angle MON, and also the longitude M′OE, from zero up to 360°.

If the point O be replaced by the centre of the earth, and if the plane of the equator be that on which the angle $\psi$ is measured, then when this angle is counted from the line drawn from this centre to the first point of Aries, the angles $\psi$ and $\phi$ will denote respectively the *right ascension* and *declination* of the planet. When these formulæ are applied to the apparent motion of the sun about the earth, we have $\alpha = 0$, $\gamma$ will express the obliquity of the ecliptic, and $\theta + \omega$ will be

the longitude of this star, hence, denoting it by $\lambda$, we shall have

$$\sin\phi = \sin\gamma\ \sin\lambda, \quad \text{tang}\,\psi = \cos\gamma\ \text{tang}\,\lambda,$$

and, consequently$(l)$,

$$\sin\phi = \frac{\sin\gamma\ \text{tang}\,\psi}{\sqrt{\cos^2\gamma + \text{tang}^2\psi}}$$

The greatest northern and southern declinations correspond to $\lambda = 90$ and $\lambda = 270°$, and are $\pm\gamma$ This angle $\gamma$ is also equal to the angle which the axis of rotation of the earth makes with the perpendicular to the plane of the ecliptic, it is subject to a slight inequality called the *nutation*, the period of which is about eighteen years, and which at its *maximum* amounts only to 9″4 Its mean value, at the beginning of this century, was

$$\gamma = 23°,27',55'',$$

it diminishes by 0″45692 for each year

224 In what precedes, no reference has been made to the force which acts on each planet, in fact, all the circumstances of its motion have been determined by data furnished by observation, and without having any recourse to the principles of dynamics, it is now requisite to determine the laws of this force as they have been already stated in No 217

It follows from the first law of Kepler, that the force which retains each planet in its orbit, is constantly directed towards the centre of the sun, although this necessary consequence of the proportionality of the areas to the times has been already deduced in No 155 from the equations of motion, it will not be deemed superfluous here to give a synthetic demonstration of it

Let $M_1M$ (fig 54) be the side of the trajectory described by the moveable in an indefinitely small portion of time $\tau$. When it arrives at the point $M$, if no force acts on this moveable, it will describe in a portion of time equal to $\tau$, a part $Mm$ of $MT$, the production of $M_1M$, and equal to $MM_1$, but, in consequence of

the force which acts on it, in this second instant, it is transferred to another point M′. Let MK be the direction of this force at the point M; we may suppose, that during the time $r$ it remains parallel to itself; hence, if we draw the line $m$M′, it will be parallel to MK (No 148). Now, if C be the fixed centre about which the radius vector describes areas proportional to the times, the triangles $M_1CM$ and MCM′, which are the areas described in two equal portions of time, will be equal to each other. But the triangles $M_1CM$ and MC$m$ are also equal to each other, since their vertices are at the same point C, and their bases $M_1M$, M$m$, are equal and exist on the same line; consequently, the triangles MC$m$ and MCM′ are equivalent; and as they exist on the same base MC, the line M′$m$ which joins their vertices must be parallel to this base; hence the line MK, parallel to $m$M′, coincides with MC. Therefore, in each point M of the trajectory, MK the direction of the force, will be that of the radius vector MC.

Conversely, if the force which acts on the moveable at any point M, is directed along MC, the right line $m$M′ will be parallel to this radius vector, the two triangles M′CM and MC$m$ will be equal, and, consequently also, the two triangles M′CM and $M_1CM$. The areas described by the radius vector about the point C, in two consecutive instants, being equal, and this being the case for the entire trajectory, it follows, that if the force which acts on the moveable be constantly directed to this point, the areas described in equal times will be equal, and, in any time whatever, proportional to these times.

225. Let, as in No 218, M be the position of the planet at the end of the time $t$ (fig 52), and let, as in preceding numbers, $r$ denote the radius vector OM, and $\theta$ the angle MOB; moreover, if $x$ and $y$ represent the two rectangular coordinates OP and PM, referred to the axes O$x$ and O$y$, of which the first passes through the perihelion, we shall have

$$x = r \cos\theta, \quad y = r \sin\theta, \quad x^2 + y^2 = r^2.$$

Also, let R denote the accelerating force, acting on the planet, the intensity of which is unknown.

We have seen that this force acts along the radius vector, and, because the concavity of the trajectory is turned towards the sun, it is directed from M to the point O, therefore, the cosines of the angles which it makes with $x$ and $y$ produced, are $-\frac{x}{r}$ and $-\frac{y}{r}$, consequently, the equations of the motion will be

$$\frac{d^2x}{dt^2} = -\frac{Rx}{r}, \quad \frac{d^2y}{dt^2} = -\frac{Ry}{r}, \qquad (1)$$

$v$ being the velocity at the point M, we shall have

$$v^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2},$$

and, by differentiating,

$$\tfrac{1}{2}d\ v^2 = \frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy,$$

consequently, if equations (1) be multiplied by $dx$ and $dy$ respectively, and then added together, we shall obtain, by observing that $xdx + ydy = rdr$,

$$\tfrac{1}{2}d\ v^2 = -Rdr$$

But, when the motion is performed in an ellipse, we have (No. 222)

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a},$$

$\mu$ being made equal to $\frac{4\pi^2a^3}{T^2}$ Consequently, we shall obtain,

$$R = \frac{\mu}{r^2},$$

from which it appears, that the force which acts on each planet, varies in the inverse ratio of the square of the distance from the centre of the sun

The magnitude of this force at the unit of distance is $\mu$, if $\mu'$ be what it becomes for any other planet, of which the semi-axis major and time of revolution are represented by $a'$ and $T'$, we shall have

$$\mu' = \frac{4\pi^2 a'^3}{T'^2}$$

But, by the third law of Kepler, we have

$$T^2 : T'^2 :: a^3 : a'^3,$$

from which results

$$\frac{a^3}{T^2} = \frac{a'^3}{T'^2}, \quad \mu = \mu',$$

consequently, at the unit of distance, and, generally, at the same distance from the sun, the accelerating force R is the same for two different planets

The motive force of each planet is therefore independent of its particular nature, and, like the weight of a body at the surface of the earth, is proportional to its mass It varies from one planet to another, according to the same law, as it does from one position to another of the same planet, and if two planets were situated at the same distance from the sun, and then remitted to themselves, without any initial velocity, they would descend towards this star with the same velocity, and reach it in the same interval of time

Thus it appears, that the three laws of Kepler completely determine the force which retains the planets in their orbits; the law of the areas proportional to the times, shews that this force is constantly directed towards the centre of the sun, that of the elliptic motion, or the expression for the velocity, which is deduced from this law, combined with the preceding, proves that its intensity varies for the same planet in the inverse ratio of the square of the distances from the sun, finally, from the law of the squares of the times of revolutions proportional to the cubes of the greater axes, we infer that at equal distances from the centre of this star, the intensity of the motive force is pro-

portional to the mass of each planet, and independent of its particular nature

226 Newton extended to the motion of comets about the sun, and of the satellites about their respective primaries, the laws of Kepler, and the consequences that follow from them, with respect to the force which acts on these moveables

Comets in their motion differ only from planets in this, that they are not constantly visible, in consequence of the great distance of their aphelia, which renders the determination of their orbits extremely difficult Notwithstanding this, there are three comets, of which the orbits and times of revolution are known almost as accurately as those of the planets With respect to other comets, astronomers have computed their motion in an approximate manner, by assuming for their trajectory, in the small extent for which they are visible, a parabola of which the focus exists in the centre of the sun, and supposing always that the areas described by the radii vectores about this point, are proportional to the times for each comet. This case is embraced under the preceding formulæ for elliptic motion, by making

$$a = \infty, \quad a(1-e) = b,$$

$b$ denoting the perihelion distance OB, which is a finite quantity

The masses of the comets are very small compared with those of the planets, and they seem to be of an entirely different nature($m$) By the third law of Kepler, the motive forces of two comets, or of a comet and a planet, at the same distance from the sun, are to each other as their respective masses, and their accelerating forces are equal, this is likewise true for several satellites of the same planet, but not for satellites of two different planets, or for a satellite and a planet, for the law of the squares of the times of the revolutions proportional to the cubes of the greater axes of the orbits, obtains only for bodies which revolve about the *same* centre, in the sequel we will

shew the relation which exists between the motive forces of two satellites belonging to two different planets, or between those of a satellite and a planet.

It ought to be observed here, that within the last few years, the laws of elliptic motion have been extended to those double stars, in which a periodic motion of one of the stars about the other, has been recognized, and their relative positions, computed by means of these laws, agree with their observed positions as accurately as could have been expected$(n)$

227 Let us now consider the changes which the resistance of an extremely rare ether diffused through the heavenly regions would produce in the elliptic motion of the planets about the sun If these bodies are not perfectly spherical, this circumstance, combined with the friction of the fluid against their surface would cause the centre of gravity to deviate from the plane of its orbit However we will at present abstract from the consideration of these circumstances, and form the equations of the motion of this point, from the consideration of a central force varying in the inverse ratio of the square of the distance, combined with a tangential force arising from the resistance of the medium

Let this resistance be supposed, as in the motion of projectiles in the air, to be proportional to the square of the velocity, to the density of the medium, and to the surface of each planet, the accelerating force which results will be in the inverse ratio of the mass of the moveable If $ds$ represents the velocity of the moveable, then we can denote this force by $\rho\ \frac{ds^2}{dt^2}$, $\rho$ being a very small coefficient, and for the same planet proportional to the density of the medium. Since this force acts always in a direction contrary to the velocity of the moveable, if the principal force directed towards the centre of the sun be denoted by $\mu$ at the unit of distance, and by $\frac{\mu}{r^2}$ at the distance $r$, equations (1) will be replaced by the following$(o)$,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}&=-\rho\frac{ds}{dt}\frac{dx}{dt},\\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}&=-\rho\frac{ds}{dt}\frac{dy}{dt}\end{aligned}\right\}\qquad(2)$$

If we make use of polar coordinates, we, may without difficulty, deduce from them(p)

$$\left.\begin{aligned}\frac{d\,(dr^2+r^2d\theta^2)}{dt^2}-2\mu d.\frac{1}{r}&=-2\rho\frac{(dr^2+r^2d\theta^2)}{dt^2}ds,\\ d.r^2d\theta&=-\rho r^2d\theta ds,\end{aligned}\right\}\qquad(3)$$

which are the equations into which the preceding may be transformed.

228. Equations (2) become, when their second members are neglected,

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0,\quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0,\qquad(4)$$

and equations (3) are reduced to

$$\frac{d\,(dr^2+r^2d\theta^2)}{dt^2}-2\mu d.\frac{1}{r}=0,\quad d.r^2d\theta=0\qquad(5)$$

We can satisfy equations (5) by means of formulæ (a), (b), (c), of No. 220; however, as these formulæ contain only two constant arbitraries, $a$ and $e$, they are not the complete integrals of these equations, but if we consider that equations (5) do not contain explicitly the variables $\theta$ and $t$, but only their differentials $d\theta$ and $dt$, it is evident that the formulæ of the number cited will still satisfy these equations, by adding to $t$ and $\theta$, certain constants. In this manner, the complete integrals of equations (5), and, consequently, of equations (4), will be expressed by the following system of formulæ,

$$\left.\begin{aligned}r&=a\,(1-e\cos u),\\ nt+\varepsilon-\omega&=u-e\sin u,\\ \operatorname{tang}\tfrac{1}{2}(\theta-\omega)&=\sqrt{\frac{1+e}{1-e}}\operatorname{tang}\tfrac{1}{2}u,\end{aligned}\right\}\qquad(a)$$

$a$, $e$, $\varepsilon$, $\omega$, being four constant arbitraries, and $n$ a constant connected with $a$ by the equation

$$a^3 n^2 = \mu,$$

which results from $\dfrac{2\pi}{\mathrm{T}} = n$, $\dfrac{4\pi^2 a^3}{\mathrm{T}^2} = \mu$, by the elimination of $\mathrm{T}$.

When the variable $u$ is cypher, the value of $r$ is a *minimum*, which is the case at the perihelion B (fig. 52). In the case of $u = 0$, we have $\theta = \omega$, so that in this case $\theta$ will represent the angle MOE, measured from the right line OF, which makes the angle BOE $= \omega$, with OB. The value of $\theta$, expressed in a series, will be of the form

$$\theta = nt + \varepsilon + \theta_1,$$

in which $\theta_1$ denotes its periodic part, arranged according to the sines of the increasing multiples of $nt + \varepsilon - \omega$. This angle $\theta$ will be the true longitude of the planet in the plane of its orbit, at the end of $t$ any time whatever; $nt + \varepsilon$ will express its mean longitude at the same instant, $\varepsilon$ its mean longitude at the epoch from which the time $t$ is reckoned, and $\omega$ the longitude of its perihelion.

229. This being established, when the complete integrals of a system of differential equations, such as equations (4), are known, the integrals of another system of differential equations, such as equations (2), which only differ from the first by very small terms, may be deduced from them by a method that has been most successfully applied by geometers to different questions in celestial mechanics, and of which we proceed to explain the principles, for the sake of the problem which we are at present occupied with. The values of $x$ and $y$ which satisfy equations (4), are of the form,

$$x = f(t, a, e, \varepsilon, \omega), \quad y = \mathrm{F}(t, a, e, \varepsilon, \omega),$$

$f$ and F denoting given functions. In order that these values may likewise satisfy equations (2), $a$, $e$, $\varepsilon$, $\omega$, should be considered as new variables, which it is required to determine

But as the number of these unknown is four, while equations (2) are only two in number, we may assume arbitrarily two auxiliary equations, and, in consequence, make

$$\left.\begin{aligned}\frac{df}{da}da+\frac{df}{de}de+\frac{df}{d\varepsilon}d\varepsilon+\frac{df}{d\omega}d\omega=0,\\ \frac{dF}{da}da+\frac{dF}{de}de+\frac{dF}{d\varepsilon}d\varepsilon+\frac{dF}{d\omega}d\omega=0,\end{aligned}\right\}\quad\text{(b)}$$

that is, in other words, the parts of $dx$ and $dy$, which arise from the variations of $a$, $e$, $\varepsilon$, $\omega$, are made equal to zero By this means, the complete values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simply

$$\frac{dx}{dt}=\frac{df}{dt},\quad \frac{dy}{dt}=\frac{dF}{dt}$$

By differentiating again, we obtain,

$$\frac{d^2x}{dt^2}=\frac{d^2f}{dt^2}+\frac{d^2f}{dtda}\,\frac{da}{dt}+\frac{d^2f}{dtde}\cdot\frac{de}{dt}+\frac{d^2f}{dtd\varepsilon}\,\frac{d\varepsilon}{dt}+\frac{d^2f}{dtd\omega}\,\frac{d\omega}{dt},$$

$$\frac{d^2y}{dt^2}=\frac{d^2F}{dt^2}+\frac{d^2F}{dtda}\cdot\frac{da}{dt}+\frac{d^2F}{dtde}\cdot\frac{de}{dt}+\frac{d^2F}{dtd\varepsilon}\,\frac{d\varepsilon}{dt}+\frac{d^2F}{dtd\omega}\,\frac{d\omega}{dt}.$$

Now, by hypothesis, the preceding values of $x$ and $y$ satisfy equations (4), $a$, $e$, $\varepsilon$, $\omega$, being considered as constant arbitraries, consequently we have

$$\frac{d^2f}{dt^2}+\frac{\mu x}{r^3}=0,\quad \frac{d^2F}{dt^2}+\frac{\mu y}{r^3}=0,$$

hence, if the complete values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ be substituted in equations (2), we shall have

$$\left.\begin{aligned}\frac{d^2f}{dtda}da+\frac{d^2f}{dtde}de+\frac{d^2f}{dtd\varepsilon}d\varepsilon+\frac{d^2f}{dtd\omega}d\omega=-\rho\frac{ds}{dt}\frac{dx}{dt}dt,\\ \frac{d^2F}{dtda}da+\frac{d^2F}{dtde}de+\frac{d^2F}{dtd\varepsilon}d\varepsilon+\frac{d^2F}{dtd\omega}d\omega=-\rho\frac{ds}{dt}\frac{dy}{dt}dt,\end{aligned}\right\}\quad\text{(c)}$$

and thus, by means of the four equations (b) and (c), we can determine $a$, $e$, $\varepsilon$, $\omega$, in functions of $t$.

230 In general, this substitution of four differential equations of the first order, in place of the two equations (2) of the second order, will not be attended with any advantage But as the values of $da$, $de$, $d\varepsilon$, $d\omega$, which are deduced from equations (b) and (c), have for a factor $\rho$ the coefficient of the resistance, which is a very small quantity, the variable parts of $a$, $e$, $\varepsilon$, $\omega$, will be also extremely small; and if the square of $\rho$ be neglected, $a$, $e$, $\varepsilon$, $\omega$, may be considered as constant, in the expressions of $da$, $de$, $d\varepsilon$, $d\omega$, hence the determination of the variable parts of $a$, $e$, $\varepsilon$, $\omega$, may be reduced to quadratures By the method of successive approximations, the values of these quantities may be thus obtained, arranged according to the powers of $\rho$, and continued to whatever degree of accuracy we please; however, we shall stop at the first power of $\rho$, in the present investigation($q$)

If in equations (a), the variable values of $a$, $e$, $\varepsilon$, $\omega$, be substituted, they will make known, as in the case of elliptic motion, the values of $r$ and $\theta$ in functions of the time The trajectory which in this case is described, will be still an ellipse, but its elements continually vary. If at each instant, the constant ellipse which satisfies the values of the elements at this same instant, be constructed, the ordinates $x$ and $y$, and their differentials $dx$ and $dy$ will be common, in virtue of equations (b), to this ellipse, and to the trajectory, which consequently will be the osculating curve of all the constant ellipses For the same reason, the velocity of the moveable and its components will have the same expressions in the elliptic motion, and in the motion altered by the resistance of the medium, and will be determined by formulæ (d) of No 222.

231 As we have identically

$$nt = \int ndt + \int tdn,$$

by comprising $\int tdn$ in the unknown $\varepsilon$, the second equation ($a$) may be written as follows,

$$\int ndt + \varepsilon - \omega = u - e \sin u, \qquad (d)$$

Then, if in the equations of elliptic motion, there be substituted for the constants $a, e, \varepsilon, \omega$, their variable values, we should at the same time replace $nt$, by the integral $\int ndt$, which is supposed to vanish when $t = 0$ The quantity $n$ that it involves, may be deduced from $a$, by means of the formula

$$n = \frac{\sqrt{\mu}}{a\sqrt{a}},$$

which is furnished by the equation $a^3 n^2 = \mu$ of No. 228 This integral $\int ndt$ expresses the mean motion of the planet (No 219), altered by the resistance of the medium, and thus, the differential of the mean motion will be $ndt$, in the disturbed motion as well as in the elliptic motion.

At the perihelion, the angle $\theta - \omega$ is either cypher, or a multiple of 360°, and this will be also the case with the angle $u$, in virtue of the first equation (a), therefore, in the time which lapses during two consecutive passages of the planet through its perihelion, the quantity $\int ndt + \varepsilon - \omega$ will be increased by 360°, this will enable us to determine this interval, when $n$, $\varepsilon$, $\omega$, are given in functions of $t$ The time of a revolution, or the interval which lapses between two returns of the planet to the same *fixed* point, will in like manner be that in which its true longitude $\theta$ is increased by 360°(r).

232 Equations (b) and (c) may be replaced by equivalent ones, from which the values of $da$, $de$, $d\varepsilon$, $d\omega$, can be inferred with greater facility For this purpose, it is to be observed, that if any equation

$$\phi\,(nt, r, \theta, a, \varepsilon, \omega) = 0,$$

obtains in the case of elliptic motion, it will likewise subsist when the motion is altered by the resistance of the medium, $a$, $e$, $\varepsilon$, $\omega$, being regarded as variables determined by equations (b) and (c), and $\int ndt$ being substituted instead of $nt$ Therefore, the differential of the function $\phi$ will be equal to cypher,

whether it be taken in the first case with respect to $nt$, $r$, $\theta$, or in the second case, with respect to $\int ndt$, $r$, $\theta$, $a$, $e$, $\varepsilon$, $\omega$ Now, as $r$ and $\theta$ are functions of $x$ and $y$, their differentials are the same in the two cases, in virtue of equations (b), consequently, if in the complete differential of $\phi$, the part $\frac{d\phi}{d\ nt}\ d\ nt + \frac{d\phi}{dr}$ $dr + \frac{d\phi}{d\theta} d\theta$, which is separately equal to cypher, be suppressed, we shall have

$$\frac{d\phi}{da} da + \frac{d\phi}{de} de + \frac{d\phi}{d\varepsilon} d\varepsilon + \frac{d\phi}{d\omega} d\omega = 0.$$

Now, if in equation (2), of No 218, $\theta - \omega$ be substituted in place of $\theta$, there results

$$r + re \cos(\theta - \omega) = a(1 - e^2),$$

and, if this be differentiated in the manner already pointed out, we shall have

$$r \cos\theta d\ e\cos\omega + r\sin\theta d\,.\,e\sin\omega = d\,.\,a(1 - e^2). \quad \text{(e)}$$

If the first equation (a) and equation (d) be also differenced, we obtain

$$(1 - e\cos u)\, da - a\cos u\, de + ae\sin u\, du = 0,$$

$$d\varepsilon - d\omega + \sin u\, de - (1 - e\cos u)\, du = 0,$$

$u$ being regarded as function of $a$, $\varepsilon$, $e$, $\omega$

By eliminating $du$ between these two($s$) equations, we obtain

$$(1 - e\cos u)^2 da + a(e - \cos u)\, de + ae\sin u\,(d\varepsilon - d\omega) = 0.$$

But, if in the formulæ

$$\cos u = \frac{1 - \text{tang}^2 \frac{1}{2} u}{1 + \text{tang}^2 \frac{1}{2} u}, \quad \sin u = \frac{2\,\text{tang}\,\frac{1}{2} u}{1 + \text{tang}^2 \frac{1}{2} u},$$

we substitute for tang $\frac{1}{2}u$, its value which is given by the third equation (a), we have($t$)

$$\cos u = \frac{e + \cos(\theta - \omega)}{1 + e\ \cos(\theta - \omega)}, \quad \sin u = \frac{\sqrt{1 - e^2}\sin(\theta - \omega)}{1 + e\ \cos(\theta - \omega)},$$

by means of which, the preceding equation becomes

$$\frac{(1-e^2)\,da}{1+e\cos(\theta-\omega)} - a\cos(\theta-\omega)\,de + \frac{ae\sin(\theta-\omega)}{\sqrt{1-e^2}}(d\varepsilon - d\omega) = 0. \quad \text{(f)}$$

What has been stated with respect to the equation $\phi = 0$, may likewise be applied to the case in which the function $\phi$ contains the first differentials of $r$ and $\theta$. Thus, in the case of elliptic motion, we have

$$\frac{dr^2 + r^2 d\theta^2}{dt^2} - \frac{2\mu}{r} = -\frac{\mu}{a},$$

$$r^2 d\theta = \sqrt{\mu a(1 - e^2)}\,dt,$$

$\sqrt{\mu a}$ being substituted in place of $a^2 n$ in the expression for $r^2 d\theta$ of No 220(v) Now, since the differentials $dr$, $d\theta$, as also $r$ and $\theta$, remain the same when $a$, $e$, $\varepsilon$, $\omega$, become variable, it follows that these two equations will also subsist on this hypothesis, this being so, if their complete differentials be compared with equations (3) of No 228, we can infer(u)

$$\left.\begin{aligned} d\,\frac{1}{a} &= 2\rho\left(\frac{2}{r} - \frac{1}{a}\right)ds, \\ d\,\sqrt{a(1-e^2)} &= -\rho\sqrt{a(1-e^2)}\,ds. \end{aligned}\right\} \quad \text{(g)}$$

Now, the values of $da$, $de$, $d\varepsilon$, $d\omega$, may easily be inferred from the four equations (e), (f), (g), by substituting for $r$ its value, namely,

$$r = \frac{a(1 - e^2)}{1 + e\ \cos(\theta - \omega)},$$

in order to express them in functions of the angle $\theta$ solely, we find(t)

$$\left.\begin{aligned} da &= -\frac{2\rho a}{1-e^2}\left[1+2e\cos(\theta-\omega)+e^2\right]ds, \\ de &= -2\rho\left[e+\cos(\theta-\omega)\right]ds, \\ ed\omega &= -2\rho\sin(\theta-\omega)\,ds, \\ d\varepsilon &= +\frac{2\rho e\sin(\theta-\omega)\left[\sqrt{1-e^2}-e^2-e\cos(\theta-\omega)\right]}{\left[1+e\cos(\theta-\omega)\right]\left(1+\sqrt{1-e^2}\right)}, \end{aligned}\right\} \quad (h)$$

The value of $ds$ which should be substituted in these formulæ, is

$$ds = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}\,d\theta,$$

and, by substituting in this expression the value of $r$, we obtain

$$ds = \frac{a(1-e^2)\sqrt{1+2e\cos(\theta-\omega)+e^2}}{\left[1+e\cos(\theta-\omega)\right]^2}\,d\theta$$

The second members of these equations should be integrated on the supposition that $a$, $e$, $\varepsilon$, $\omega$, are constant, as has been already stated, and when the coefficient $\rho$ is given in a function of $r$, and consequently of $\theta$, the values of the variables $a$, $e$, $\varepsilon$, $\omega$, may be deduced by the method of quadratures, or by reduction into a series, which values should be substituted in the equations of elliptic motion.

233 If the excentricity $e$ be a very small fraction, formulæ (h) reduced to their principal part, become

$$da = -2\rho a^2 d\theta, \quad de = -2\rho a\cos(\theta-\omega)\,d\theta,$$
$$ed\omega = -2\rho a\sin(\theta-\omega)\,d\theta, \quad d\varepsilon = 2\rho ae\sin(\theta-\omega)\,d\theta,$$

and the coefficient $\rho$ may be regarded as constant Hence, by integrating and denoting the variable parts of $a$, $e$, $\varepsilon$, $\omega$, by $\delta a$, $\delta e$, $\delta\varepsilon$, $\delta\omega$, we shall have

$$\begin{aligned} \delta a &= -2\rho a^2\theta, \\ \delta e &= -2\rho a\sin(\theta-\omega), \\ e\delta\omega &= 2\rho a\cos(\theta-\omega), \\ \delta\varepsilon &= -2\rho ae\cos)\,\theta-\omega). \end{aligned}$$

If the corresponding part of $n$, or of $\dfrac{\sqrt{\mu}}{a\sqrt{a}}$ be expressed by $\delta n$, so that we may have

$$\delta n = -\frac{3\sqrt{\mu}}{2a^2\sqrt{a}}\,\delta a,$$

there will result

$$\delta n = 3\rho a n\theta.$$

Hence it appears, that the effect of the resistance of an extremely rare medium, on the motion of a planet in an orbit of very small excentricity, will be to cause the greater axis to decrease indefinitely, also to increase $n$ the angular velocity, and to produce in each of the three quantities $e$, $\omega$, $\varepsilon$, an inequality of which the period is the same as that of the planet's revolution Not only is the angular motion more and more accelerated, but also the absolute velocity, for it is very nearly equal to $an$, its increment is therefore $a\delta n + n\delta a$, which is a positive quantity and equal to $\rho a^2 n\theta$.

If the excentricity be altogether neglected, we have

$$r = a, \quad \theta = \int n dt + \varepsilon;$$

therefore, if $\delta r$ and $\delta\theta$ denote the parts of the radius vector, and of the longitude which are produced by the action of the resisting medium, we shall have to the same degree of approximation,

$$\delta r = -2\rho a^2\theta, \quad \delta\theta = \tfrac{3}{2}\rho a\theta^2.$$

In consequence of this continual diminution of the radius vector, which amounts to $4\pi\rho a^2$ at the end of each revolution of the planet, this body must eventually fall on the surface of the sun

Finally, if there exists in the regions of free space, an ether which affects the motion of the heavenly bodies, its influence will be particularly sensible in accelerating the motion of the comets, on account of the extreme smallness of their mass, and because that, every thing else being the same, the coefficient $\rho$ is in the inverse ratio of the mass of the body. And, in fact,

we have not been able as yet to recognize any traces of an ether which resists the motion of the planets and satellites, but, according to the calculations of M. Enke, this resistance appears to have an appreciable influence on the motion of the comet recently discovered, the time of whose revolution is about 1200 days.

## III. *Motion of a material Point acted on by a central Force.*

234. The problem which we now proceed to consider is the inverse of that of the preceding section, there the trajectory and the law of motion being supposed to be furnished by observation, it was proposed to determine, in magnitude and direction, the force which produced this motion, in the present case, a constant force directed towards a fixed centre, and given in a function of the distance of the moveable from this point, is supposed to be applied to this moveable, and it is required to determine from thence the trajectory and the law of the motion.

This curve DMB (fig 55) will be contained in the plane passing through the fixed centre C, and through DA, the direction of the initial velocity. Let two rectangular axes C$x$, C$y$, of which the first passes through D, the point from which the body commences its motion, be drawn in this plane, through the point C, and let them be the axes of the coordinates. At the end of the time $t$, which is reckoned from the departure at D, let the moveable be supposed to be in M, and let $x$ and $y$ denote its coordinates CP and PM, $r$ its radius vector CM, R its accelerating force directed from M towards C, and given in a function of $r$, the equations of the motion will be

$$\frac{d^2x}{dt^2} = -\frac{x}{r}\text{R}, \quad \frac{d^2y}{dt^2} = -\frac{y}{r}\text{R}, \qquad (1)$$

if the force R acts in the direction of the production of CM, it will only be necessary to change the signs of their second

members. We can immediately deduce from these equations, the two following integrals of the first order,

$$xdy - ydx = cdt, \quad \frac{dx^2 + dy^2}{dt^2} = -2\int \text{R}dr + b,$$

in which $b$ and $c$ are the constant arbitraries introduced by the integration, and if $\theta$ denotes the angle MC$x$, so that we may have

$$x = r\cos\theta, \quad y = r\sin\theta,$$

these integrals will become

$$r^2 d\theta = cdt, \quad \frac{dr^2 + r^2 d\theta^2}{dt^2} = -2\int \text{R}dr + b, \tag{2}$$

if the values of $dt$ and $d\theta$ be deduced from these equations, they will be of the form

$$dt = fr\,dr, \quad d\theta = \text{F}rdr,$$

which it will be only necessary to integrate either accurately or by approximation.

If $dt$ be eliminated between equations (2), we obtain

$$c^2\left(\frac{d\frac{1}{r}}{d\theta}\right)^2 + \frac{c^2}{r^2} + 2\int \text{R}dr = b, \tag{3}$$

for the differential equation of the trajectory

If $v$ represents the velocity of the moveable at the point M, we shall have

$$v^2 = b - 2\int \text{R}dr; \tag{4}$$

and $\delta$ denoting the angle that its direction makes with MC, its components will be

$$v\cos\delta = -\frac{dr}{dt}, \quad v\sin\delta = \frac{rd\theta}{dt},$$

in the respective directions of MC, the radius vector, and of MH perpendicular to MC.

The two constants $b$ and $c$, and those which are introduced by the integration of the values of $dt$ and $d\theta$, can be determined

by means of the initial position and velocity of the moveable. For this purpose, let $\gamma$ represent the initial distance CD, $\alpha$ the angle CDA which may be either acute or obtuse, and $h$ the height due to the initial velocity, so that this velocity may be $\sqrt{2gh}$, $g$ denoting the gravity. If the integral $\int R dr$, which occurs in the preceding formulæ vanishes, when $r = \gamma$, we shall have, as is evident from the general expression for $v^2$,

$$b = 2gh$$

In virtue of the equation $r^2 d\theta = c\,dt(y)$, the value of $v \sin \epsilon$ is the same thing as $\frac{c}{r}$, consequently, we shall have

$$c = \gamma \sqrt{2gh} \sin \alpha$$

As to the two other constant arbitraries, they are determined by the supposition that $\theta = 0$, and $r = \gamma$, when $t = 0$, and thus, the problem will be completely resolved.

235. When the force R is proportional to the distance $r$, the variables $x$ and $y$ are separated in equations (1), and there is no occasion to recur to polar coordinates, or to equations (2). In fact, let $k$ denote the value of R which corresponds to $r = \gamma$, and

$$R = \frac{kr}{\gamma},$$

will be its general value. Equations (1) will become

$$\frac{d^2x}{dt^2} = -\frac{kx}{\gamma}, \quad \frac{d^2y}{dt^2} = -\frac{ky}{\gamma},$$

and then complete integrals will be (z)

$$x = A \cos t \sqrt{\frac{k}{\gamma}} + A' \sin t \sqrt{\frac{k}{\gamma}},$$

$$y = B \cos t \sqrt{\frac{k}{\gamma}} + B' \sin t \sqrt{\frac{k}{\gamma}},$$

A, A′, B, B′, being the four constant arbitraries that are intro

duced by the integration. In order to determine them, we have, by the preceding suppositions,

$$x = \gamma, \quad y = 0, \quad \frac{dx}{dt} = -\sqrt{2gh}.\cos a, \quad \frac{dy}{dt} = \sqrt{2gh}.\sin a,$$

when $t = 0$, hence it follows that

$$\text{A} = \gamma, \quad \text{A}'\sqrt{\frac{k}{\gamma}} = -\sqrt{2gh}\cos a,$$

$$\text{B} = 0, \quad \text{B}'\sqrt{\frac{k}{\gamma}} = \sqrt{2gh}.\sin a,$$

and, consequently,

$$x = \gamma\left(\cos t\sqrt{\frac{k}{\gamma}} - \sqrt{\frac{2gh}{k\gamma}}\cos a \sin t\sqrt{\frac{k}{\gamma}}\right),$$

$$y = \gamma\sqrt{\frac{2gh}{k\gamma}}.\sin a \sin t\sqrt{\frac{k}{\gamma}}.$$

It appears from these formulæ, that the revolutions of the moveable about the point C, will be isochronous, and their common duration equal to $2\pi\sqrt{\frac{\gamma}{k}}(a')$. We can deduce from them

$$\gamma \sin a \sin t\sqrt{\frac{k}{\gamma}} = y\sqrt{\frac{k\gamma}{2gh}},$$

$$\gamma \sin a \cos t\sqrt{\frac{k}{\gamma}} = x.\sin a + y\cos a,$$

hence we obtain $(b')$

$$\frac{k\gamma}{2gh}y^2 + (x\sin a + y\cos a)^2 = \gamma^2\sin^2 a,$$

for the equation of the trajectory, which is, evidently, an ellipse of which the centre is at the point C. This ellipse becomes a circle when $a = 90°$, and $k\gamma = 2gh$. In this case, the motion is uniform, for by the expressions for $x$ and $y$, we have

$$\frac{dx}{dt} = -\sqrt{\gamma k}\sin t\sqrt{\frac{k}{\gamma}}, \quad \frac{dy}{dt} = \sqrt{\gamma k}\cos t\sqrt{\frac{k}{\gamma}},$$

hence the velocity $v$ is equal to $\sqrt{\gamma k}$. The central force R and the centrifugal force $\frac{v^2}{\gamma}$ are constant, and respectively equal to $k$.

If the force R be repulsive instead of being attractive, as has been supposed, $k$ must be changed into $-k$ in the preceding formulæ. The trajectory will be then an hyperbola, and the body in its motion will never return a second time to the same point.

236 Let the force R be supposed to be proportional to the inverse cube of the distance, and, consequently, that

$$\mathrm{R} = \frac{k\gamma^3}{r^3},$$

$k$ being always its value at the point D.

In this hypothesis, we shall have,

$$2\int \mathrm{R}dr = k\gamma\left(1 - \frac{\gamma^2}{r^2}\right),$$

because the integral vanishes when $r = \gamma$.

If we take into account the values of $b$ and $c$, and make

$$\frac{\gamma}{r} = z, \quad \frac{\gamma d\frac{1}{r}}{d\theta} = \frac{dz}{d\theta},$$

equation (3) will become

$$\frac{dz^2}{d\theta^2} + \left(1 - \frac{k\gamma}{2gh\ \sin^2 a}\right)z^2 = \frac{1}{\sin^2 a} - \frac{k\gamma}{2gh\sin^2 a}.$$

As the coefficient of $z^2$ may be either positive or negative, let

$$1 - \frac{k\gamma}{2gh\ \sin^2 a} = \pm n^2,$$

consequently, we have($c'$)

$$\frac{dz^2}{d\theta^2} \pm n^2z^2 = \cot^2 a \pm n^2,$$

from which we obtain

$$n d\theta = \frac{n dz}{\sqrt{\cot^2 a \pm n^2 \mp n^2 z^2}}$$

When the superior signs are employed, we shall have($d'$)

$$n\theta = \text{arc}\left(\sin = \frac{nz}{\sqrt{\cot^2 a + n^2}}\right) - \text{arc}\left(\sin = \frac{n}{\sqrt{\cot^2 a + n^2}}\right),$$

and, when the inferior signs are made use of,

$$n\theta = \log \frac{nz + \sqrt{\cot^2 a - n^2 + n^2 z^2}}{n + \cot a},$$

taking care to observe, that $r = \gamma$ and $z = 1$ when $\theta = 0$

From the first value of $n\theta$, we deduce

$$nz = \cot a \sin n\theta + n \cos n\theta$$

The *maximum* of $z$, or the *minimum* of $r$, corresponds to the value of $\theta$, deduced from the equation $dz = 0$, or($e'$)

$$\tan n\theta = \frac{1}{n} \cot a,$$

for which we obtain

$$z = \frac{\gamma}{r} = \sqrt{1 + \frac{1}{n^2} \cot^2 a},$$

Beyond this value of $\theta$, the distance of the moveable from the point c will increase indefinitely, and its radius vector $r$ will be infinite, for the least value of $\theta$ deduced from the equation $z = 0$, or($f'$)

$$\text{tang}\, n\theta = - n \,\text{tang}\, a,$$

and to attain this value of $\theta$ an infinite time is required The value of $t$ as a function of $\theta$, may be obtained without any difficulty, by substituting in the first equation (2), in place of $r$, the value of $\frac{\gamma}{z}$.

When the value of $n\theta$ is a logarithmic function, we shall have, by passing to numbers, and denoting the base in Napier's system of logarithms by $e$,

$$nz + \sqrt{\cot^2 a - n^2 + n^2 z^2} = (n + \cot a)\, e^{n\theta},$$

hence we obtain ($g'$)

$$z = \frac{\gamma}{r} = \frac{1}{2n}(n + \cot a)\, e^{n\theta} + \frac{1}{2n}(n - \cot a)\, e^{-n\theta},$$

it appears from this expression, that the value of $r$ diminishes indefinitely, consequently, the curve described by the moveable body about the point C, will be a spiral, and it will not reach the centre until after it has performed an infinite number of revolutions.

If, in order to simplify, we make $a = 90°$, we shall have

$$r = \frac{2\gamma}{e^{n\theta} + e^{-n\theta}},$$

for the equation of this spiral. The first equation (2) becomes

$$\sqrt{2gh}\, dt = \frac{4\gamma d\theta}{(e^{n\theta} + e^{-n\theta})^2},$$

and, by integrating, we obtain

$$nt\sqrt{2gh} = \gamma\frac{(e^{n\theta} - e^{-n\theta})}{e^{n\theta} + e^{-n\theta}}.$$

237. Let us suppose, for the last example, that the force R varies in the inverse ratio of the square of the distance, which is the law that obtains in nature, in which case we have

$$R = \frac{k\gamma^2}{r^2}, \quad \int R dr = k\gamma\left(1 - \frac{\gamma}{r}\right);$$

$k$ being the intensity of this force at the point D, where the preceding integral is supposed to be equal to nothing.

If we make

$$\frac{1}{r} = \rho, \quad 2k\gamma - b = \beta,$$

equation (3) of the trajectory will become

$$c^2\frac{d\rho^2}{d\theta^2} = 2k\gamma^2\rho - c^2\rho^2 - \beta\,,$$

from which we obtain($h'$)

$$d\theta = \frac{cd\rho}{\sqrt{\frac{k^2\gamma^4}{c^2} - \beta - \left(\frac{k\gamma^2}{c} - c\rho\right)^2}}$$

Hence if we integrate and denote the constant arbitrary by $\omega$, we shall have

$$\theta = \omega + \text{arc}\left(\cos = \frac{k\gamma^2 - c^2\rho}{\sqrt{k^2\gamma^4 - c^2\beta}}\right),$$

by means of which we can obtain

$$k\gamma^2 r = c^2 - r\sqrt{k^2\gamma^4 - c^2\beta}\cos(\theta - \omega), \qquad \text{(a)}$$

$\omega + \pi$ being substituted in place of $\omega$, in order that $\omega$ may be the value of $\theta$ which corresponds to the least value of $r$, that is to say, to the point of the trajectory where the moveable is nearest to c

In order to obtain the equation of this curve in rectangular coordinates, let

$$x' = r\ \cos(\theta - \omega), \quad y' = r\ \sin(\theta - \omega),$$

$x'$ and $y'$ being the rectangular coordinates of the moveable referred to the axes c$x'$ and c$y'$, so that $x'$c$x = \omega$, we shall have evidently

$$x'^2 + y'^2 = r^2,$$

and if the two members of equation (a) of the trajectory be raised to the square, it will become($i'$)

$$k^2\gamma^4 y'^2 + \beta c^2 x'^2 = c^4 - 2c^2 x'\sqrt{k^2\gamma^4 - c^2\beta}$$

Now, it is evident from inspection of this expression, that it belongs to a conic section, which will be an ellipse, a parabola, or hyperbola, according as the constant $\beta$ is positive, cypher, or negative. It appears also, that in all cases, the point c will be a focus of this curve, for, in the equation (a), the radius vector $r$ is a linear function of the abscissa $x'$, now in

the three conic sections, this is only the case, when the origin of the coordinates is one of the foci.

Since $b = 2gh$, we shall have

$$\beta = 2k\gamma - 2gh\,,$$

hence it follows, that the sign of $\beta$, and consequently, the nature of the conic section described by the moveable, depends only on the initial distance and velocity of projection, and not at all on the direction of this velocity, so that if several material points commence to move from the same point C with equal velocities, all of them will describe conic sections of the same nature, whatever may be their initial directions. If for example $k = g$, the curve described will be an ellipse, a parabola, or hyperbola, according as the height due to the initial velocity is less than CD, equal to this distance, or greater than it.

238 In the case of the ellipse, equation (a) shews that the greatest and least values of $r$ correspond to $\theta = \omega + \pi$ and $\theta = \omega$, if we denote them by $a(1+e)$ and $a(1-e)$ respectively, then $a$ is the semi-axis major, and $e$ the excentricity, and we shall have

$$(k\gamma^2 - \sqrt{k^2\gamma^4 - c^2\beta})\,a\,(1+e) = c^2,$$

$$(k\gamma^2 + \sqrt{k^2\gamma^4 - c^2\beta})\,a\,(1-e) = c^2,$$

or, what is the same thing,

$$\beta a\,(1+e) = k\gamma^2 + \sqrt{k^2\gamma^4 - c^2\beta},$$

$$\beta a\,(1-e) = k\gamma^2 - \sqrt{k^2\gamma^4 - c^2\beta}.$$

If we add these equations together, and if we also multiply their corresponding members, we shall obtain

$$\beta a = k\gamma^2, \quad \beta a^2\,(1-e^2) = c^2\,,$$

and, by substituting for $\beta$ and $c$ their values,

$$\beta = 2\,(k\gamma - gh), \quad c = \gamma\sqrt{2gh}\sin\alpha,$$

we obtain

$$\left.\begin{aligned} 2(k\gamma - gh)\,a &= k\gamma^2, \\ \gamma k\sqrt{1-e^2} &= 2\sqrt{gh(k\gamma - gh)}\sin\alpha, \end{aligned}\right\} \qquad \text{(b)}$$

by means of which, the semi-axis major, and the excentricity, may be obtained. The angle $\omega$ is determined by making, at the same time, $\theta = 0$ and $r = \gamma$ in equation (a) Thus, when the position, initial velocity and direction of the moveable are given, the dimensions of the ellipse, and the position of its greater axis, will be completely determined With respect to its motion on this curve, that is known by formulæ (a) (b) (c) of No 220 It appears from formula (4) of No 234, that the square of the velocity at any instant is expressed by the equation

$$v^2 = 2gh - 2k\gamma + \frac{2k\gamma^2}{r},$$

or, what comes to the same thing,

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \qquad \text{(c)}$$

on account of the value which we have found for $a$, and by making $k\gamma^2 = \mu$, so that here, as in the formula of No. 225, $\mu$ expresses the intensity of the central force at the unit of distance

239. As the motions of comets are observed to approximate, during the time of their apparition, to that of a body moving in a parabola, it will not be irrelevant to discuss here this particular species of motion Since we have in this case $\beta = 0$, or $k\gamma = gh$, equations (b) give $a = \infty$ and $e = 1$, which is in fact the case in the parabola Formula (c) becomes

$$v^2 = \frac{2\mu}{r},$$

and if $u$ denotes the velocity in a circle, of which the radius is equal to $r$, we shall have in virtue of the same formula,

$$u^2 = \frac{\mu}{r},$$

consequently, at equal distances from the sun, the velocity of a comet is to that of a planet, which describes a circle, as $\sqrt{2}$ to 1.

In general, if the two members of the last of equations (b) be raised to the square, and if they then be respectively multiplied by those of the first, we shall obtain

$$k\gamma a(1-e)(1+e) = 2gh\gamma \sin^2 a,$$

hence, if $p$ denotes the least distance of the comet from the sun, so that($k'$)

$$p = a(1-e),$$

and if we make $k\gamma = gh$ and $e = 1$, we shall have

$$p = \gamma \sin^2 a;$$

from this it appears, that the perihelion distance is determined, when the initial distance and angle of projection are known

If, in equation (a), we make $\beta = 0$, and $k\gamma = gh$, it will become, by substituting for $c^2$ its value $2gh\gamma^2 . \sin^2 a$,

$$r = 2\gamma \sin^2 a - r\cos(\theta - \omega),$$

hence there results

$$r = \frac{2p}{1+\cos(\theta-\omega)}, \tag{d}$$

for the equation of the trajectory. If we make $\theta = 0$, and $r = \gamma$, we obtain($l'$)

$$\gamma(1+\cos\omega) = 2p, \quad \cos\tfrac{1}{2}\omega = \sin a,$$

by means of which, we can determine the angle $\omega$ that the radius vector drawn to the perihelion, makes with the radius drawn to the point from which a body is projected.

If we substitute for the values of $c$ and $r$ in the first equation (2) of No. 234, and if, in order to abridge, we make

$$\frac{\gamma\sqrt{gh}\sin a}{p^2} = n,$$

there results

$$\frac{4\,d\theta}{[1+\cos(\theta-\omega)]^2} = n\sqrt{2}\,.\,dt,$$

and because

$$1+\cos(\theta-\omega) = 2\cos^2\tfrac{1}{2}(\theta-\omega),$$

if we make

$$\theta-\omega = 2\psi, \quad d\theta = 2\,d\psi,$$

we shall have

$$\frac{d\psi}{\cos^4\psi} = \frac{n\,dt}{\sqrt{2}},$$

hence by integrating and denoting the constant arbitrary by $\varepsilon$, we obtain($m'$)

$$(3+\text{tang}^2\psi)\,\text{tang}\,\psi + \varepsilon = \frac{3nt}{\sqrt{2}}.$$

In order to determine this constant, it is to be observed that we have, at the same time,

$$t=0, \quad \theta=0, \quad \psi = -\tfrac{1}{2}\omega,$$

and because $\cos\frac{1}{2}\omega = \sin a$, there results

$$\varepsilon = (3+\cot^2 a)\cot a$$

Naming $t'$ the interval which lapses from the moment the body begins to move until it passes through the perihelion, we shall have at once

$$t=t', \quad \theta=\omega, \quad \psi=0,$$

and, consequently,

$$t' = \frac{\varepsilon\sqrt{2}}{3n}.$$

This being so, if $\tau$ denotes the time reckoned from the instant of this passage, so that $t = t'+\tau$, we shall have

$$[3+\text{tang}^2\tfrac{1}{2}(\theta-\omega)]\,\text{tang}\,\tfrac{1}{2}(\theta-\omega) = \frac{3nt}{\sqrt{2}}, \qquad (e)$$

and if this equation, which is of the third degree with respect to $\text{tang}\,\frac{1}{2}(\theta-\omega)$, be resolved, we shall obtain $\text{tang}\,\frac{1}{2}(\theta-\omega)$

in a function of $\tau$, and, consequently, we shall have the values of $r$ and $\theta$ at any given instant, the time $\tau$ is evidently positive after the body has passed through the perihelion and negative previously to this passage

Because

$$gh\gamma = k\gamma^2 = \mu, \quad \sqrt{\gamma}\sin a = \sqrt{p},$$

the preceding value of $n$ is the same as

$$n = \frac{\sqrt{\mu}}{p\sqrt{p}},$$

therefore, it appears from the equation $a^3n^2 = \mu$ of No 228, that $n$ expresses the mean angular velocity of a planet, of which the greater semi-axis is equal to $p$, and if $\imath$ denotes that of the earth, and $l$ its semi-axis major, so that we may have

$$\imath = \frac{\sqrt{\mu}}{l\sqrt{l}},$$

we shall obtain$(n')$

$$n = \frac{\imath l\sqrt{l}}{p\sqrt{p}},$$

for the value of $n$

240 From the preceding analysis it appears, that if the determination of the motion of a comet be considered as a *mere problem of dynamics*, and if, in consequence, its initial position, its direction, and velocity, are supposed to be known, we can, from these data, determine $p$ the distance of the summit of the parabola from its focus, the instant of the passage of the body through this summit, or the value of $t'$, and the position of the axis which depends on the angle $\omega$, the velocity of the comet, and its position on its trajectory at any instant whatever, are known by means of equations (c), (d), (e), and as the plane of this curve is that which passes through the centre of the sun, and through the direction of the initial velocity, it follows that the motion is completely determined But the *astronomical* problem is altogether different When

a comet is first discovered, neither the plane of its orbit, its distance from the sun, its velocity, nor its direction at the instant of its apparition, are furnished by observation, so that if its position at this instant be taken for its point of departure, the constant quantities $\gamma, h, a$, will not be given as in the preceding problem The question then consists in deducing from observations, the five following quantities, namely, the inclination of the orbit, and the longitude of its ascending node on the plane of the ecliptic, (which two quantities will determine the plane of the orbit,) the longitude of the perihelion, and its distance from the sun, (from which two, the position of the orbit in its plane may be known,) and lastly, the time corresponding to the passage of the comet through its perihelion. When these five unknown quantities are determined, equations (c), (d), (e), represent, as in the preceding numbers, the motion of the comet in its plane Now, from each observation of the comet, we can deduce its right ascension and declination, therefore, three observations furnish six data, and, consequently, six equations, which are more than sufficient to determine the five unknown quantities mentioned above, and this circumstance enables us to replace two of these equations by that combination of them which is most proper to diminish the influence of the errors of observations.

Having thus deduced approximate values of the five elements adverted to above, from three observations made at the epoch of their apparition, subsequent observations serve to correct these first values, and to verify formulæ (d) and (e).

We can only in this treatise thus generally advert to this problem, which is one of pure astronomy, of which different solutions have been proposed

## CHAPTER VII.

### DIGRESSION ON UNIVERSAL ATTRACTION

241. *The material points of all bodies attract each other mutually, in the direct ratio of the masses, and in the inverse of the square of the distances.*

This great law of nature, which was discovered by Newton, is a necessary consequence of observation and calculation In fact, it is shewn in the *exposition of the system of the world*, how, by setting out from experiment, we are conducted, without assuming any hypothesis, by an uninterrupted train of rigorous reasoning, to the principle of *universal gravitation*. The developments of this principle constitute the especial object of the *celestial mechanics.* We shall restrict ourselves, in this chapter, to a brief exposition of its principal consequences.

242. The force that retains the planets in their orbits is the resultant of the attraction, which all the material points of the sun exercise on all those of each planet. Considering the smallness of the dimensions of the sun and the planets relatively to the distances which intervene between them, it is evident that these attractions may be regarded, with an approximation sufficiently near to the truth, as equal and parallel forces in the entire extent of each planet, their resultant is then equal to their sum, and, the distance remaining the same, the motive force of each planet is proportional to the product of its mass into that of the sun, the circumstance of the form of these bodies being very nearly spherical, renders this conclusion still more exact, when their mutual distance is assumed to be that of their centres of gravity, (No. 99 )

In order to express the intensity of this force numerically, let a certain distance, for example, that of the sun from the earth, be assumed as the linear unit, also let a determinate mass and interval of time be selected as the respective units of these two descriptions of quantities, and finally, let the unit of force be, as in No. 118, the constant accelerating force, which produces, in the unit of time, a velocity equal to the unit of length. Let us now suppose two bodies, whose masses are respectively equal to that which has been selected to represent unity, to be distant from each other by a quantity equal to the linear unit, then if $f$ denotes the attractive force of one of these two bodies on the other, that is to say, the numerical ratio of its intensity to that of the force which has been selected to represent unity, M and $m$ being the masses of the sun and planet, the motive force of the planet will be $f\text{M}m$ at the unit of distance, and at any other distance $r$, it will be $\frac{f\text{M}m}{r^2}$.

The magnitude of the quantity which we have denoted by $f$, depends on the attractive power with which matter is endowed, when the mass and distance are equal, this power is the same for all bodies, nothing hitherto observed leads us to suppose that it increases or diminishes with the time, and we have every reason to think that it has been and will remain constantly the same

243. The motive force of the mass M, due to the attraction of $m$, is also represented by $\frac{f\text{M}m}{r^2}$, so that the reaction of each planet on the sun, is equal and contrary to the action of this star on the planet, but the motive force $\frac{f\text{M}m}{r^2}$, acting on the two masses M and $m$, will impress on them, in each instant, infinitely small velocities, which are reciprocally proportional to these masses, or, in other words, their accelerating forces are $\frac{fm}{r^2}$ and $\frac{f\text{M}}{r^2}$ Hence it follows, that if these bodies are remitted, with-

out any initial velocities, to their mutual attraction, they will move towards each other, passing over, in the same time, spaces which are in the inverse ratio of their masses, and they will meet at their common centre of gravity, which we know divides their initial distance in the inverse ratio of their masses. In general, if a planet is projected in any direction whatever in space, and if it be required to determine its apparent motion about the centre of the sun, considered as a *fixed* point, we must conceive that in each instant, there is impressed on this star, an infinitely small velocity, equal and contrary to that which it receives from the attraction of the planet; but, in order that the *relative* motion of these two bodies be not altered, we should, at the same time, impress this velocity on the planet, which is the same thing, as if there was applied to it an accelerating force equal and contrary to that of the sun; therefore, in the motion which we are considering, the accelerating force of the planet $m$ will be constantly directed towards the sun M, and equal to the sum of the two forces $\frac{f\text{M}}{r^2}$ and $\frac{fm}{r^2}$, hence if it is represented by $\frac{\mu}{r^2}$, as in No 225, we should assume

$$\mu = f(\text{M} + m)$$

Therefore, this value ought to be substituted in the different equations of elliptic motion which have been given in the preceding numbers, and the equation

$$\frac{4\pi^2 a^3}{\text{T}^2} = \mu$$

of that number will then give

$$\frac{\text{T}^2}{a^3} = \frac{4\pi^2}{f(\text{M} + m)}, \tag{1}$$

T being always the time of the planet's revolution, and $a$ the semi-axis major of its orbit.

It appears, therefore, that the ratio $\frac{\text{T}^2}{a^3}$, inasmuch as it depends on $m$, the mass of the planet, will not be the same for

two planets when their masses are unequal However, as the observations which establish the third law of Kepler prove, that this ratio, if not exactly, is, at least, very nearly constant, it follows, that the masses of the planets are very small relatively to that of the sun, and this is the reason why $\frac{T^2}{a^3}$ the ratio of the square of the time to the cube of the mean distance varies very little, in passing from one planet to another In fact, the mass of Jupiter, which is the most considerable of them all, is less than the thousandth part of the mass of the sun

244. It is on this account that the mutual attraction of the planets produces only *perturbations*, which are either very slow, or very inconsiderable, in the elliptic motion produced by the attraction of the sun In fact, the masses of two planets being $m$ and $m_1$, the motive force directed from the one to the other, is expressed by $\frac{fmm_1}{\rho^2}$ at the distance $\rho$, therefore, the accelerating force of $m$ arising from the attraction of $m_1$, will be $\frac{fm_1}{\rho^2}$, and as the distance $\rho$ never becomes very small relatively to $r$ the distance of $m$ from the sun, it follows, that if $m_1$ be a very small fraction of $\mathrm{M}$, the motion of $m$ produced by the solar attraction, ought to be very little modified by the attraction of $m_1$

The planetary perturbations may, therefore, be determined by the method of the variation of constant arbitraries, which has been already explained in No 229 They are of two different species The one consists of *periodic inequalities*, which are, for the most part, very small, and of which the periods comprise multiples of the revolutions of the disturbing and disturbed planet, that *in general* are inconsiderable. However, when their mean motions are nearly commensurable, these periods may become much longer, and the inequalities much more sensible Thus, as the mean motions of Saturn and

Jupiter are very nearly in the proportion of 2 to 5, Laplace proved, that an inequality results from the mutual attraction of these two planets, whose period is 929 years, and of which the *maximum* is about 48′ in the longitude of Saturn, and very nearly 20′ in that of Jupiter.

The other perturbations of the planets are—1st, the progressive motions of the perihelia and nodes of their orbits, in which these points traverse the entire circumference, in periods of such length, that they may exceed thousands of centuries, 2ndly, the secular variations which affect the excentricities and inclinations of these orbits, and also the mean longitudes of the planets, the periods of these are similar to the preceding, and their amplitudes, though contained within very narrow limits, are not yet very well known.

But while these different elements of elliptic motion simultaneously vary in virtue of the planetary attraction, it is very remarkable, that this force produces no change in the greater axes of the orbits and the mean motions of the planets, which will be the same at all epochs, as also the times of revolutions, that are connected with the greater axes by equation (1). Nevertheless, the secular variations of the mean longitudes produce corresponding ones in the intervals between two consecutive returns to the same fixed point, they are insensible in the motion of the planets, but this is not the case in the motion of the satellites, and particularly in that of the moon, which, on this account, is accelerated from century to century. As the accelerating force which arises from the attraction of a planet $m_1$, on another planet $m$, is independent of the mass of $m$, and proportional to the mass $m_1$, it is easy to conceive how the perturbations due to this force, and which are observed in the motion of $m$ about the sun, may enable us to determine the ratio of the mass $m_1$ to that of this star. Thus, for example, by means of the great inequality in the motion of Saturn produced by the action of Jupiter, we find that the

mass of this last planet is equal to the $\frac{1}{1070}$th of that of the sun.

In the next number we will point out another means of estimating the mass of those planets, which are accompanied by one or several satellites. The comets, on account of the smallness of their masses, do not produce any appreciable effect on the planets, but their motions are deranged by the attractions of the planets, and their perturbations, which have a considerable influence on the epochs of the reappearance of each comet, that is to say, on the interval of time comprised between two consecutive passages through its perihelion, are also determined by the method of No 229

245 Let $m'$ and $m$ denote the masses of a satellite and its primary, and $r'$ the distance of their centres The motive force of the satellite, directed towards the centre of the planet, will be expressed by $\frac{fmm'}{r'^2}$ at this distance $r'$, the coefficient $f$ being the same as in the former expressions. The accelerating force of the satellite in its *apparent* motion about the planet, will be expressed by $\frac{\mu'}{r^2}$, in which

$$\mu' = f(m + m')$$

If $a'$ denotes the semi-axis major of the orbit of the satellite, and $\mathrm{T}'$ the time of its revolution, we shall obtain, by applying equation (1) to its motion,

$$\frac{\mathrm{T}'^2}{a'^3} = \frac{4\pi^2}{f(m + m')},$$

and if these two equations be respectively divided by each other, in order to eliminate the coefficient $f$, there will result,

$$\frac{\mathrm{T}^2}{\mathrm{T}'^2}\frac{a'^3}{a^3} = \frac{m + m'}{\mathrm{M} + m}.$$

Now, with the exception of the moon, the masses of the satellites are very small relatively to those of their respective

primaries, for example, the mass of a satellite of Jupiter is not the ten thousandth part of that of this planet, therefore, in this last equation, we may put $m$ in place of $m + m'$, and as $a$, $a'$, T, T', are given by observation, this equation will enable us to determine the ratio $m$ to M. It was in this way Newton found that the mass of Jupiter was the $\frac{1}{1067}$th of that of the sun, which differs very little from the fraction $\frac{1}{1070}$, that has been since obtained by another method

246. If a planet is attended by several satellites, their mutual attraction, combined with the inequality of the action of the sun on each satellite and on the planet, produces in the elliptic motions of the satellites, perturbations analogous to those we have already adverted to in the case of the planets The perturbations which arise from the reciprocal action of the satellites, will enable us to determine the ratio that their masses bear to that of the planet, whose attraction produces their elliptic motion But as the moon is the only satellite which revolves about the earth, we cannot apply this method to determine its mass, however it may be obtained from other considerations, one of which is the action of this satellite on the waters of the ocean, and which we now proceed to detail.

Let C (fig 56) be the centre of the earth, A that of the moon, M any point whatever of the terrestrial spheroid, also let

$$\text{CA} = a, \quad \text{AM} = \rho, \quad \text{CM} = r,$$

then if we denote the angle ACM by $\lambda$, we shall have

$$\rho^2 = a^2 - 2ar\cos\lambda + r^2,$$

and if from the point M, the perpendicular MB be let fall on the line AC, we shall also have

$$\text{MB} = r\sin\lambda, \quad \text{AB} = a - r\cos\lambda.$$

At the point M, the value of the accelerating force arising from the attraction of the moon, and acting in the direction MA, will be $\frac{fm'}{\rho^2}$, $m'$ denoting the mass of this satellite and $f$ the

same coefficient as in the preceding expressions. The components of this force in the direction of the perpendicular ML and of MD parallel to AC, will be therefore

$$\frac{fm'r \sin\lambda}{\rho^3}, \quad \frac{fm'a}{\rho^3} - \frac{fm'r\cos\lambda}{\rho^3}.$$

If in these expressions we substitute for $\rho$ its value given above, we may neglect the square of $r$, the radius of the earth, as it is about the sixtieth part of $a$; hence if we make $(a)$

$$\frac{fm'r \sin\lambda}{a^3} = \phi, \quad \frac{2fm'r\cos\lambda}{a^3} = \phi',$$

the two components of the lunar attraction will be $\phi$ and $\frac{fm'}{a^2} + \phi'$. Therefore, all the points of the earth will be solicited in a direction parallel to CA, by a constant force equal to $\frac{fm'}{a^2}$, and besides, by the forces $\phi$ and $\phi'$, of which the resultant varies in magnitude and direction, from one point M to another, and at the centre C it vanishes. Now, it is evident, that in virtue of the force $\frac{fm'}{a^2}$, the *entire* mass of the earth is urged towards the moon, by a motion which is common to all its parts, without the points of the fluid parts undergoing any change in their relative position; therefore, the flowing and ebbing of the sea, produced by the action of the moon, arise from the forces $\phi$ and $\phi'$ applied to different points of the ocean. If M be the mass of the sun, and $a$ its distance from the centre of the earth, and if, moreover, $\mu$, $\psi$, $\psi'$, denote what $\lambda$, $\phi$, $\phi'$, become relatively to this star, we shall also have

$$\psi = \frac{f\mathrm{M}r \sin\mu}{a^3}, \quad \psi' = \frac{2f\mathrm{M}r\cos\mu}{a^3},$$

for the components of the force arising from the action of the sun, which contribute to produce the phenomena of the tides.

If they be compared with the forces $\phi$ and $\phi'$, it will appear, that for a point of the ocean, whose radius vector $r$ makes the same angle $\lambda$ or $\mu$ with the radius vector of the moon's or or sun's orbit, the actions of these two stars, which produce the oscillations of the sea, are to each other as their masses, divided by the cubes of their respective distances from the centre of the earth. Now, we may conceive that, when every thing else is the same, the magnitudes of these oscillations must be to each other, as the corresponding forces; if, therefore, $\omega$ represents the ratio of the lunar to the solar tide, in the same place of the earth and for similar positions of these two stars, we shall have(*b*)

$$\frac{m'}{a^3} = \frac{\omega M}{a^3},$$

in which equation we assume, that $a$ and $a$ denote the mean distances of the moon and sun from the earth, hence we obtain

$$\frac{m'}{m} = \omega \frac{a^3}{a^3} \frac{M}{m},$$

$m$ denoting the mass of the earth.

As the lunar and solar tides may be perfectly distinguished from each other, by means of the different laws to which they are respectively subject, their ratio in each place of the earth may be determined. From a mean of a great number of observations made in the harbour of Brest, we obtain

$$\omega = 2{,}3533,$$

for the value of this ratio. The distance $a$ is very nearly 400 times the distance $a$, and the mass M is also, as we shall see immediately, very nearly 355000 times the mass of $m$. By substituting these values in the preceding formula, we find the mass of the moon to be equal to the $\frac{1}{75}$th of that of the earth. Besides the oscillations of the fluid part of the earth, the actions of the sun and moon produce also in the motion of the terrestrial spheroid, about its centre of gravity, in conse-

quence of its not being perfectly spherical, perturbations, the nature of which will be investigated, when the motion of rotation of solids comes to be treated of.

247 We may remark here, that if each of the forces $\phi$ and $\phi'$ be resolved in the direction of ME, the production of the radius vector CM, their sum will be $\phi' \cos\lambda - \phi \sin\lambda$; so that its value is(*c*)

$$(2\cos^2\lambda - \sin^2\lambda)\frac{fmr}{75a^3}$$

This is the diminution of gravity at the point M, produced by the action of the moon Now, if M be supposed to exist on the surface of the earth, we have $fm = gr^2$ very nearly, $g$ denoting the gravity at this point, moreover, $2\cos^2\lambda - \sin^2\lambda$ is a maximum when $\lambda = 0$, in which case its value is equal to 2. Consequently, the greatest value of this diminution of gravity will be $\frac{2gr^3}{75a^3}$, a quantity which is very nearly equal to the eight thousandth part of $g$, on the supposition that the ratio $\frac{a}{r}$ is equal to 60 Therefore, in order that the influence of the action of the moon, on the length of a pendulum that vibrates seconds should be appreciable, the approximation ought to be continued as far as the second decimal beyond the hundredth thousandth place, at which point we generally stop in expressing the measure of its length This influence will produce, in the measure of the time, an inequality depending on the motion of the moon, which, when a *maximum*, does not amount to half a hundredth of a second in a day.

248 If we do not take into account the centrifugal force that arises from the rotation of the earth, the weight which is observed at its surface is the resultant of the attractions that all the points of the spheroid exercise on each material point, which resultant depends solely on the position and mass of this point, and not at all on the nature of the body to which it belongs, and this, in fact, has been fully confirmed by ex-

periment. The intensity of this force should diminish according as we ascend above the surface of the earth; this diminution is indicated by observations of the pendulum made at different elevations Besides, the terrestrial gravity, diminished in the ratio of the square of the radius of the earth to the square of the radius of the lunar orbit, should be equal to the accelerating force, which retains the moon in its orbit. But, as the distance of this satellite is very nearly sixty times the radius of the earth, it follows, that if the moon fell from a state of rest, it would move towards the earth in a *minute*, through the same space that any body would in a *second* fall through, in a vacuo, at the surface of the earth This quantity is, in fact, the versed sine of the arc which the moon describes in a minute in its orbit, or very nearly the square of this arc divided by the diameter of this curve, and as the circumference of the orbit is 60 times that of the earth, it follows, that the quantity in question is equal to 40 millions of metres, multiplied by $\frac{60\pi}{n^2}$, $n$ denoting the number of minutes contained in the time of a revolution of the moon Therefore, it appears from the numerical value of $g$, which is determined by experiments made with the pendulum, that this product must be very nearly equal to $4^m,90$, it has been found, in fact, equal to $4^m,88$, $n$ being assumed equal to 39343(*d*); and the difference will be still less, if we take into account the various circumstances, the consideration of which we omitted in order to simplify the demonstration. It follows from this, that the terrestrial weight is only a particular case of universal attraction, and, on this account, this general force has been likewise termed *the weight*, or *universal gravitation*

249 As the earth deviates very little from the spherical form, the attraction which it exercises on a point of its surface is very nearly $\frac{fm}{r^2}$, which is the expression for that of a sphere, in which $m$ denotes its mass, $r$ its radius, and $f$ the coefficient of universal

attraction. This approximate value should be altogether exact for points which exist on a certain *parallel*, and, it appears from the theory of the attraction of spheroids, which differ little from spheres, that this parallel is that of which the square of the sine of the latitude is $\frac{1}{3}(e)$. On this parallel, the measure of the gravity is $9^m,79386$ (No. 193); but, in order that it may be accurately equal to the terrestrial attraction, it should be previously increased by the vertical component of the centrifugal force, which component is, under this parallel, equal to the fraction $\frac{2}{3 \cdot 289}$ of the gravity (No. 178).

Consequently if we make

$$g = (9^m,79386)\left(1 + \frac{2}{3 \cdot 289}\right) = 9^m,81645,$$

we may regard this value of the gravity, thus corrected, as equal to the attraction of the earth, and assume

$$g = \frac{fm}{r^2}.$$

By multiplying the members of this equation by the corresponding members of equation (1) of No. 243, applied to the motion of the earth about the sun, we shall obtain

$$\frac{m}{\mathrm{M} + m} = \frac{g\mathrm{T}^2 r^2}{4\pi^2 a^3},$$

which formula enables us to determine the ratio of the mass of the earth to that of the sun.

If we conceive a right angled triangle, of which the base is the radius of the earth, and whose height is its distance from the sun, then the small angle at the sun opposite to the base is the *parallax* of the sun, which can be determined either directly by astronomical observations, or theoretically by a certain inequality produced in the motion of the moon by the action of the sun, and which has been termed the *parallactic* inequality. The magnitude of the parallax depends on the the radius of the earth, and on the distance of the sun from

the earth; for the mean distance $a$, and for the radius drawn to the parallel of which the square of the sine of the latitude is $\frac{1}{3}$, its value is $8'',60$. Consequently, we have

$$\frac{r}{a} = \text{tang}.8'',60, \quad a = (23984)\, r.$$

If we assume the compression of the earth equal to $\frac{1}{290}$, we have under this same parallel,

$$r = 6364551^{m},$$

for the value of the radius $r$. The time of its revolution about the sun, expressed in seconds, is

$$\text{T} = (86400)\,(365{,}256374).$$

By means of these values, and that of $g$, in which it is also assumed that the second is the unit of time, we find

$$m = \frac{\text{M}}{354592}.$$

250. The sun is a sphere, of which the radius is 110 times that of the earth; therefore, as the ratio of the volumes of these two bodies is known, and also that of their masses, we may infer at once that of their mean densities. It appears by this method, that the density of the sun is, very nearly, the fourth part of that of the earth. At the surface of this star, the attraction is expressed by

$$\frac{f\text{M}}{\text{R}^2},$$

R representing its radius. And since

$$\text{R} = 110\, r, \quad g = \frac{fm}{r^2},$$

this quantity is equal to

$$\frac{g\text{M}}{(110)^2 m},$$

which, by substituting for $\frac{\text{M}}{m}$ its numerical value, becomes

(29,5) $g$. As the duration of the sun's rotation on its axis is $25^d,5$, the centrifugal force at its equator is only the sixth part of this force at the equator of the earth. Therefore, if we neglect the diminution which it produces in the weight at the surface of the sun, it is evident that the weight of a body at this surface is 29 times and a-half the weight of the same body at the surface of the earth, and that bodies descend there through 135 metres in the first second of their fall.

If equation (1) of No 243 be applied successively to the earth and to any other planet, the quantities $m$, $a$, $\text{T}$, relatively to the earth, being supposed to become $m_1$, $a_1$, $\text{T}_1$, with respect to the planet, we shall obtain from it, by eliminating $f$,

$$\frac{a_1^3}{a^3} = \frac{\text{M} + m_1}{\text{M} + m} \cdot \frac{\text{T}_1^2}{\text{T}^2},$$

If the value of $a$ be known from an observation of the solar parallax, and if likewise, $m$ the mass of the earth, and $\text{T}$ the duration of the sidereal year be known, this equation will enable us to determine $a_1$ the semi-axis major of any planet, when $m_1$ its mass, and $\text{T}_1$ the time of its revolution are known. In the method given in No 245 for determining this mass, it is only supposed that an approximate value of the semi-axis major is known.

251 The attraction which a considerable mass, such as a high mountain, exercises at the earth's surface, causes heavy bodies to deviate from the vertical direction, consequently, the production of the *plumb-line* will not in such a case meet the heavens in the *zenith*. It will be deflected from it, in contrary directions, at the two opposite sides of the mountain, so that if every thing corresponds on one side and the other, with respect to the form of the mountain, and the quantity of the deviation of the plumb-line from the vertical, the angular distance of the two stars through which its production passes, will be double of its deviation ($f$). This effect has been observed at Peru and in Scotland, but, because the masses of the highest

mountains are very small, relatively to the mass of the earth, the deviations in question are also inconsiderable, and amount only to a small number of seconds We proceed to give an example of the calculation of the deviation of the plumb-line, caused by the attraction of a given mass

Let A (fig 57) be the centre of a homogeneous sphere suspended at the extremity of an inextensible and inflexible thread, of which the other extremity is attached to the fixed point C, also let O be the centre of another homogeneous sphere which acts on the first. The thread CA will be drawn from the vertical CB, without, however, deviating from the vertical plane which passes through this line and CO, and when it is in equilibrio, the resultant of the weight of the first sphere and of the attraction of the second, should pass through the fixed point C; now, these two forces will be applied to the point A, the one in the vertical direction AD, and the other along the line AO, and their tendency is to make the thread CA turn in opposite directions about the point C In order, therefore, that their resultant should pass through the point O, it is necessary that their moments with respect to this same point, should be equal (No 46), consequently, if P and Q denote respectively the weight of the first sphere and the entire attraction of the second, and if $p$ and $q$ denote the perpendiculars CE and CF, let fall from the point C on the productions of DA and OA, we shall have

$$Pp = Qq,$$

for the equation of equilibrium, by means of which the unknown deviation BCA can be determined. If $x$ denotes this angle, $\gamma$ the given angle BCO, $a$ and $c$ the distances CA and CO which are also given, and $y$ the unknown distance AO, we shall have

$$y^2 = a^2 + c^2 - 2ac\cos(\gamma - x),$$

and, besides,

$$\sin \text{COA} = \frac{a\sin(\gamma - x)}{y}, \quad q = \frac{ac\sin(\gamma - x)}{y}, \quad p = a\sin x.$$

Likewise if $m$ denotes the mass of the earth, $m_1$ that of the attracted sphere, and $m'$ that of the attracting sphere, the motive forces P and Q will have for values

$$P = \frac{fmm_1}{r^2}, \quad Q = \frac{fm'm_1}{y^2},$$

$f$ and $r$ denoting, as before, the coefficient of universal attraction, and the radius of the earth, also, if $\rho$ be the mean density of the earth, $\rho'$ that of the attracting sphere, and $r'$ its radius, we shall have

$$m' = \frac{m\rho' r'^3}{\rho r^3}$$

By means of these different values, the equation $Pp = Qq$ will be changed into ($g$),

$$\rho r y^3 \sin x = \rho' r'^3 c \sin (\gamma - x),$$

in which it is only necessary to substitute the value of $y$ given above, to determine that of $x$

If we assume that CA, the length of the plumb-line, is very small with respect to the distance CO, which is generally the case, we may neglect $a$ relatively to $c$ in the values of $y$, and we shall have simply $y = c$, hence there results

$$\frac{\sin x}{\sin (\gamma - x)} = \frac{\rho' r'^3}{\rho r c^2}$$

The density $\rho'$ and the radius $r'$ of the attracting sphere remaining the same, the value of $x$, which is deduced from this equation, will be so much the greater, according as the distance $c$ is less, and the angle $\gamma$ approaches to a right angle, and as $c$ can never be less than the radius $r'$, it follows, that the *greatest* deviation of the plumb-line, which the attraction of a given sphere can produce, will be obtained by assuming $c = r'$ and $\gamma = 90$, from which suppositions we deduce

$$\text{tang}\, x = \frac{\rho' r'}{\rho r}$$

If, for example, we suppose $\rho' = \rho$, and that it were required to know the magnitude of the radius $r'$, when the deviation $x$ amounted to one second, we would have $r' = r$ tang $1''$, and, because $2\pi r$ the circumference of the earth is 40 millions of metres, it follows that $r = 30^m,856..$ Hence it appears, that a homogeneous sphere, of which the radius is about 31 metres, and whose density is equal to the mean density of the earth, produces only a deviation of one second, at the most, in the direction of the plumb-line, and in order that it may produce this, it is necessary that it touch the inferior extremity of the string, and that its centre should be situated in the horizontal plane passing through this extremity.

252 This mean density of the earth, which has been inferred from the deviation of the plumb-line produced by the attraction of mountains, has been estimated at four or five times that of water Cavendish, who determined it by the attraction exercised by two leaden spheres, eight inches in diameter, which he rendered sensible by means of *the balance of torsion*, found it equal to five times and a-half this density Without entering herein to all the details of this beautiful experiment, the different precautions which it requires, and the computations that must be made in order to produce an exact result, we shall only briefly indicate the principal points of these computations The balance of torsion is the most exact instrument which we have for measuring very small forces Coulomb, to whom we are indebted for its invention, has chiefly employed it, in measuring the attractive and repulsive forces of electrified bodies, and, on this account, it is also known in physics, by the name of the *electric balance* It consists principally of a very fine vertical metallic thread, attached to a fixed point and having a *horizontal(h)* lever suspended at its extremity This lever consists of a slender rod ACA′ (fig 58) divided into two equal parts at the point C, to which the thread is attached, and it is terminated by two spheres of a small diameter, whose centres are A and A′. From the point C as centre, and with a

radius equal to CA, let a horizontal circle BAB′A′ be described, and let its circumference be divided into a great number of equal parts When the lever turns about the point C, its extremities A and A′ will trace this circumference, and the points of division to which they correspond at each instant, will make known the arcs that they have described As long as the suspended thread which terminates at the point C, is not twisted, the lever quiesces in a certain position Suppose that then it exists on the line BCB′, if we cause it to deviate from this line, and make it assume any other position ACA′, the suspended thread will be twisted on itself, and this torsion will tend to bring the lever back towards the line BCB′ Let us suppose that, in order to retain it in the direction ACA′, there is applied to its two extremities equal and opposite forces, directed in the horizontal plane, and perpendicular to its length, the common value of these two forces will be the measure of the force of torsion with which they constitute an equilibrium Now, Coulomb has proved by experiment, that, when the thread of suspension remains the same, this force of torsion is proportional to the angle BCA; therefore, if we take the right angle to represent unity, and $h$ to denote the force of torsion which corresponds to this angle, $\theta$ the angle BCA, this force will be equal to $h\theta$, in the position ACA′ of the lever, thus, in this position, the torsion of the suspended thread is equivalent to two horizontal forces, equal to $h\theta$, applied at the points A and A′ perpendicularly to ACA′, and which tend to bring back the lever to the line of repose BCB′

This being premised, let us cause two equal homogeneous spheres, consisting of the same matter, and symmetrically placed on opposite sides of the line BCB′, to approach the lever, and let O and O′ be their centres, situated in the horizontal plane which contains the lever, equally distant from C, and existing on OCO′, the line drawn through this point The attraction of these two bodies will cause the lever to deviate from the line BCB′, and, because every thing is similar about

the centre C, the line ACA will turn about this point, which will remain immoveable. According as the lever deviates from the line of repose the force of torsion will increase, and there exists a position in which this force constitutes an equilibrium with the attraction of the two spheres; but as the lever attains this position with an acquired velocity, it passes beyond it, and oscillates on one side and another in the same manner as a horizontal pendulum. The duration of an entire oscillation may be known from observation, and by comparing the length of this pendulum with that of a common pendulum, which vibrates in the same time, the ratio of the force of attraction of each sphere to gravity may be inferred; and, consequently, the ratio of the mass of this sphere to that of the earth. It is easy to obtain the equation, by means of which this ratio can be determined, as we now proceed to shew.

253. Since the two spheres whose centres are in A and A′, are solicited by the same forces, and have the same motion about the fixed point C, it will be sufficient to consider the motion of the centre of one of them, the point A for example; if, therefore, as in the preceding problem,

$$CA = a, \quad CO = c, \quad BCO = \gamma;$$

and if, as before, $m'$ denotes the mass of the attracting sphere, and $f$ the coefficient of universal attraction, we shall have

$$z^2 = a^2 + c^2 - 2ac\cos(\gamma - \theta),$$

$z$ denoting the distance AO, and $\theta$ the angle ACB at the end of any time $t$, also the accelerating force arising from the attraction in the direction AO will be $\dfrac{fm'}{z^2}$. If this force be resolved into two other forces, one in the direction of the production of CA, and the other perpendicular to CA, this last component will be equal to $\dfrac{fm'}{z^2}\sin CAO$, that is to say, to $\dfrac{fm'c}{z^3}\sin(\gamma - \theta)$, by substituting for sin CAO its value deduced from the triangle COA.

If from this component, which acts in the direction of a tangent to the trajectory, we take the force of torsion $h\theta$, which is directly opposed to it, we shall have, as the arc BA described by the sphere is equal $a\theta$,

$$\frac{ad^2\theta}{dt^2}=\frac{fm'c}{z^3}\sin(\gamma-\theta)-h\theta,$$

for the equation of the motion (No. 152).

As the attraction of the mass $m'$ is a very small force, the angle $\theta$, by which the lever ACA′ deviates from the line of repose, is very small. Denoting the distance BO, or the value of $z$ which corresponds to $\theta=0$, by $b$, in which case we have

$$b^2=a^2+c^2-2ac\cos\gamma,$$

there results by developing according to the powers of $\theta$(i)

$$\frac{\sin(\gamma-\theta)}{z^3}=\frac{\sin\gamma}{b^3}-[(a^2+c^2)\cos\gamma-2ac-ac\sin^2\gamma]\frac{\theta}{b^5}+\&c$$

Therefore, if in order to abridge, we make

$$[(a^2+c^2)\cos\gamma-2ac-ac\sin^2\gamma]\frac{fm'c}{b^5}+h=g',$$

$$\frac{fm'c\sin\gamma}{b^3}=\beta g',$$

and if powers of $\theta$ higher than the first are neglected, the equation of the motion will become

$$a\frac{d^2\theta}{dt^2}=g'\beta-g'\theta,$$

hence, by integrating, we obtain

$$\theta=\beta+k\cos\left(t\sqrt{\frac{g'}{a}}+k'\right),$$

$k$ and $k'$ being two arbitrary constants.

It appears from this value of $\theta$, that the least and greatest deviations of the lever ACA′, reckoning from the line BCB′, will be $\beta-k$ and $\beta+k$, and, if the line DCD′ be drawn in

such a manner, that the angle BCD may be equal to $\beta$, the lever will perform on each side of this line, equal and isochronous oscillations, whose amplitude will be equal to the constant $k$, the angle $\beta$ is determined by measuring the least and greatest deviation of the lever, and then, taking for this angle, half the sum of these extreme values of $\theta$ The line DCD′ that answers to $\theta = \beta$, is the position of the lever in which it would remain in equilibrium, if it attained to it, without any acquired velocity The duration of each entire oscillation of the lever, on one side and the other of this line, will be the time in which the angle $t\sqrt{\frac{g'}{a}} + k'$ increases by 180°; therefore, if we denote it by T, we shall have

$$T = \pi\sqrt{\frac{a}{g'}},$$

and this duration T will be likewise given by observation

Now, if $g$ denotes the force of gravity, and $l$ the length of the simple pendulum, which makes infinitely small oscillations in the time T, we have (No. 182),

$$T = \pi\sqrt{\frac{l}{g}};$$

consequently,

$$ga = g'l,$$

and hence, because,

$$g = \frac{fm}{r^2}, \quad g' = \frac{fm'c \sin\gamma}{\beta b^3},$$

we shall have, finally,

$$\frac{m'}{m} = \frac{\beta a b^3}{c l r^2 \sin\gamma}$$

$m$ being the mass of the earth and $r$ its radius.

As all the quantities contained in this formula are known in each experiment, it will enable us to determine the ratio of the mass $m'$ to that of the earth, and as besides, the volumes

of these two bodies and the density of $m'$ are known, we can deduce from it the mean density of the earth

254. It is demonstrated in the Celestial Mechanics of Laplace, that in order to insure the stability of the equilibrium of the sea, it is necesssary and sufficient that the mean density of the earth should surpass that of water It is in consequence of this condition being fulfilled, that the forces arising from the simultaneous actions of the sun and moon produce only very small oscillations, if this was not the case, and if the earth, for example, while its mean density continued to be what it is, was covered over by a sea of mercury, the action of the least forces extraneous to the terrestrial spheroid would produce in this fluid a progressive motion, in consequence of which it would traverse the entire surface of the earth.

It may be also proved, from various considerations, that the density of the concentrical strata of the terrestrial spheroid must increase as we approach the centre, hence it follows, that its mean density must surpass that of the superficial stratum, a condition which is found in fact to be established, for with the exception of the metals, that constitute a very small part of this stratum, the density of the other materials of which it is composed, are all of them much less than five times and a-half the density of water

But it should be observed, that this increase of density does not imply the existence of substances entirely different from those which are observed at the surface, and of which the actual density is excessively great, it may be assumed, that all the strata of the earth are composed of the same kind of matter, a little compressible, or of a variety of different substances, as is the case at the surface, and on this hypothesis, which appears to be the most natural, their increase of density would arise from the condensation, produced by the pressure of those above them, and which continually augments from the surface to the centre.

In the interior of the earth, the law of attraction depends

on the unknown law of the densities, beyond the surface, it varies on the production of each radius, very nearly in the inverse ratio of the square of the distance from the centre and it experiences at the same time, from one radius to another, a variation proportional to the square of the cosine of the angle which each radius makes with the axis of figure of the terrestrial spheroid(*k*) It follows from a consideration of this last variation, that at equal distances from the centre of the earth, the force applied to the centre of the moon, and arising from the attraction of this spheroid, is not the same in all directions of the radius vector, so that this force may be considered as composed of two others, the one arising from the spherical part of the earth, and which may be regarded as constant, when the distance from the centre of the earth remains the same, the other is due to the protuberance, or excess of matter at the equatorial parts of the earth, and this varies with the direction of the radius with respect to the axis of the poles Laplace has determined the small inequality in longitude and in latitude, which this second force produces in the motion of the moon, assuming that its magnitude depends on the compression of the earth, and comparing it with that furnished by observation, the compression of the earth is found to be equal to $\frac{1}{305}$, which differs very little from that which results from taking the mean of a great number of measures of the pendulum and of degrees of the meridian(*l*)

At the surface of the earth, the variation of gravity arising from that of the attraction and of the centrifugal force, follows the same law as at any distance whatever from the centre, that is to say, it is, as has been already stated, (No 178), proportional to the square of the cosine of latitude When it is proposed to verify this law by taking the lengths of pendulums which vibrate seconds, we should take care not to make the observations near to a mountain, for the *horizontal* component of the attraction of the mountain will cause the pendulum to deviate from the vertical, in its position o[…]

equilibrium, at the same time that the vertical component of this force diminishes the gravity, and consequently, the length of the simple pendulum, that vibrates seconds Even when this cause of anomaly is avoided, it is still found, that in certain places, the length of the pendulum which vibrates seconds deviates from the law given by theory, this must be owing to the density of the earth in those places, being for a considerable depth and extent, greater or less than the general density of the superficial stratum, hence arises an increase or diminution of the total gravity, and, consequently, of the length of the simple pendulum, which is proportional to its intensity The pendulum may, in this point of view, be considered as a geological instrument, that indicates, by its anomalies, variations of great extent in the nature of the strata

Finally, it should be observed, that the law of the decrease of gravity, proportional to the square of the cosine of latitude, as we proceed from the pole to the equator, supposes that we assume for the surface of the earth, the production of the level of the seas, and as the elevations of those places on land, where observations have been made, above this level, are different, the observed lengths should be reduced to those which they would have at this level itself in each vertical. This reduction is commonly made by increasing the gravity and the length of the pendulum which vibrates seconds, in the ratio of the square of the distance of the place of observation from the centre of the earth, to the square of this same distance diminished by the height of this place above the level of the sea, it thus appears that the attraction of the stratum of earth comprised between the surface of the continent and the production of the sea is neglected. But this correction is too great by nearly one-half, as is evident from the following considerations

255. Let AM′B (fig 59) be the surface of a continent, DAMBE the level of the sea and its production, C the centre

of the earth, also let M′ be the place of observation, and M the point where the radius CM′ meets this production, let M′M, the height of the point M′ above the surface of the sea, be represented by $h$, (this can be determined either by levelling, or by barometrical measurements) If M′ was *very near* to the sea, the gravity would be a little *diminished*, and its direction a little deranged, because the density of the water is less than that of the earth, but for the present we shall suppose that this is not the case, we shall also assume that the surface of the earth about M′ is horizontal, or sensibly perpendicular to the radius CM′, and that its density is uniform. The question then is to compute the attraction exercised at the point M′, by the stratum AM′BM elevated above the level of the sea In this computation, it is not necessary to take into account either the curvature of this stratum, or the variation of its thickness, i e, in other words, we may consider the thickness of this stratum as constant and equal to $h$, through the entire extent in which its thickness can be sensible Let $c$ denote the radius of this extent, and $\rho'$ the density of the stratum

This being premised, let K be any point whatever of the attracting stratum, $z$ and $y$ its distances from the surface of the earth and from its radius CM′, and let two cylindrical surfaces be described which may have MM′ for their common axis, and whose radii may be $y$ and $y + dy$. The volume comprised between these two surfaces will have $2\pi y dy$ for its base and $dz$ for its height, and if it be decomposed into horizontal rings, whose thickness is infinitely small, the volume of the ring corresponding to the point K, will be $2\pi y\, dy\, dz$, and its mass $2\pi\rho' y\, dy\, dz$ The attraction of this ring on a material point situated at M′, will be reduced to a force acting in the direction MM′, which will be equal to the sum of the vertical components of the attractions of all its points, and since for any point K, we have

$$\text{KM}' = \sqrt{y^2 + z^2}, \quad \cos \text{KM}'\text{M} = \frac{z}{\sqrt{y^2 + z^2}},$$

the expression for the accelerating force arising from the attraction of the entire ring, will be

$$\frac{2\pi f\rho' yz\,dy\,dz}{(y^2+z^2)^{\frac{3}{2}}},$$

$f$ always representing the coefficient of universal attraction. Consequently, in order to obtain the attraction of the stratum in question, this formula should be integrated from $z=0$ to $z=h$, and from $y=0$ to $y=c$, which gives $(m)$

$$k' = 2\pi f\rho'\left(c+h-\sqrt{(c^2+h^2)}\right),$$

$k'$ denoting this force But in general, the vertical thickness of the attracting stratum is small, relatively to its horizontal radius, therefore, if $h^2$ be neglected with respect to $c^2$, we shall have simply,

$$k' = 2\pi\rho f' h.$$

Let $k$ denote the attraction exercised at the point M by the part of the earth which is terminated by the level of the sea, and $r$ the radius CM, this attraction will become at the point M′

$$\frac{kr^2}{(r+h)^2}$$

Denoting the weight and the vertical component of the centrifugal force at the point M by $g$ and $\gamma$, respectively, and by $g'$ and $\gamma'$ at the point M′, we shall have therefore

$$g = k-\gamma, \quad g' = \frac{kr^2}{(r+h)^2}+k'-\gamma'$$

Developing the first term of the value of $g'$ according to the powers of $h$, and then taking $g'$ from $g$, we obtain, by neglecting the square of $h$, and the small difference $\gamma'-\gamma$, $(n)$

$$g-g' = \frac{2kh}{r}-k'$$

Since the factor $\frac{h}{r}$ is very small, we may assume $k=g'$ in

the first term of this formula, in the value of the small quantity $k'$, we may likewise suppose

$$\frac{4\pi\rho f r}{3} = g',$$

$\rho$ denoting the mean density of the earth, and then assuming its volume equal to $\frac{4\pi r^3}{3}$, there will result

$$k' = \frac{3\rho' h g'}{2\rho r},$$

and, consequently, (o)

N

$$g = g'\left(1 + \frac{2h}{r} - \frac{3\rho' h}{2\rho r}\right)$$

Therefore, it appears from this, that it is by the factor comprised between these parentheses, and not by the factor $1 + \frac{2h}{r}$, as is usually done, that we ought to multiply the weight $g'$, which has place on the continent at the height $h$ above the level of the sea, in order to reduce it to this level. In general, we may estimate $\rho'$ as equal to the half of $\rho$, and, consequently, assume $1 + \frac{5h}{4r}$ for this factor (p) At Paris, $h$, the elevation of the point of the observatory where the barometer is placed, is about 63 metres, hence it follows, that the gravity and length of the pendulum which vibiates seconds, is less there than at the level of the sea in the ratio of one to 1,0000125

# BOOK THE THIRD.

## STATICS

## SECOND PART.

### CHAPTER I.

#### OF THE EQUILIBRIUM OF A SOLID BODY.

256. There is no body whatever that is not more or less compressible, and which does not consequently change its form, when it is subjected to the action of the forces which constitute an equilibrium. But when the solid body which we now proceed to consider has assumed the suitable form, then the points of application of the forces which solicit it, may be regarded as a system of an invariable form, and it is to this state, that the coordinates of these different points, which occur in the equations of equilibrium and are assumed to be known, refer.

Let $M$, $M'$, $M''$, &c., be this system of material points. In the case of each point, there are seven quantities to be considered, namely, its three coordinates, the force which solicits it, and the three angles which determine its direction. Let $P$ denote the force which is applied to the point $M$, $MD$ its direction (fig. 60), $x$, $y$, $z$, the three coordinates $OG$, $GH$, $HM$ of the point $M$ referred to the rectangular axes $Ox$, $Oy$, $Oz$; $\alpha$, $\beta$, $\gamma$, the angles, either acute or obtuse, which the line $MD$ makes

with the parallels to these axes drawn through the point M. Relatively to the other points M′, M″, &c., the analogous quantities are represented by the same letters with corresponding accents.

This being premised, previously to investigating the conditions of equilibrium of the given forces P, P′, P″, &c , we shall proceed to transform this system of forces into three others, of which one will be the resultant of forces parallel to the axis $oz$, another of forces parallel to the axis $oy$, and acting in the plane of the axes of $x$ and $y$, and the third of forces acting in the direction of the axis $Ox$.

257. Let each of the forces P, P′, P″, &c , be decomposed, without changing its point of application, into three forces parallel to the axes of $x$, $y$, $z$ , $P\cos\alpha$, $P'\cos\alpha'$, $P''\cos\alpha''$, &c , will be the forces parallel to the axis $ox$ , $P\cos\beta$, $P'\cos\beta'$, $P''\cos\beta''$, &c., the forces parallel to the axis $oy$, $P\cos\gamma$, $P'\cos\gamma'$, $P''\cos\gamma''$, &c., the forces parallel to the axis $oz$ , the given forces may at once be replaced by these three groups of parallel forces.

We can, without at all altering the system of forces under consideration, apply to the same point, two equal and parallel forces Therefore, at the point M, let two forces $g$ and $-g$, equal and directly opposed, be applied, parallel to the axis of $z$ Let the force $g$ which acts in the direction MC, be compounded with the force $P\cos\alpha$ acting in the direction MA parallel to $ox$, and let ME be the direction of their resultant, and K the point where its production meets the plane of the axes of $x$ and $y$ , if its point of application be transferred to the point K, and if it be then decomposed into two forces parallel to the axes of $x$ and $z$, the forces $P\cos\alpha$ and $g$ will be reproduced, but the force $P\cos\alpha$ is now directed along the projection of its first direction on the plane of the axes of $x$ and $y$, and the force $g$ is applied perpendicularly to this plane, at the point K of this projection, the coordinates of which are easily determined.

In fact, H being the projection of M on the plane of the axes of $x$ and $y$, its coordinates will be $x$ and $y$, and $y$ and $x$ — KH will be those of the point K, since these two points belong to the same parallel to the axis of $x$. Now, since in the rectangle KNMH, its diagonal KM is the direction of the resultant of the forces $g$ and P $\cos\alpha$, which act in the direction of the sides KN and KH, we have

$$\text{KH} \quad \text{HM} \;.\; \text{P} \cos \alpha \quad g,$$

hence we obtain, because HM $= z$,

$$\text{KH} = \frac{z\,\text{P} \cos \alpha}{g},$$

consequently, the coordinates of K, the point of application of the force $g$ in the plane of the axes of $x$ and $y$, are

$$y \text{ and } x - \frac{z\,\text{P} \cos \alpha}{g}.$$

By performing the same operations on the forces P $\cos\beta$ and $-g$, the first will be transferred on the plane of the axes of $x$ and $y$, along the projection of its first direction, and the coordinates of the new point of application of the force $-g$, in this same plane, will be

$$y + \frac{z\text{P} \cos \beta}{g} \text{ and } x.$$

If in the same manner, all the forces P$'\cos\alpha'$, P$''\cos\alpha''$, &c., P$'\cos\beta'$, P$''\cos\beta''$, &c., be transferred in the plane of the axes of $x$ and $y$, each of these forces will act along the projection on this plane, of its primitive direction, which may be either above or below this same plane, and besides, there will be as many couple of forces, $g'$ and $-g'$, $g''$ and $-g''$, &c., as there are points M$'$, M$''$, &c. The expressions for the coordinates of the points of application of these last forces, in the plane of the axes of $x$ and $y$, may be inferred from those which express those of the forces $g$ and $-g$, by accenting the letters $x$, $y$, $z$, $g$, P, $\alpha$, $\beta$.

258. Now, if a similar operation be performed on the forces $P\cos a$, $P'\cos a'$, $P''\cos a''$, &c, parallel to the axis of $x$, and comprised in the plane of the axes of $x$ and $y$, they will also be transformed into two groups of forces, one of which will be composed of forces parallel to the axis $Oy$, and the other of forces acting in the direction of the axis $Ox$

Thus, let two forces parallel to $Oy$, and represented by $h$ and $-h$, be applied to the point H (fig 61), at which the force $P\cos a$ acts in the direction HF, and let the force $h$, acting in the direction HB, be compounded with the force $P\cos a$, then if the point of application of their resultant be transferred to the point Q, where the production of its direction HK meets the axis $Ox$; and if it be decomposed into two forces in the rectangular directions $Qx$ and $Qy$, the forces $P\cos a$ and $h$ will be reproduced at this point Q, moreover we shall have

$$\text{GQ} \quad \text{GH} \quad P\cos a \quad h,$$

and, because $\text{OG} = x$ and $\text{GH} = y$, we shall obtain

$$\text{OQ} = x - \frac{y\,P\cos a}{h},$$

for the abscissa of the point Q

Therefore, the force $P\cos a$, the direction of which was HF, will be replaced by a force $P\cos a$, acting in the direction of the axis $Ox$, and two forces $h$ and $-h$, perpendicular to this axis, and applied to the points Q and G, the positions of which are known The same will be the case for the other forces $P'\cos a'$, $P''\cos a''$, &c., parallel to the axis of $x$, and comprised in the plane of the axes of $x$ and $y$, for these will be in like manner replaced by the forces $P'\cos a'$, $P''\cos a''$, &c, acting in the direction of the line $Ox$, and by the couples of forces $h'$ and $-h'$, $h''$ and $-h''$, &c, parallel to the axis $Oy$

259. It appears, therefore, that by means of these two successive operations, the given forces will be transformed, as has been stated, into three groups of forces, acting in the di-

rection of the axis of $x$, in that of a perpendicular to this axis, and comprised in the plane of $x$ and $y$, and in the direction of a perpendicular to this plane.

In this transformation, any of the forces such as P will be replaced by six others, namely,—1st, the three forces $P\cos\gamma$, $g$, $-g$, parallel to the axis of $z$, the coordinates of whose points of application on the plane of the axes of $x$ and $y$, and referred to the axes O$x$ and O$y$, will be, for the first, $x$ and $y$, for the second, $x - \frac{zP\cos\alpha}{g}$ and $y$, for the third, $x$ and $y + \frac{zP\cos\alpha}{g}$.

2ndly The two forces $P\cos\beta - h$ and $h$, parallel to the axis of $y$, comprised in the plane of the axes of $x$ and $y$, and which may be supposed to be applied to the axis of $x$, the first at the distance $x$ from the point O, the second at the distance $x - \frac{yP\cos\alpha}{h}$

3rdly The force $P\cos\alpha$ acting in the direction of the axis of $x$, the point of application of which may be transferred to O.

260 It is easy now to form the equations of equilibrium of the given forces P, P′, P″, &c, or of the three groups of forces which have been substituted for them

It ought, however, in the first place to be remarked, that this equilibrium cannot exist, unless it obtains separately for each of these three groups of forces In fact, if the forces parallel to the axis of $z$ do not destroy each other's effect, and if, notwithstanding, the equilibrium of all the forces was possible, we could, without disturbing this equilibrium, *fix* a line traced in the plane of the axes of $x$ and $y$, but then the forces comprised in this plane will be destroyed either because they will meet this fixed axis, or because they will be parallel to it It is, therefore, permitted to suppress them, and if this is done, the equilibrium will be deranged, contrary to hypothesis, since there is nothing to prevent the forces perpendicular to the plane of the axes of $x$ and $y$, from causing the solid body to revolve about the fixed axis, consequently,

the equilibrium will be impossible, unless these last forces separately destroy each other's effects. In the same manner it may be shewn that the equilibrium cannot subsist between the forces comprised in the plane of the axes of $x$ and $y$, unless the forces parallel to the axis of $y$ mutually destroy each other's effects For if it obtained, and this condition was not at the same time satisfied, a point might be fixed in the axis of $x$, which would destroy the effect of all forces acting in the direction of this line, nothing would then prevent the forces perpendicular to this line, from making the system to turn about this point, so that the equilibrium would be destroyed by fixing a point in the system, which is absurd This being established, it is necessary, in the first place, (No. 57), if the solid body, which is considered, be entirely free, in order for the equilibrium of the parallel forces $P\cos\gamma$, $P'\cos\gamma'$, $P''\cos\gamma''$, &c, $g$ and $-g$, $g'$ and $-g'$, $g''$ and $-g''$, &c, that their sum should be cypher, hence we have

$$P\cos\gamma + P'\cos\gamma' + P''\cos\gamma'' + \&c, = 0$$

It is likewise necessary, that the sums of their moments with respect to the plane of the axes of $x$ and $z$, and to that of the axes of $y$ and $z$, which are parallel to these forces, should be also equal to nothing. Now, with respect to the first plane, we have

$$yP\cos\gamma + y'P'\cos\gamma' + y''P''\cos\gamma'' + \&c,$$

for the value of the sum of the moments of the forces, $P\cos\gamma$, $P'\cos\gamma'$, $P''\cos\gamma''$, &c., that of the moments of the forces $g$, $g'$, $g''$, &c, is

$$gy + g'y' + g''y'' + \&c,$$

and the sum of the moments of the forces $-g$, $-g'$, $-g''$, &c, is, in consequence of the expressions of the coordinates of their points of application,

$$-g\left(y + \frac{zP\cos\beta}{g}\right) - g'\left(y' + \frac{z'P'\cos\beta'}{g'}\right) - \&c,$$

therefore, by adding together these three sums and concinnating, we obtain

$$\text{P}\,(y\cos\gamma - z\cos\beta) + \text{P}'\,(y'\cos\gamma' - z'\cos\beta') + \&\text{c.}, = 0\,;$$

and in the same manner, if the sum of the moments of the same forces, with respect to the plane of the axes of $y$ and $z$ be formed, and put equal to cypher, we shall have

$$\text{P}\,(x\cos\gamma - z\cos\alpha) + \text{P}'\,(x'\cos\gamma' - z'\cos\alpha') + \&\text{c}, = 0.$$

With respect to the forces $\text{P}\cos\beta - h$, $\text{P}'\cos\beta' - h'$, $\text{P}''\cos\beta'' - h''$, &c, and $h, h', h''$, &c, parallel to the axis of $y$, as they are comprised entirely in the plane of the axes of $x$ and $y$, there are only two equations of equilibrium (No. 57), it will suffice, therefore, that their sum should be equal to cypher, which will give

$$\text{P}\cos\beta + \text{P}'\cos\beta' + \text{P}''\cos\beta'' + \&\text{c}, = 0,$$

and, that the sum of their moments with respect to the plane of the axes of $y$ and $z$, should be also equal to cypher Now, with respect to this plane, the sum of the moments of the first forces is

$$x\,(\text{P}\cos\beta - h) + x'\,(\text{P}'\cos\beta' - h') + \&\text{c}, = 0\,,$$

that of the moments of the forces $h, h', h''$, &c, is, at the same time,

$$h\left(x - \frac{\text{P}y\cos\alpha}{h}\right) + h'\left(x' - \frac{\text{P}'y'\cos\alpha'}{h'}\right) + \&\text{c.},$$

as is evident from the values of their distances from the axis of $y$, consequently, if the entire sum be put equal to cypher, we shall have

$$\text{P}\,(x\cos\beta - y\cos\alpha) + \text{P}'\,(x'\cos\beta' - y'\cos\alpha') + \&\text{c.}, = 0.$$

Finally, for the equilibrium of the forces acting in the direction of the axis of $x$, it will suffice if their sum be cypher, consequently,

$$\text{P}\cos\alpha + \text{P}'\cos\alpha' + \text{P}''\cos\alpha'' + \&\text{c}, = 0.$$

The preceding are the six equations which are necessary and sufficient to secure the equilibrium of a solid body entirely free, and solicited by any forces whatever.

261. If, in order to abridge, we make

$$P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' + \&c. = X,$$

$$P\cos\beta + P'\cos\beta' + P''\cos\beta'' + \&c. = Y,$$

$$P\cos\gamma + P\cos\gamma' + P''\cos\gamma'' + \&c. = Z,$$

$$P(x\cos\beta - y\cos\alpha) + P'(x'\cos\beta' - y'\cos\alpha') + \&c. = L,$$

$$P(z\cos\alpha - x\cos\gamma) + P'(z'\cos\alpha' - x'\cos\gamma') + \&c. = M,$$

$$P(y\cos\gamma - z\cos\beta) + P'(y'\cos\gamma' - z'\cos\beta') + \&c. = N,$$

these equations of equilibrium will become

$$X = 0,\ Y = 0,\ Z = 0,\ L = 0,\ M = 0,\ N = 0. \qquad (1)$$

It may be remarked here, that these quantities, L, M, N, as also Z, Y, X, may be deduced the one from the other by the rule of No 22

These six equations contain the conditions of equilibrium which belong to all systems of material points, that are entirely free, for whatever be the nature of such a system, or the mutual connexion of the material points which compose it, it is evident that, if their coordinates and the forces which solicit them remain the same, the equilibrium will not be disturbed by making their distances invariable Consequently, the equations of equilibrium of a system of an invariable form, which obtain between these quantities, must also subsist for every other system, but then they are no longer sufficient, and it is necessary to combine with them other conditions that are peculiar to each system in particular, which, as we shall see in the sequel, will enable us to determine the relative positions of its different points in the state of equilibrium

262 When all the given forces are parallel to each other, the angles which they make with each of the axes $ox$, $oy$, $oz$, are either equal or supplementary, according as these forces

act in the same or in opposite directions, they may, however, all be supposed equal, if, at the same time, those forces which act in one direction are considered as positive, and those which act in the contrary direction, negative, (No. 11), therefore in this case we shall have

$$a = a' = a'', \&c, \quad \beta = \beta' = \beta'', \&c, \quad \gamma = \gamma' = \gamma'', \&c,$$

in consequence of which the three first equations (1) are reduced to one, namely,

$$P + P' + P'' + \&c. = 0,$$

and the three others will become

$$(Px + P'x' + P''x'' + \&c)\cos\beta = (Py + P'y' + P''y'' + \&c)\cos a,$$

$$(Pz + P'z' + P''z'' + \&c)\cos a = (Px + P'x' + P''x'' + \&c)\cos\gamma,$$

$$(Py + P'y' + P''y'' + \&c)\cos\gamma = (Pz + P'z' + P''z'' + \&c)\cos\beta$$

But as in the case of the equilibrium of parallel forces, the number of equations is only three, these three last equations must be reducible to two, and in fact, if they be added together, after having been multiplied by $\cos\gamma$, $\cos\beta$, $\cos a$, respectively, there results an identical equation, so that one of these may be deduced from the two others

When all the given forces exist in the same plane, this plane may be taken for that of the axes of $x$ and $y$, in this case the angles $\gamma, \gamma', \gamma''$, &c, are right, and the coordinates $z, z', z''$, &c, are equal to cypher, which will cause the third and also the two last equations (1) to disappear. In this particular case, as in that of parallel forces, there are only three equations of equilibrium, which are

$$X = 0, \quad Y = 0, \quad L = 0.$$

263. When the given forces do not constitute an equilibrium, it may be required to know what condition they should satisfy, in order that they may have an unique resultant, and what is the value of this resultant In order to determine this question, let R denote this force, and $a$, $b$, $c$ the angles which

its direction makes with lines parallel to the axes o$x$, o$y$, o$z$, drawn through one of its points, which may be taken as its point of application, and whose coordinates parallel to these same axes are represented by $x_1$, $y_1$, $z_1$ If this force be taken in an opposite direction from that in which it acts, it will constitute an equilibrium with the given forces. Therefore equations (1) will obtain, if to P, P′, P″, &c, there be joined a force equal and contrary to R, consequently, we shall have

$$X = R\cos a,\quad Y = R\cos b,\quad Z = R\cos c, \tag{2}$$

and, besides,

$$L = R(x_1\cos b - y_1\cos a),$$

$$M = R(z_1\cos a - x_1\cos c),$$

$$N = R(y_1\cos c - z_1\cos b),$$

that is to say, in virtue of the three first equations,

$$\left.\begin{aligned} Xy_1 - Yx_1 + L &= 0,\\ Zx_1 - Xz_1 + M &= 0,\\ Yz_1 - Zy_1 + N &= 0 \end{aligned}\right\} \tag{3}$$

As the coordinates $x_1$, $y_1$, $z_1$ may belong to any point whatever of the right line along which the resultant is directed, these three last equations will be those of its projections on the three planes of the coordinates. Therefore, in order that this line may exist, these equations must be reducible to two, now, if after having multiplied them by Z, Y, X respectively, they be added together, the three variables $x_1$, $y_1$, $z_1$ will disappear, and there results

$$ZL + YM + XN = 0, \tag{4}$$

consequently, in order that the given forces may have an unique resultant, it is necessary, and it suffices, that equation (4) be satisfied, when this is the case, this force will be determined both in magnitude and direction, by equations (2)

If the three sums of the components parallel to the axes of

$x$, $y$, $z$ are respectively cypher, equation (4) will be satisfied, but then the resultant will be an infinitely small force, situated at an infinite distance from the point of application of the given forces, or more correctly, these forces are reducible to two, equal and parallel, acting in opposite directions, but not *directly* opposed, and therefore not reducible to an unique force, (No 44)

When the three sums L, M, N are respectively cypher, equation (4) will also be satisfied, and, from equations (3) it appears, that the resultant will then pass through the origin of the coordinates

264 When the condition expressed by equation (4) is not fulfilled, it may be satisfied by joining a suitable force to the given forces For greater simplicity, let it be supposed that it passes through O, the origin of the coordinates, if it be denoted by Q, and if $\lambda$, $\mu$, $\nu$ represent the angles which it makes with the axes O$x$, O$y$, O$z$, the quantities L, M, N will not be changed by the addition of this force, and the sums X, Y, Z will be increased by the terms $Q\cos\lambda$, $Q\cos\mu$, $Q\cos\nu$. Equation (4) will therefore become

$$Q\,(L\cos\nu + M\cos\mu + N\cos\lambda) + LZ + MY + NX = 0\,,$$

so that it may be satisfied in an infinite variety of different ways, by means of the force Q and of the angles $\lambda$, $\mu$, $\nu$, which determine its direction.

The intensity and position of R, the resultant of the forces Q, P, P$'$, P$''$, &c , will be determined by means of equations (2) and (3), in which $X + Q\cos\lambda$, $Y + Q\cos\mu$, $Z + Q\cos\nu$ should be substituted for X, Y, Z Consequently, the given forces P, P$'$, P$''$, &c , may be replaced by this resultant R and by a force equal and directly opposite to the force Q $(a)$, hence we may infer that when the given forces are neither in equilibrio, nor reducible to an unique force, they may always be reduced, in an infinite variety of different ways, to two forces only, which however will not exist in the same plane, for otherwise, if they

did, they would be reducible to one sole force contrary to the hypothesis. This is, moreover, rendered immediately apparent by the transformation effected in No 257, for the given forces P, P′, P″, &c, may be replaced by the resultant of the forces parallel to the axis of $z$, and by that of the forces comprised in the plane of the axes of $x$ and $y$ and we may then, without any difficulty, transform these two resultants into two other forces in an infinite variety of ways If the condition which should be satisfied in order that their directions should meet, be investigated, we will light on equation (4) relative to the existence of an unique resultant

265 From what has been now established it will be easy to determine the conditions of equilibrium of two solid bodies A and A′ (fig. 62), which being solicited by given forces, touch at a point K, and press against each other

For this purpose, let the six quantities X, Y, Z, L, M, N of No 261, refer to the body A, X′, Y′, Z′, L′, M′, N′, being what these quantities become with respect to the body A′, and let $x_1$, $y_1$, $z_1$ be the coordinates of the point K, referred to the same axes as those which occur in these different quantities; through the point K let the line HKH′ be drawn perpendicular to the plane which touches the two bodies, and let $a$, $b$, $c$ denote the angles, which KH, the part of this line comprised in A, makes with lines drawn through K parallel to the axes of $x$, $y$, $z$; all these quantities are given, and it is only necessary to form the equations of equilibrium which they ought to satisfy.

Now, the body A will exercise on A′, in the direction of KH′, an unknown pressure, which we shall denote by R, it will experience from it, at the same time, a resistance equal and contrary to this normal force If, therefore, to the given forces which act on A, there be joined a force R acting in the direction KH, we may then abstract altogether from the consideration of A′, and, in the same manner, if to the forces applied to A′ there be joined a force R acting in the direction KH′, A′ may be considered by itself as detached from A It

follows from this, and from equations (1), that the twelve following equations will be required in order to the equilibrium of these two bodies

$$\text{X} + \text{R}\cos a = 0,\quad \text{Y} + \text{R}\cos b = 0,\quad \text{Z} + \text{R}\cos c = 0,$$
$$\text{X}' - \text{R}\cos a = 0,\quad \text{Y}' - \text{R}\cos b = 0,\quad \text{Z}' - \text{R}\cos c = 0,$$
$$\text{L} + \text{R}\,(x_1\cos b - y_1\cos a) = 0,$$
$$\text{M} + \text{R}\,(z_1\cos a - x_1\cos c) = 0,$$
$$\text{N} + \text{R}\,(y_1\cos c - z_1\cos b) = 0,$$
$$\text{L}' - \text{R}\,(x_1\cos b - y_1\cos a) = 0,$$
$$\text{M}' - \text{R}\,(z_1\cos a - x_1\cos c) = 0,$$
$$\text{N}' - \text{R}\,(y_1\cos c - z_1\cos b) = 0,$$

which will be reduced to eleven by the elimination of R After that these eleven equations of equilibrium shall have been verified, one of the preceding will make known the value of R, which must be a positive quantity, in order that the two bodies may press the one against the other

From these twelve equations there results immediately

$$\text{X} + \text{X}' = 0,\quad \text{Y} + \text{Y}' = 0,\quad \text{Z} + \text{Z}' = 0,$$
$$\text{L} + \text{L}' = 0,\quad \text{M} + \text{M}' = 0,\quad \text{N} + \text{N}' = 0,$$

which may be also deduced from the conditions of equilibrium which belong to all systems entirely free, like that of the two bodies A and A′ (No 261)

In the same manner, the equations of equilibrium of any number of bodies, which press against each other, may be found, and it is easy to perceive that the number of these equations will be equal to six times that of the bodies minus the number of their points of contact ($b$)

266 The equations of equilibrium of a solid body subject to given conditions must be comprised among those of a body entirely free, for the equilibrium of this last will not be disturbed if it is subjected to these particular conditions, pro-

vided that no new equation of equilibrium be introduced by these conditions But, on the contrary, one or more of equations (1) will become superfluous, and it will be of use to determine, in the different cases that may occur, which of these equations will be necessary It is this which it is proposed to do, in the following cases that are discussed in this number, it being always supposed that the given forces P, P′, P″, &c, are replaced by the three groups of forces of No 259

1. If the solid body which is in equilibrio, contains a fixed point, we may take this point for the origin of the coordinates. The forces acting in the direction of the axis $ox$ will be destroyed by this point, this will cause the equation $X = 0$ to disappear In order that the forces parallel to the axis $oy$, and comprised in the plane of the axes of $x$ and $y$ should be in equilibrio, it is not necessary that we should have $Y = 0$, it will suffice if their resultant coincides with the axis $oy$, or that L, the sum of their moments with respect to the plane of the axes of $y$ and $z$, should be equal to cypher ($c$) Finally, for the equilibrium of the forces parallel to the axis of $z$, the equation $Z = 0$, will not be necessary, it will suffice, if their resultant coincides with the axis $oz$, this will require that the sums of their moments with respect to the planes of the axes of $y$ and $z$, and of $x$ and $z$, which are represented by $-M$ and N, may be equal to cypher Thus, in this first case, the three equations of equilibrium which will be necessary, are

$$L = 0, \quad M = 0, \quad N = 0.$$

In fact, they indicate that the given forces have an unique resultant, and that this resultant passes through the fixed point o

This force will express in magnitude and direction the pressure exerted against this point, and it will be determined by equations (2)

2. Let the solid body be retained by a fixed axis, about which it is constrained to turn, without being able to slide in

the direction of its length. Let this axis be assumed to be that of $z$, the forces parallel to this line $oz$ cannot produce any motion, and the three equations $\mathrm{Z} = 0$, $\mathrm{M} = 0$, $\mathrm{N} = 0$, relative to their equilibrium, will be no longer necessary. Neither will equations $\mathrm{X} = 0$ and $\mathrm{Y} = 0$ be required for the equilibrium of the forces comprised in the plane of the axes of $x$ and $y$, so that in this case, there will be only one sole equation of equilibrium, which will be $\mathrm{L} = 0$, that is to say,

$$\mathrm{P}\,(x\cos\beta - y\cos\alpha) + \mathrm{P}'\,(x'\cos\beta' - y'\cos\alpha') + \&c = 0 \qquad (5)$$

But if the body is free to slide along the fixed axis, it is, moreover, necessary, in order to hinder this motion, that $\mathrm{Z}$, the sum of the forces parallel to $oz$, should be equal to cypher, and in this case there will be two equations of equilibrium,

$$\mathrm{Z} = 0, \quad \mathrm{L} = 0$$

The pressure which the fixed axis will experience perpendicularly to its direction, will be the resultant of forces comprised in the plane of the axes of $x$ and $y$, determined both in magnitude and direction, by the two first equations (2), and passing through the point o in virtue of equation (5) The forces parallel to this axis will at the same time tend to make it revolve on itself.

It appears from a comparison of the quantities $\mathrm{M}$ and $\mathrm{N}$ with $\mathrm{L}$, that the equation of equilibrium about the axis $oy$ will be $\mathrm{M} = 0$, and that it will be $\mathrm{N} = 0$ about the axis $ox$. It likewise results from what has been established, that the condition of equilibrium about a fixed point consists in this, that the equilibrium obtains successively about three fixed rectangular axes, drawn arbitrarily through this point Consequently, if the equilibrium subsists about three rectangular axes which intersect in the same point, it will also obtain about every other line passing through this point.

3. If in a solid body, three or a greater number of points which do not lie in the same right line, are constrained to

exist on a fixed plane, of which the position is given, and if this plane be assumed to be that of the axes of $x$ and $y$, as the forces parallel to the axis of $z$ cannot produce any motion, the equations relative to their equilibrium will not have place, but the three equations

$$X = 0, \quad Y = 0, \quad L = 0,$$

which obtain in the case of forces comprised in the plane of the axes of $x$ and $y$, will be necessary, to prevent the body from sliding or turning parallel to this fixed plane.

The force Z will express the entire pressure which the fixed plane experiences If the body is merely placed on this plane, so that it may act, for example, as a polyhedron, one of whose faces is in contact with the plane of the axes of $x$ and $y$, the sign of Z must be such, that this force may press against this plane It is, moreover, necessary that this resultant of the forces parallel to the axis of $z$, should meet the plane of the axes of $x$ and $y$, within the area of the base of the body, otherwise it will detach it from this plane, by causing it to turn about one of the sides of this base Now, if $x_1$ and $y_1$ denote the coordinates of the point, where this resultant meets the plane of the axes of $x$ and $y$, its moments with respect to the planes of the axes of $x$ and $z$, and of $y$ and $z$, will be $Zy_1$ and $Zx_1$, they should be equal to the sums of the moments of the components with respect to the same planes, and from the values of these two sums which have been already found, (No 260), we shall have

$$Zx_1 = -M, \quad Zy_1 = N.$$

It is necessary, therefore, *in each particular case*, to shew that the values of $x_1$ and $y_1$, deduced from these equations, belong to a point *within* the base of the body, this condition of equilibrium cannot be expressed by equations, no more than that which is relative to the sign of Z

4 If the number of points of a body which is constrained

to rest on the fixed plane of the axes of $x$ and $y$ be only two, or if they are all situated on the same line, we may assume this line for the axis of $y$, the resultant Z must then meet the plane of the axes of $x$ and $y$, in a point situated on this axis, and, independently of the three equations of the preceding case, we shall have a fourth equation of equilibrium, namely, $M = 0$

5 Finally, when the solid body touches the fixed plane of the axes of $x$ and $y$ in only one point, at which, if O the origin of the coordinates be placed, it is easy to perceive that there will be five equations of equilibrium, namely,

$$X = 0, \quad Y = 0, \quad L = 0, \quad M = 0, \quad N = 0$$

The force Z will always express the pressure excited on the fixed plane at the point O, and it must consequently be affected with a suitable sign

This result coincides with that of the preceding number, for if the body A′ be supposed to be fixed and terminated by a plane which may be taken for that of the axes of $x$ and $y$, and if the point K (fig 62) be taken for the origin of the coordinates, we should make in the equations of this number, $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, $a = 90°$, $b = 90°$, by means of which we can reduce to the five preceding equations, a like number of the six equations which refer to the equilibrium of the body A. The sixth of these equations will, at the same time, become

$$R + Z = 0,$$

by supposing that $c$ is equal to cypher, or which is the same thing, that the part RH of the normal coincides with the positive axis of $z$, consequently, the pressure excited on A′, which is equal and contrary to the resistance R, will be the force Z, in magnitude and direction.

From a consideration of the preceding enumeration of the different cases of equilibrium, it is evident that the number of equations relative to a solid body constrained by fixed ob-

stacles, may be any number less than six, which is the number in the case of a body entirely free.

267. As equation (5) relative to the equilibrium about the axis of $z$ supposed fixed, contains neither the components of the given forces P, P′, P″, &c, parallel to this axis, nor the coordinates of M, M′, M″, &c, their points of application parallel to the same axis, the equilibrium will not be deranged, if these forces and their points of application are replaced by their projections on the plane of the axes of $x$ and $y$, this may be easily demonstrated *a priori*.

Let Q, Q′, Q″, &c., represent the forces P, P′, P″, &c. projected on the plane of the axes of $x$ and $y$, that is to say, resolved parallel to this plane, and transferred to the projections of the points M, M′, M″, &c on this same plane Let $q, q', q''$, &c denote the perpendiculars let fall from the origin of the coordinates, which is supposed to be fixed, on the directions of the forces Q, Q′, Q″, &c, and, for greater clearness, let the effect of the forces Q, Q′, Q″, &c be to cause the body to turn in the same direction, and the effect of Q‴, $\text{Q}^{IV}$, &c. be to make the body turn in the opposite direction For the equilibrium of all these forces, it is necessary, by No. 47, that we should have

$$\text{Q}q + \text{Q}'q' + \text{Q}''q'' - \text{Q}'''q''' - \text{Q}^{IV}q^{IV} - \&c = 0, \qquad (6)$$

$q, q', q'', q''', q^{IV}$, &c. being considered as positive quantities, as also Q, Q′, Q″, Q‴, &c, consequently, this equation must coincide with equation (5), which it is easy to verify in the following manner

Let H (fig 63) be the projection of the point M, OG and HG its coordinates $x$ and $y$, HA the direction of the force Q, $\lambda$ and $\mu$ the angles which this line makes with parallels to the axes O$x$ and O$y$, drawn through the point H. Through the point O, let two other axes O$x_1$ and O$y_1$ be drawn, the first in the direction HA, and the second perpendicular to this line, and such that the angle $y$O$y_1$, may be acute or obtuse at the

same time as $xox_1$, now if $x_1$ and $y_1$ denote the coordinates OF and FH of the point H, referred to these new axes, we shall have

$$x_1 = y\cos\mu + x\cos\lambda, \quad y_1 = y\cos\lambda - x\cos\mu.$$

But, as the perpendicular OK or $q$, let fall from the point O on HA, must be a positive quantity, we shall have

$$Q = \pm y_1 = \pm (y\cos\lambda - x\cos\mu),$$

according as the ordinate $y_1$ is positive or negative, or what comes to the same thing, from the direction which has been assigned to the axes $oy_1$, according as the force Q tends to make the body turn in one direction or the contrary, about the point O Moreover, we have

$$Q = P\sin\gamma,$$

and, besides, (No 8)

$$\cos\alpha = \sin\gamma\cos\lambda, \quad \cos\beta = \sin\gamma\cos\mu,$$

hence there will result ($d$)

$$Qq = \pm P(y\cos\alpha - x\cos\beta)$$

As, by hypothesis, the forces $Q'$ and $Q''$ tend to make the body turn in the same direction as Q, we shall have also

$$Q'q' = \pm P'(y'\cos\alpha' - x'\cos\beta'),$$
$$Q''q'' = \pm P''(y''\cos\alpha'' - x''\cos\beta''),$$

and as the other forces $Q'''$, $Q^{IV}$, &c. tend to make the body turn in the opposite direction, we shall have

$$Q'''q''' = \mp P'''(y'''\cos\alpha''' - x'''\cos\beta'''),$$
$$Q^{IV}q^{IV} = \mp Q^{IV}(y^{IV}\cos\alpha^{IV} - x^{IV}\cos\beta^{IV}),$$
&c

Therefore, in all these values, the superior signs or the inferior must be taken at the same time, and by substituting them in equation (6), it will become equation (5), which was required to be verified.

268. The body in equilibrio being always acted on by gravity, the weight of a body acting in the direction of a vertical passing through its centre of gravity, must be supposed to be included among the given forces P, P′, P″, &c. For example, if we suppose that the heavy body rests on an inclined plane, and is sustained by one sole force, (fig 64 represents a section of this body passing through the centre of gravity G, and perpendicular to the inclined plane,) the length of this plane is AB, its base BC, and its height AC. Let O, the origin of the coordinates, be in the vertical GH passing through the centre of gravity, and let the axes O$z$ and O$x$ be, the one perpendicular, and the other parallel to AB, the third axis O$y$, which is not represented in the figure, will be perpendicular to the plane of the figure. The force P will be the weight of the body, the vertical GH its direction, and HO$x$ the angle $\alpha$ Moreover, we shall have $x=0$, $y=0$, $\beta=90°$ Hence, if P′ represents the given force which sustains the heavy body, the equations of equilibrium of the third case of No 266 will be reduced to

$$\text{P} \cos\alpha + \text{P}' \cos\alpha' = 0, \quad \text{P}' \cos\beta' = 0,$$
$$\text{P}'(x' \cos\beta' - y' \cos\alpha') = 0$$

From the two last, we obtain $\beta' = 90°$, $y' = 0$, which shews that the force P′ must exist in the plane of the axes of $x$ and $z$; and, in fact, this is evidently necessary, in order that this force and the weight of the body may have an unique resultant, perpendicular to the inclined plane Let O be the point where the direction of P′ meets the vertical GH, and let OD represent this direction The angle $\alpha'$ or DO$x$ must be obtuse, in order to satisfy the first of the three preceding equations, let $\delta$ denote the acute angle DO$x'$, which the force P′ makes with the production of O$x$, so that we may have

$$\cos\alpha' = -\cos\delta.$$

The angle $\alpha$ or HO$x$ is the complement of ABC, the inclina-

tion of the plane, therefore denoting the height AC by $h$, and the length AB by $l$, we shall have

$$\cos a = \frac{h}{l},$$

hence there will result the equation of equilibrium, ($e$)

$$\frac{\mathrm{P}h}{l} = \mathrm{P}'\cos\delta,$$

by means of which one of the two quantities P′ and $\delta$ will be determined, when the other is given

For example, when the force P′ is parallel to the inclined plane, we shall have $\delta = 0$, and consequently,

$$\mathrm{P}' \quad \mathrm{P} \quad h \cdot l,$$

or, what comes to the same thing,

$$\mathrm{P}' = \mathrm{P}\sin i,$$

$i$ denoting the inclination of the plane If Q denotes the pressure that the plane experiences, and which in this case will be the weight P resolved in the direction of the perpendicular O$z$, we shall have, at the same time,

$$\mathrm{Q} = \mathrm{P}\cos i$$

269. In the preceding discussion no account has been taken of the friction, which combines its effect with that of the force P′ parallel to the inclined plane, in preventing the body from sliding along it If this force P′ vanishes, the friction alone may retain the body, as long as the inclination $i$ does not attain a certain limit It this limit be $\lambda$, i e what $i$ becomes when the equilibrium commences to give way, and if at this instant the friction is the fraction $f$ of the pressure Q, the force $f$Q must be in equilibrio with P sin $\lambda$, the component of the weight of the body resolved parallel to the inclined plane Consequently, we shall have at the same time

$$\mathrm{Q} = \mathrm{P}\cos\lambda, \quad f\mathrm{Q} = \mathrm{P}\sin\lambda,$$

hence we obtain

$$f = \text{tang}\,\lambda,$$

which will enable us to determine the value of $f$, by observing $\lambda$ the angle at which the body commences to move, this has been termed *the angle of friction*

It is proved by experiment, that, every thing else being the same, at the instant the equilibrium gives way, the friction is proportional to the pressure, so that the coefficient $f$ and the angle $\lambda$ are independent of the pressure $Q(f)$, and consequently of the weight P. This coefficient depends on the nature of the body and the smoothness of the surfaces, it has been also observed, that it does not attain its *maximum* value until the body and the plane have been some time in contact, (which time varies with the nature of the body), and that it is only when this *maximum* is attained, that the friction is proportional to the pressure Assuming this experimental law to be correct, it follows, that if several bodies of the same nature, and whose surfaces have the same degree of smoothness, are placed on a horizontal plane, and if after the lapse of a certain time this plane is gradually inclined, all these bodies will commence to slide at the same inclination $\lambda$, whatever be their weights, and the extent of their surfaces in contact with the plane.

270 When a body is placed on a horizontal plane the force with which, in consequence of its weight P, the plane is pressed, is distributed among the points of support of this plane, but when their number is more than three, this distribution seems at first view to be indeterminate The difficulty which appears to occur in this case we now proceed to examine.

For greater clearness, let this horizontal plane be supposed to be the surface of a table, the feet of which are vertical. Let two rectangular axes $ox$ and $oy$ (fig. 65) be drawn in this plane, let C be the projection of the centre of gravity of the body on this plane, and A, A′, A″, &c. the points of this

plane where the legs of the table meet it, and let $x_1$ and $y_1$, $x$ and $y$, $x'$ and $y'$, $x''$ and $y''$, &c denote the coordinates of the points C, A, A′, A″, &c. referred to the axes $ox$ and $oy$ In order that the table may not be overturned, the point C must lie within the polygon A A′ A″ A‴, &c This condition being satisfied, the weight P, applied to the point C, may be resolved into a number of parallel forces acting in the direction of gravity, and passing through the points of support A, A′, A″, &c, which forces are the loads that are supported by the legs of the table If Q, Q′, Q″, &c, denote these unknown loads, by the theory of parallel forces, we shall have

$$P = Q + Q' + Q'' + \&c$$
$$Px_1 = Qx + Q'x' + Q''x'' + \&c$$
$$Py_1 = Qy + Q'y' + Q''y'' + \&c$$

Now, if there are only three points of support, A, A′, A″, these three equations will be sufficient to determine the loads Q, Q′, Q″, but if there be a greater number than three, the problem will be indeterminate, and the values of all the unknown pressures, minus three, may be assumed to be what we please, provided that for these three, the values which result for them are positive This indetermination would in fact obtain, if the table was rigorously inflexible, but this is never the case and, though this flexibility may be supposed ever so little, still it will so far cause a slight displacement, the effect of which will be, that the table will be unequally pressed in its different parts Now, the figure that it will assume, and, the force with which it will be pressed, in each point, will depend not only on the weight P, but also on the number and disposition of the points of support A, A′, A″, &c., and both the one and the other, as also the pressure which has place in each of these points, will be completely determinate in each particular case However, this determination is an extremely difficult problem, the general solution

of which belongs to the department of mathematical physics, and has not yet been given. We shall therefore restrict ourselves here to one sole observation which is suggested by this subject, namely, *that in nature every thing is necessarily determinable, and that when any thing appears to us indetermined, it is because we have not taken into account some one of the data of the problem*, i. e, *some property of matter, like the degree of flexibility in the table in question*

## CHAPTER II

### THEORY OF MOMENTS

271. THE moments which we propose to treat of in this chapter are those that have been already considered in No 42, where the moment of a force P was defined to be the product of this force and of $p$ the perpendicular let fall from the centre of the moments on its direction Therefore, if C be this centre (fig 66), and if the force P be represented by the line MA taken on its direction, its moments will be expressed by twice the triangle CAM, which has this force for its base, and its vertex at the point C From this it appears, that the theorem of No 46, relative to the moment of the resultant of two forces, is a geometrical proposition extremely easy to demonstrate

In fact, if MA and MB be the two components, MD the diagonal of the parallelogram MADB will be their resultant; and if the point C be without the angle AMB and its vertically opposite, it is easy to demonstrate that the triangle CMD is equal to the sum of the triangles CMA and CMB For in the first place, we have

$$\text{CMD} = \text{CMA} + \text{CAD} + \text{MAD},$$

and if a perpendicular CE be let fall from the point C on the line MB, this will meet DA which is parallel to MB in F, and we shall have

$$\text{CMB} = \tfrac{1}{2}.\text{MB CE}, \quad \text{CAD} = \tfrac{1}{2}.\text{AD}.\text{CF}.,$$

and since

$$\text{MB} = \text{AD}, \quad \text{CF} = \text{CE} - \text{EF},$$

there will result

$$\text{CAD} = \text{CMB} - \text{MAD},$$

and, consequently,

$$\text{CMD} = \text{CMA} + \text{CMB}$$

which was required to be proved.

In the figure, the line EF is supposed to be the difference of the perpendiculars CE and CF; but it may be equal to their sum, and it is easy to modify the preceding demonstration so as to apply it to this case. In the same manner it may be proved, that, when the point C lies within the angle AMB, or its vertically opposite, the triangle CMD is equal to the difference between the triangles CMA and CMB.

272. Through the centre of moments (fig 67), let any plane whatever be drawn, and let the right line AB, which represents the force P in magnitude and direction, be projected on this plane, also let Q be the force represented by A′B′ the projection of AB, the moment of the force P will be twice the triangle CAB, and that of the force Q twice the triangle CA′B′, consequently, the centre of moments remaining the same, the moment of the projection of a force on a plane passing through this point, is the projection of the moment of this force on this same plane.

If H and K denote respectively the moment of the force P and of its projection Q, and if the perpendiculars CD and CE be erected to the planes of these respective moments, $\delta$ the angle contained between these perpendiculars will be equal to the inclination of H on K, and by No. 10 we shall have

$$K = H \cos \delta.$$

The force P remaining the same, the angle $\delta$ and the moment H will vary with the position of the point C on the line CE, but if the position of this line is not altered, the product $H \cos \delta$ will not vary, for K or the triangle CA′B′ will only be displaced parallel to itself, without undergoing any change in its magnitude

273. In place of one force P, let any number of forces P, P′, P″, &c, be considered If H, H′, H″, &c, be their moments with respect to the point C (fig. 68), $\delta, \delta', \delta''$, &c, the angles which the perpendiculars CD, CD′, CD″, &c, to the planes of these moments make with the same axis CL, Q, Q′, Q″, &c, the projections of P, P′, P″, &c., on the plane drawn through

the point C, and perpendicular to this axis, K, K′, K″, &c., the projections of H, H′, H″, &c., on this same plane, we shall have

$$\text{K} = \text{H} \cos \delta, \quad \text{K}' = \text{H}' \cos \delta', \quad \text{K}'' = \text{H}'' \cos \delta'', \ \&\text{c}.$$

If it is merely proposed to determine the areas of the projections from knowing those of the projected surfaces, the inclinations $\delta, \delta', \delta''$, &c., must be considered as acute angles, but in the applications which we shall make of the projections of the moments, these angles will be regarded either as acute or obtuse, or in other words, it being agreed on to consider a certain direction as positive, and the opposite as negative, we will assume for the lines CD, CD′, CD″, &c., the parts of the perpendiculars to the planes of the moments H, H′, H″, &c., which make acute or obtuse angles with the axis CE, according as Q, Q′, Q″, &c., the projections the forces of P, P′, P″, &c., tend to produce a revolution in a certain direction previously agreed on, or the contrary. Thus, in the figure, the angles DCE, D′CE, D″CE, being acute, and the angles D‴CE, $\text{D}^{IV}$CE, &c., being obtuse, this implies, that the forces Q, Q′, Q″, tend to produce a revolution in the same direction, and the forces Q‴, $\text{Q}^{IV}$, &c., in the opposite direction. As the lines CD″ and CD‴ are the production the one of the other, this indicates that the forces P″ and P‴ are both comprised in the same plane, passing through the point C, but that they, as likewise their projections Q″ and Q‴, tend to produce revolutions in opposite directions.

Denoting the sum of the values, whether positive or negative, of K, K′, K″, &c., by S, we shall have,

$$\text{S} = \text{H} \cos \delta + \text{H}' \cos \delta + \text{H}'' \cos \delta'' + \&\text{c}.,$$

and, abstracting from the consideration of the sign, S will be the sum of the moments of the forces Q, Q′, Q″, &c., which tend to produce a revolution in one direction *minus* the sum of the moments of those which tend to produce a revolution in the opposite direction; therefore, by the theorem of No. 47, the quantity $\pm$ S will express the moment of their re-

sultant, which will tend to produce a revolution in the direction of the forces that refer to the acute angles $\delta$, $\delta'$, $\delta''$, or to the obtuse angles $\delta'''$, $\delta^{IV}$, &c, according as the preceding value of s will be positive or negative

If all the lines CD, CD$'$, CD$''$, &c., are at the same time changed into their productions at the other side of C, the angles $\delta$, $\delta'$, $\delta''$, &c., will be changed into their supplements, and s will be changed into $-$ s. This will also be the case when for the axis CE there is substituted its production CE$'$.

Each of the parts which make up the sum s, and consequently s itself, will be independent of the position of the point C on the axis CE, it will depend only on the system of forces P, P$'$, P$''$, &c, on the position of this axis, and on its direction perpendicular to the plane of projection Henceforth this quantity s will denote the moment of the forces P, P$'$, P$''$, &c, with respect to the axis CE.

274 From this definition it appears, that the three quantities L, M, N, of No 261, will be the moments of the forces P, P$'$ P$''$, &c, with respect to the axes of the positive coordinates of their points of application.

In order to prove this, let Q be the projection of the force P on the plane of the axes of $x$ and $y$, and $q$ the perpendicular let fall from the origin of the coordinates on its direction, so that the value of its moment with respect to this point may be Q$q$. Let us suppose that the force Q acts from A towards B (fig 69), and that AC and AD are the coordinates $x$ and $y$ of its point of application A, referred to the rectangular axes O$x$ and O$y$ Likewise let $\lambda$ and $\mu$ be the angles BAC$'$ and BAD$'$ which the force Q makes with the productions of $x$ and $y$, Q $\cos\lambda$ and Q $\cos\mu$ will be the components acting in the direction of AC$'$ and AD$'$, and $y$Q $\cos\lambda$, $x$Q $\cos\mu$, will be their moments with respect to the point O, it appears from the figure, that they will tend to make the system to turn in opposite directions, and that the force Q tends to produce a revolution in the direction of Q $\cos\mu$, consequently, we shall have

$$Qq = xQ\cos\mu - yQ\cos\lambda$$

It is easy to perceive from a consideration of the different positions which the point A can have, and of the different directions which may be assigned to the force Q, that this equation will subsist, whatever be the signs of $x$, $y$, $\cos\lambda$, $\cos\mu$, provided that the force Q transferred to the point E or F, where its direction meets the axes of the coordinates $x$ or $y$, tends to make $ox$ the axis of the positive coordinates $x$s to turn within the angle of the positive $x$s and $y$s, and, consequently, $oy$, the axis of the positive $y$s, without this angle, as is indicated by the sagittæ $s$ and $s'$ If the contrary be the case, that is to say, if the force Q, thus transferred, tends to make the axis of the positive $y$s to turn within the angle of the positive $x$s and $y$s, and, consequently, the axis of the positive $x$s without this angle, we shall have

$$Qq = yQ\cos\lambda - xQ\cos\mu$$

whatever may be the signs of $x$, $y$, $\cos\lambda$, $\cos\mu$.

It follows from this, that if S be the moment of the forces P, P$'$, P$''$, &c, with respect to the axis of the positive $z$s, and if the angles $\delta$, $\delta'$, $\delta''$, &c, of the preceding number are considered to be acute or obtuse, according as the projections Q, Q$'$, Q$''$, &c, of these forces tend to make the axis of the positive $x$s to turn within the angle contained between the positive coordinates $x$ and $y$, or without this angle, we shall have

$$S = Q(x\cos\mu - y\cos\lambda) + Q'(x'\cos\mu' - y'\cos\lambda')$$
$$+ Q''(x''\cos\mu'' - y''\cos\lambda'') + \&c.,$$

$x'$, $y'$, $\lambda'$, $\mu'$, $x''$, $y''$, $\lambda''$, $\mu''$, &c., being what $x$, $y$, $\lambda$, $\mu$, become relatively to the forces Q$'$, Q$''$, &c.

Moreover, if $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$, $\gamma'$, $\alpha''$, $\beta''$, $\gamma''$, &c, be the angles which the directions of the forces P, P$'$, P$''$, &c, make with parallels to the axes of $x$, $y$, $z$, we shall have

$$Q = P\sin\gamma, \quad Q' = P'\sin\gamma', \quad Q'' = P''\sin\gamma'', \text{ \&c.},$$

$$\cos\alpha = \sin\gamma\cos\lambda, \ \cos\alpha' = \sin\gamma'\cos\lambda', \ \cos\alpha'' = \sin\gamma''\cos\lambda'', \text{ \&c },$$

$$\cos\beta = \sin\gamma\cos\mu, \ \cos\beta' = \sin\gamma'\cos\mu', \ \cos\beta'' = \sin\gamma''\cos\mu'', \text{ \&c.},$$

and from these values, it is evident that the expression for S will coincide with the value of L of No 261($a$) Hence, L is the moment of the forces P, P′, P″, &c., with respect to the axis of the positive *zs*, and, according as it is positive or negative, this system of forces will cause the plane of the axes of the positive *xs* and *zs*, to turn within the solid angle of the positive coordinates, or without this angle.

Now, if the axes of the positive *zs*, *xs*, *ys*, be substituted for those of the positive *xs*, *ys*, *zs*, L will be changed into M, it follows, therefore, that M is the moment of the forces P, P′, P″, &c., with respect to the axis of the positive *ys*, and that according as it is positive or negative, this system of forces will tend to make the plane of the axes of the positive *zs* and *ys* to turn about this axis, within the solid angle of the positive coordinates, or without this angle. In like manner, if the axes of the positive *ys*, *zs*, *xs*, be substituted for those of the positive *zs*, *xs*, *ys*, M will be changed into N, consequently, N will be the moment of the forces P, P′, P″, &c., with respect to the axis of the positive *xs*, and, according as this moment is positive or negative, this system will tend to make the plane of the positive *ys* and *xs* to turn about this axis, within the solid angle of the positive coordinates, or without this angle

The three quantities L, M, N, are consequently, as has been stated, the moments of the same system of forces with respect to the three axes of the positive coordinates of their points of application, and the signs of their values, such as they are written in No. 261, refer to a known direction of rotation, about each axis supposed to be fixed.

275. It appears from the preceding number, that the first value of $Qq$ is the same thing as($b$)

$$\text{Q}q = x\,\text{P} \cos\beta - y\,\text{P} \cos\alpha$$

Hence if H denotes the moment of P with respect to the origin of the coordinates, and $\delta$ the angle contained between a part of the perpendicular to the plane of this moment and the axis of the positive $zs$, we shall have (No 272)

$$\text{H} \cos\delta = \text{P}\,(x \cos\beta - y \cos\alpha)\,,$$

which implies, that this part of the perpendicular to the plane of H, is that which makes an angle with the axis of the positive $zs$, that is acute, or obtuse, according as the quantity included between the brackets is positive or negative If $\delta_1$ and $\delta_2$ be the angles which the same part of this perpendicular makes with the axes of the positive $ys$ and $xs$, we shall have in like manner

$$\text{H} \cos\delta_1 = \text{P}\,(z \cos\alpha - x \cos\gamma),$$
$$\text{H} \cos\delta_2 = \text{P}\,(y \cos\gamma - z \cos\beta)$$

If, therefore, in order to abridge, we make

$$(x \cos\beta - y \cos\alpha)^2 + (z \cos\alpha - x \cos\gamma)^2 + (y \cos\gamma - z \cos\beta)^2 = p^2,$$

there results, ($p$ being regarded as a positive quantity,)

$$\text{H} = \text{P}p,$$

because

$$\cos^2\delta + \cos^2\delta_1 + \cos^2\delta_2 = 1\,,$$

consequently, we shall have

$$\cos\delta = \frac{1}{p}(x \cos\beta - y \cos\alpha),$$

$$\cos\delta_1 = \frac{1}{p}(z \cos\alpha - x \cos\gamma),$$

$$\cos\delta_2 = \frac{1}{p}(y \cos\gamma - z \cos\beta)\,,$$

by means of which the three angles $\delta, \delta_1, \delta_2$, can be determined without any ambiguity The angle $\delta$ will be acute or obtuse, according as the sign of $x \cos\beta - y \cos\alpha$ is positive

or negative, and the angles $\delta_1$ and $\delta_2$, according to the signs of $z\cos\alpha - x\cos\gamma$, and $y\cos\gamma - z\cos\beta$.

It is easy to verify these formulæ In fact, if the equation of the plane which comprehends the origin of the coordinates and the force P, be denoted by

$$Au + Bv + Cw = 0;$$

$u, v, w$, being any coordinates whatever, then if for these we substitute $x, y, z$, the coordinates of the point of application of this force, this equation will become

$$Ax + By + Cz = 0,$$

moreover, the equations of a right line drawn through the origin of the coordinates and parallel to this force, will be

$$v\cos\alpha = u\cos\beta, \quad w\cos\alpha = u\cos\gamma;$$

and as this parallel is likewise comprised in the plane which we are considering, there results from it this second equation of condition

$$A\cos\alpha + B\cos\beta + C\cos\gamma = 0$$

From these two equations we deduce($c$)

$$C = \frac{A(x\cos\beta - y\cos\alpha)}{y\cos\gamma - z\cos\beta},$$

$$B = \frac{A(z\cos\alpha - x\cos\gamma)}{y\cos\gamma - z\cos\beta},$$

and by substituting these values in the equation of the plane, it becomes

$$u(y\cos\gamma - z\cos\beta) + v(z\cos\alpha - x\cos\gamma) + w(x\cos\beta - y\cos\alpha) = 0.$$

Now, by known formulæ (No 17), the cosines of $\delta, \delta_1, \delta_2$, the angles that the normal to this plane makes with the axes of $u, v, w$, which are likewise those of $x, y, z$, will have for values the formulæ which it was required to verify.

In virtue of the equation $H = Pp$, the quantity $p$ is the perpendicular let fall from the origin of the coordinates on the

direction of the force P. This may be also verified without any difficulty, by taking the foot of this perpendicular for the point of application of P, for, denoting the radius vector of this point, which will then be this perpendicular, by $r$, and the angles which its direction makes with the axes of $x$, $y$, $z$, by $\lambda$, $\mu$, $\nu$, we shall have

$$x = r\cos\lambda, \quad y = r\cos\mu, \quad z = r\cos\nu,$$

and if these values be substituted in that of $p^2$, in consequence of equations (Nos 6 and 9), namely,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$
$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1,$$
$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 0,$$

we shall find($d$)

$$p^2 = r^2, \text{ or } p = r.$$

276 The moments of the same system of forces with respect to different axes, possess remarkable properties, which being an immediate consequence of those of the projections of plane surfaces on different planes, we now proceed to investigate.

Let O$x$, O$y$, O$z$, be three rectangular axes which intersect in the point O (fig 70) Through this point let three other rectangular axes O$x'$, O$y'$, O$z'$, be also drawn. In order to determine the directions of these new axes with respect to the first, let

$$x\mathrm{O}x' = \alpha, \quad y\mathrm{O}x' = \beta, \quad z\mathrm{O}x' = \gamma,$$
$$x\mathrm{O}y' = \alpha', \quad y\mathrm{O}y' = \beta', \quad z\mathrm{O}y' = \gamma',$$
$$x\mathrm{O}z' = \alpha'', \quad y\mathrm{O}z' = \beta'', \quad z\mathrm{O}z' = \gamma'',$$

$\alpha$, $\beta$, $\gamma$, $\alpha'$, &c , being considered as nine given angles, which may be either acute or obtuse. Their cosines will be connected together by six equations, for with respect to each of the three lines O$x'$, O$y'$, O$z'$, we shall have evidently,

$$\left.\begin{aligned}\cos^2\alpha+\cos^2\beta+\cos^2\gamma&=1,\\ \cos^2\alpha'+\cos^2\beta'+\cos^2\gamma'&=1,\\ \cos^2\alpha''+\cos^2\beta''+\cos^2\gamma''&=1,\end{aligned}\right\}\quad(1)$$

and because $x'oy'$, $x'oz'$, $y'oz'$, are respectively right angles, we shall likewise have

$$\left.\begin{aligned}\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'&=0,\\ \cos\alpha\cos\alpha''+\cos\beta\cos\beta''+\cos\gamma\cos\gamma''&=0,\\ \cos\alpha'\cos\alpha''+\cos\beta'\cos\beta''+\cos\gamma'\cos\gamma''&=0\end{aligned}\right\}\quad(2)$$

By means of the nine angles $\alpha$, $\alpha'$, $\alpha''$, &c., we can reciprocally determine the directions of the first axes $ox$, $oy$, $oz$, with respect to the second $ox'$, $oy'$, $oz'$, in which case we shall have

$$\left.\begin{aligned}\cos^2\alpha+\cos^2\alpha'+\cos^2\alpha''&=1,\\ \cos^2\beta+\cos^2\beta'+\cos^2\beta''&=1,\\ \cos^2\gamma+\cos^2\gamma'+\cos^2\gamma''&=1,\end{aligned}\right\}\quad(3)$$

and, besides,

$$\left.\begin{aligned}\cos\alpha\cos\beta+\cos\alpha'\cos\beta'+\cos\alpha''\cos\beta''&=0,\\ \cos\alpha\cos\gamma+\cos\alpha'\cos\gamma'+\cos\alpha''\cos\gamma''&=0,\\ \cos\beta\cos\gamma+\cos\beta'\cos\gamma'+\cos\beta''\cos\gamma''&=0;\end{aligned}\right\}\quad(4)$$

these equations are evidently equivalent to the six preceding, and may be substituted for them

If $a$ denotes the area of a plane surface bounded by any outline whatever, and situated in a plane passing through the point o, and if a perpendicular OD be erected at this point to this plane, then by making

$$x\text{OD}=q,\quad y\text{OD}=q',\quad z\text{OD}=q'',$$

these three angles, which may be either acute or obtuse, will determine the direction of OD and that of the plane of $a$; if each of these three be changed into their respective supplements, the right line OD will also be changed into its production, but the plane of $a$ will remain the same as before.

Also, if the projections of $a$ on the planes of $yoz$, $xoz$, $xoy$, be denoted by $p, p', p''$, we shall have (No 10)

$$p = a \cos q, \quad p' = a \cos q', \quad p'' = a \cos q''.$$

Finally, if $b$ denotes the projection of $a$ on any fourth plane, as for instance the plane $y'oz'$, and if $c$ denotes the angle $x'$OD, we shall also have

$$b = a \cos c,$$

and, by formula (2) of No. 9,

$$\cos c = \cos q \cos \alpha + \cos q' \cos \beta + \cos q'' \cos \gamma, \qquad (5)$$

hence we infer,

$$b = p \cos \alpha + p' \cos \beta + p'' \cos \gamma, \qquad (6)$$

by means of which the projection of an area $a$ on any plane whatever, will be determined, when its projections on three rectangular planes are known

As equation (5) obtains only when the signs of the cosines which it contains are taken into account, it follows that it is also necessary to have regard in equation (6), to the signs of the projections $p, p', p''$, and to treat them as positive or negative, according as OD the perpendicular to the plane of $a$, makes acute or obtuse angles with the axes $ox$, $oy$, $oz$

277. This being established, let any number of plane areas whatever, such as $a, a', a''$, &c., existing in different planes, be projected on the three planes $xoy$, $xoz$, $yoz$, and let the sum of the projections on each of these planes be taken, respect being had to their signs in the manner above specified, then if the three sums, which are obtained in this manner, be denoted by A, A', A'', and if in like manner B denotes the sum of the projections of $a, a', a''$, &c , on the plane $y'oz'$, by forming for each of these areas, an equation similar to equation (6), we shall obtain, by adding together all these equations,

$$\text{B} = \text{A} \cos \alpha + \text{A}' \cos \beta + \text{A}'' \cos \gamma$$

If again the sum of the projections of $a, a', a''$, &c., on the plane $x'oz'$ be denoted by B$'$, it is evident that the value of B$'$ may be deduced from that of B, by substituting the axis $oy'$ perpendicular to this plane, for the axis $ox'$ perpendicular to the plane $y'oz'$, that is to say, by substituting in the preceding formula, $\alpha', \beta', \gamma'$, for $\alpha, \beta, \gamma$, by this means we obtain

$$\text{B}' = \text{A} \cos \alpha' + \text{B} \cos \beta' + \text{C} \cos \gamma'.$$

If, in like manner, the sum of the projections of $a, a', a''$ &c., on the plane $x'oy'$ be denoted by B$''$, its value may be deduced from that of B, by substituting $\alpha'', \beta'', \gamma''$, in place of $\alpha, \beta, \gamma$, from which there will result

$$\text{B}'' = \text{A} \cos \alpha'' + \text{B} \cos \beta'' + \text{C} \cos \gamma''.$$

We can likewise deduce conversely from these values of B, B$'$, B$''$, and by having regard to equations (3) and (4),

$$\left.\begin{aligned} \text{A} &= \text{B} \cos \alpha + \text{B}' \cos \alpha' + \text{B}'' \cos \alpha'', \\ \text{A}' &= \text{B} \cos \beta + \text{B}' \cos \beta + \text{B}'' \cos \beta'', \\ \text{A}'' &= \text{B} \cos \gamma + \text{B}' \cos \gamma' + \text{B}'' \cos \gamma''. \end{aligned}\right\} \quad (7)$$

From a consideration of these different equations, it is evident, that the projections of plane surfaces on different planes, are subject to the same laws as those of right lines on other right lines.

278. If the sum of the squares of the values of B, B$'$, B$''$, be taken, there results, by equations (3) and (4),

$$\text{B}^2 + \text{B}'^2 + \text{B}''^2 = \text{A}^2 + \text{A}'^2 + \text{A}''^2, \quad (8)$$

from which it appears, that the sum of the squares of these three quantities B, B$'$, B$''$, does not vary with the direction of the three rectangular planes of projectio to wl ich they are referred In the particular case, in which all the areas $a, a', a''$, &c , exist in the same plane, this sum is in fact the square of the entire area $a + a' + a'' +$ &c , and, if for example, this plane be assumed to be that of the axes $oy$ and $oz$, we shall have

$$A = a + a' + a'' + \&c., \quad A' = 0, \quad A'' = 0.$$

In this case all the areas $a, a', a''$, &c., lie in the same plane, but if they should exist in different planes, then it is easy to prove, that this sum is equal to $\sqrt{A^2 + A'^2 + A''^2}$, in fact, by equation (8) we have

$$B = \sqrt{A^2 + A'^2 + A''^2 - B'^2 - B''^2},$$

from which it appears, that the sum B which varies from one plane of projection to another, is greatest when $B' = 0, B'' = 0$; in which case it is equal to $\sqrt{A^2 + A'^2 + A''^2}$, this is, therefore, the expression for the greatest sum of the projections on the same plane, of the plane areas $a, a', a''$, &c., considered as existing in different planes.

279 The plane $y'oz'$ that answers to this greatest projection, possesses remarkable properties, which will be adverted to in the sequel of this treatise. It is easy to determine its position by means of the equations $B' = 0$, $B'' = 0$, which characterize it.

In fact, equations (7) are then reduced to

$$A = B \cos \alpha, \quad A' = B \cos \beta, \quad A'' = B \cos \gamma,$$

from which we obtain

$$\cos \alpha = \frac{A}{\sqrt{A^2 + A'^2 + A''^2}},$$

$$\cos \beta = \frac{A'}{\sqrt{A^2 + A'^2 + A''^2}},$$

$$\cos \gamma = \frac{A''}{\sqrt{A^2 + A'^2 + A''^2}}.$$

Hence, when $A, A', A''$, the sums of the projections on any three rectangular planes $yox, xoz, xoy$, selected arbitrarily, are known, the direction of $y'oz'$, the plane of greatest projection, may be immediately ascertained, by means of the angles $\alpha, \beta, \gamma$, which determine the line $ox'$ perpendicular to this plane With respect to its *absolute* position in space, that is evidently unde-

termined, for the projections of each of the areas $a$, $a'$, $a''$, &c, and, consequently, the sum of these projections, are the same on all parallel planes

280 The sum of the projections of the areas $a$, $a'$, $a''$, &c., on all planes equally inclined to the plane of greatest projection, is always the same

In order to demonstrate this, let the plane perpendicular to the line OD be taken, let C denote the sum of the projections of $a, a', a''$, &c, on this plane, $q$, $q'$, $q''$, the angles which this line OD makes with the axes O$x$, O$y$, O$z$, &c, $c$ the angle $x'$OD, which measures the inclination of this plane on that of the greatest projection By what has been established in No 277, we shall have

$$\mathrm{C} = \mathrm{A} \cos q + \mathrm{A}' \cos q' + \mathrm{A}'' \cos q''$$

Hence, by substituting for A, A', A'', their respective values B $\cos \alpha$, B $\cos \beta$, B $\cos \gamma$, we shall have

$$\mathrm{C} = \mathrm{B} (\cos \alpha \cos q + \cos \beta \cos q' + \cos \gamma \cos q''),$$

that is, in virtue of formula (5), $\mathrm{C} = \mathrm{B} \cos c$, and, substituting for B its value,

$$\mathrm{C} = \sqrt{\mathrm{A}^2 + \mathrm{A}'^2 + \mathrm{A}''^2} \cos c,$$

consequently, the value of C, is the same for all planes, which make the same angle $c$, with $y'$O$z'$, the plane of greatest projection

This value diminishes according as the angle $c$ approaches to 90°, and it is cypher for all planes perpendicular to $y'$O$z'$

281 In order to apply these propositions relative to the projections on plane surfaces to the theory of moments, it is only necessary to suppose that the areas $a$, $a'$, $a''$, &c, are the doubles of triangles which have for their common vertex the point O, and for bases the lines, which represent, in magnitude and direction, the forces P, P', P'', &c, that have been considered above Their moments L, M, N, with respect to the

axes $oz$, $oy$, $ox$, of the positive coordinates of their points of application (No 274) will be then the sum of the projections of $a, a', a''$, &c , on the planes $xoy$, $xoz$, $yoz$ The following consequences result from the propositions which have been now demonstrated

1st Denoting by E the moment of the forces P, P′, P″, &c , with respect to an axis passing through the point O, and that makes with the axes $ox, oy, oz$, the angles $\varepsilon, \varepsilon', \varepsilon''$, which may be either acute or obtuse, we shall have

$$\text{E} = \text{N} \cos \varepsilon + \text{M} \cos \varepsilon' + \text{L} \cos \varepsilon''$$

2ndly. Among all the directions which the axis of the moment E can assume about the point O, there is one for which this moment is the greatest possible and equal $\sqrt{\text{L}^2+\text{M}^2+\text{N}^2}$ With respect to every other axis passing through the point O, and perpendicular to that of the greatest moment, the moment E is cypher, and it is equal to $\sqrt{\text{L}^2+\text{M}^2+\text{N}^2}\cos\delta$, relatively to an axis which makes the angle $\delta$ with that of the greatest moment

3rdly Finally, if $\alpha, \beta, \gamma$, denote the angles that the axis of the greatest moment passing through the point O, makes with $ox, oy, oz$, the axes of the moments N, M, L, and if G represents the magnitude of this greatest moment, we shall have

$$\cos\alpha = \frac{\text{N}}{\text{G}}, \quad \cos\beta = \frac{\text{M}}{\text{G}}, \quad \cos\gamma = \frac{\text{L}}{\text{G}},$$

and, at the same time,

$$\text{G} = \sqrt{\text{L}^2+\text{M}^2+\text{N}^2},$$

hence it follows, that if on the axes $ox, oy, oz$, we take, reckoning from the point O, lines proportional to the moments N, M, L, and if the parallellopiped, of which these lines are the adjacent sides, be completed, the length of its diagonal will represent the magnitude of the greatest moment, and this line will be the axis of this principal moment

Euler was the person who first announced these remarkable theorems They establish a complete analogy between the composition of moments and that of forces, an analogy

which arises from this, that forces being represented by right lines, moments are expressed by plane surfaces, which are projected on different planes, in the same manner as lines are projected on different lines (No. 277)

282. The point o and the system of forces P, P′, P″, &c , being given, their greatest moment G is termed the *principal moment* of these forces If all these forces be transferred parallel to themselves, to this point o, their resultant R will have for components parallel to the axes o$x$, o$y$, o$z$, respectively, the three quantities X, Y, Z, of No 261 The consideration of this resultant and of the principal moment, furnishes us with a very simple means of stating the results of the preceding chapter.

In order to the equilibrium of the forces P, P′, P″, &c , applied to a solid body entirely free, it suffices if the resultant R and the principal moment G are respectively equal to cypher, for since

$$R^2 = X^2 + Y^2 + Z^2, \quad G^2 = L^2 + M^2 + N^2,$$

the equations $R = 0$, $G = 0$, imply that the six equations of equilibrium of No 261 have place

Hence it is easy to infer, that when one system of forces is in equilibrio with another, it is necessary, and it suffices, 1st, that the resultants R of the forces of the two systems should be respectively equal and contrary, 2ndly, that for the same point o, their principal moments should be equal, and should refer to two axes, drawn in opposite directions, or of which the one is the production of the other The resultant R and its direction, the principal moment and the direction of its axis, will remain the same, in all the transformations which the same system of forces can undergo, and, generally, for any two systems of equivalent forces.

If $a$, $b$, $c$ be the angles which the force R makes with the axes o$x$, o$y$, o$z$, we shall have

$$\cos a = \frac{X}{R}, \quad \cos b = \frac{Y}{R}, \quad \cos c = \frac{Z}{R}.$$

Likewise, if $\omega$ be the angle comprised between its direction and the axis of the principal moment, we shall have, ($\alpha$, $\beta$, $\gamma$, being the angles which this axis makes with $ox$, $oy$, $oz$),

$$\cos\omega = \cos a \cos\alpha + \cos b \cos\beta + \cos c \cos\gamma,$$

or, what comes to the same thing,

$$\cos\omega = \frac{\text{XN} + \text{YM} + \text{ZL}}{\text{RG}}$$

It follows, therefore, that the condition of an unique resultant which is expressed (No 263) by the equation

$$\text{XN} + \text{YM} + \text{ZL} = 0,$$

consists in this, that the axis of the principal moment G, and the direction of the resultant R, should be at right angles to each other This is what can in fact be verified by observing, that if the forces P, P′, P″, &c , in their actual position, have an unique resultant, this force must be equal and parallel to R, and that its moment with respect to the point O, must also be the principal moment G, so that the axis of the principal moment is then perpendicular to this resultant transferred to the point O in a direction parallel to itself, but this reasoning does not suffice to prove *conversely* that, when the preceding equation obtains, the given forces have an unique resultant

283 If the point O be transferred to any other point $O_1$, and if $x_1$, $y_1$, $z_1$, be the coordinates of the point O, with respect to the axes $ox$, $oy$, $oz$, and if $L_1$, $M_1$, $N_1$, be what L, M, N, become relatively to this point $O_1$, the values of these last quantities may be obtained from the first (No. 261), by substituting $x-x_1$, $y-y_1$, $z-z_1$, in place $x$, $y$, $z$, from which there will result ($e$)

$$\left.\begin{aligned} L_1 &= L + Xy_1 - Yx_1, \\ M_1 &= M + Zx_1 - Xz_1, \\ N_1 &= N + Yz_1 - Zy_1 \end{aligned}\right\} \quad (a)$$

From these formulæ it appears that when P, P′, P″, &c. are

reduced to equal and parallel forces, which though acting in contrary directions, still are not directly opposed to each other, in which case we have $X = 0$, $Y = 0$, $Z = 0$, the quantities $L_1$, $M_1$, $N_1$ are independent of the coordinates of the point $O_1$, so that the magnitude of the principal moment and the direction of its axis do not vary with the position of this point In fact it is evident, that, whatever be the position of the point $O_1$, the axis of the principal moment of the two parallel forces which may be substituted for the given forces $P$, $P'$, $P''$, &c, is perpendicular to their plane, and we know from other considerations (No 48) that the sum of the moments of these two forces, which will be the principal moment of the given forces, is a constant quantity

In every other case, the principal moment varies with the position of the point $O_1$, and if it were required to determine the point, or series of points, for which this moment is a *minimum*, by making generally

$$G_1^2 = L_1^2 + M_1^2 + N_1^2$$

we shall have

$$G_1^2 = (L + Xy_1 - Yx_1)^2 + (M + Zx_1 - Xz_1)^2 + (N + Yz_1 - Zy_1)^2$$

If in order to determine the *minimum* value, its three partial differences with respect to $x_1$, $y_1$, $z_1$, be put equal to cypher, and if we observe that

$$R^2 = L^2 + M^2 + N^2,$$

three equations result, which it is easy to write under the following form, (*f*)

$$R^2x_1 = X(Xx_1 + Yy_1 + Zz_1) + YL - ZM,$$
$$R^2y_1 = Y(Xx_1 + Yy_1 + Zz_1) + ZN - XL,$$
$$R^2z_1 = Z(Xx_1 + Yy_1 + Zz_1) + XM - YN$$

Now, if these three equations be multiplied by $X$, $Y$, $Z$ respectively, and then added together, there will result an

identical equation, it follows, therefore, that one of them is a consequence of the other two, and as the coordinates $x_1, y_1, z_1$ do not exceed the first dimension, they appertain to a right line, which is therefore the locus of the centres of the moments, with respect to which the principal moment is a *minimum*. It is evidently unnecessary to examine whether the preceding equations determine a *maximum* or a *minimum*, for the value of G is not susceptible of being a *maximum*, inasmuch as it increases indefinitely with the variables $x_1, y_1, z_1$.

284. If the quantity $Xx_1 + Yy_1 + Zz_1$ be eliminated between the preceding equations taken successively two by two, there results ($g$)

$$\left.\begin{aligned} Xy_1 - Yx_1 + L &= Z\frac{(NX + MY + LZ)}{R^2}, \\ Zx_1 - Xz_1 + M &= Y\frac{(NX + MY + LZ)}{R^2}, \\ Yz_1 - Zy_1 + N &= X\frac{(NX + MY + LZ)}{R^2}, \end{aligned}\right\} \quad (b)$$

which are the equations of the projections on the three planes of the coordinates, of the loci of the centres of the principal *minima* moments.

By squaring, and then adding them together, we obtain

$$G_1 = \frac{NX + MY + LZ}{R} \quad (c)$$

for the value of the principal *minimum* moment, which is consequently the same for all the centres $O_1$.

Denoting the angles that the axis of the moment $G_1$ makes with parallels to the axes $Ox$, $Oy$, $Oz$, drawn through the point $O_1$, by $\alpha_1, \beta_1, \gamma_1$, we shall have

$$\cos\alpha_1 = \frac{N_1}{G_1}, \quad \cos\beta_1 = \frac{M_1}{G_1}, \quad \cos\gamma_1 = \frac{L_1}{G_1},$$

wherever the centre of the moments may be, and, from equa-

tions ($a$), ($b$), ($c$), there will result in particular, for a point $o_1$ which belongs to the right line determined by equations ($b$)

$$\cos\alpha_1 = \frac{X}{R}, \quad \cos\beta_1 = \frac{Y}{R}, \quad \cos\gamma_1 = \frac{Z}{R},$$

which shews that the axes of all the principal *minima* moments, whose common value is given by formula (c), are parallel to each other and to the direction of the force R

When the given forces have an unique resultant, it is evident that the value of $G_1$ is the least of all when the point $o_1$ exists on its direction, in which case this value is cypher Conversely, if the value of $G_1$ is cypher with respect to any point $o_1$, it follows that the given forces $P$, $P'$, $P''$, &c have an unique resultant passing through this point, for if they can be reduced to two forces not existing in the same plane, one of them may be made to pass through the point $o_1$, and thus reduce their principal moment to that of the other force, which would not be cypher, contrary to the hypothesis From this it follows, that the condition which is necessary and sufficient in order that the given forces may have an unique resultant consists in this, that their principal moment may be equal to cypher. This moment being then a *minimum*, the condition in question will, in consequence of formula (c), be expressed by the equation

$$LZ + MY + NX = 0,$$

and as the point o, to which it refers, belongs to this resultant, the equations of the line in the direction of which it acts will be, in virtue of equations (b),

$$Xy_1 - Yx_1 + L = 0,$$
$$Zx_1 - Xz_1 + M = 0,$$
$$Yz_1 - Zy_1 + N = 0$$

These results coincide with those of No 263, that have been obtained from other considerations

## CHAPTER III

### EXAMPLES OF THE EQUILIBRIUM OF A FLEXIBLE BODY.

#### I *Equilibrium of the Funicular Polygon*

285. EVERY system of cords connected together by fixed knots, or merely run into rings which can slide along these cords, is termed a *funicular machine* Any number of strings may terminate at the same knot, but for greater simplicity we will suppose that at each knot there are never more than three strings, and, in the first place, we will exclude the consideration of moveable rings

Let then A and B (fig 71) be the two extremities of a cord perfectly flexible, and of any length whatever, and, M, M', M'', &c, being different points of this cord, let the strings MC, M'C', M''C'', &c, drawn in the direction in which the given forces P, P', P'', &c act, be attached to these points, likewise to the point M let there be applied a given force H, acting in the direction of the string MA, and to the last of the points M, M', M'', &c, let another given force K, directed towards the point B, be applied In the state of equilibrium, this flexible cord will constitute a polygon, the summits of which will be the points A, M, M', M,''. . B, and it is termed a *funicular polygon* The problem to be solved with respect to it, is to find the conditions, which the given forces H, P, P', P'', . K, must satisfy, in order that this equilibrium may be possible, and to determine the figure of the polygon which suits this state

In order to find these conditions, this self-evident principle must be assumed, namely, that when the equilibrium obtains, each of the strings MM', M'M'', &c, must be drawn at its

two extremities by equal forces, acting in the directions of its productions; for if these two forces had not the same directions as the string, nothing would prevent them from deflecting it; and, unless they were equal and contrary, they would cause the string to move in its direction.

It follows at once, that the resultant of the two forces H and P, applied to the point M, must coincide with MD the production of the string M′M. We may therefore transfer the point of application of this force to the point M′, which exists on its direction (No. 41); if it be then compounded with the force P′ applied to this point, this second resultant, which will be in fact that of the three forces H, P, P′, must coincide with M′D′, the production of the string M″M′; and, consequently, it may be transferred to the point M″. If, in the same manner, the resultant of this force and of P″, which acts at this same point M″, be taken, the force which draws the string M″M‴ at its extremity M″, and which should act in the direction of its production M″D″, will be obtained. This force is evidently the resultant of the forces H, P, P′, P″; by similar reasoning it may be shewn, that the force which draws the same string at its extremity M‴, and which must coincide with its other production M‴D‴, is the resultant of the forces P‴, P$^{IV}$, K; therefore these two resultants are equal, and directly opposed to each other; and consequently, the resultant of all the given forces H, P, P′, P″, K, must be equal to cypher. We would evidently arrive at the same result, if the forces which act at the two extremities of any other side of the polygon were considered.

Hence, the forces applied to the funicular polygon should be such, that when they are transferred to the same point respectively parallel to themselves, they may constitute an equilibrium. This furnishes us, as we know, with three equations, between the magnitudes of these forces and the angles which their directions make with three rectangular axes drawn through this point. These equations are (No. 35)

$$\left.\begin{aligned} \text{H}\cos a + \text{K}\cos e + \text{P}\cos \alpha + \text{P}'\cos \alpha' + \&\text{c.} = 0, \\ \text{H}\cos b + \text{K}\cos f + \text{P}\cos \beta + \text{P}'\cos \beta' + \&\text{c} = 0, \\ \text{H}\cos c + \text{K}\cos g + \text{P}\cos \gamma + \text{P}'\cos \gamma' + \&\text{c} = 0, \end{aligned}\right\} \quad (a)$$

$a$, $e$, $\alpha$, $\alpha'$, &c denoting the angles relative to one of the axes, $b$, $f$, $\beta$, $\beta'$, &c. the angles relative to another axis, and $c$, $g$, $\gamma$, $\gamma'$, &c those which refer to the third

286. When the forces H, P, P′, P″, K, and the directions of the strings along which they act, do not satisfy these equations, an equilibrium cannot be effected between them by means of the funicular polygon, whatever be the figure it is made to assume, but whenever these equations are satisfied, we may assign to the polygon a figure suitable to the existence of the equilibrium. As the magnitudes and directions of the forces H, P, P′, P″, . . . K, are given, this figure is necessarily determinate, and its construction results from the series of compositions of forces which we proceed now to point out.

In fact, the directions of the strings MA and MC along which the forces H and P act, being known, the magnitude and direction of their resultant will be determined. On the production of this direction, reckoning from the point M, let there be taken the given length of the side MM′, and then to the point M′ let there be applied the resultant of H and P acting in the direction of the line M′M, and the force P′ acting in the given direction of the string M′C′ Let the resultant of these two forces be taken, and on the production of its direction, let there be laid off, reckoning from the point M′, the given length of the side M′M″, and at the point M″, let a construction similar to that indicated for the point M′ be performed, and then at the point M″ let there be applied this last resultant in the direction of the side M″M′, and the force P″ in the given direction of the string M″C″, these two forces must then be compounded into one, and, on the production of the resultant, the given length of the side M″M′ should be taken

This operation should be continued until we arrive at the

last of the knots M, M′, M″, &c, if in the present case we suppose the last of them to be $M^{IV}$, then $M^{IV}B$ will be the last side of the polygon. Its direction is known, being that of the last force K, which by hypothesis is given. It is therefore necessary that the produced direction of the resultant of the two forces applied to the point $M^{IV}$, in the direction of the side $M'''M^{IV}$, and in the direction of the string $M^{IV}C^{IV}$, should coincide with the given direction of the side $M^{IV}B$. This is, in fact, what will always be the case, for, by our construction, the force acting in the direction of $M^{IV}M'''$, is the resultant itself of the five forces H, P, P′, P″, P‴, transferred to the point $M^{IV}$ parallel to their directions, and if it be compounded with the force $P^{IV}$, acting in the direction $M^{IV}C^{IV}$, the resultant of all the given forces, except the force K, will be obtained, and, in consequence of equations (a), which are supposed to be satisfied, this resultant is equal and directly opposite to the force K (No. 35)

If through the point A, the three axes to which the angles $a, e, a', a'$, &c, $b, f, \beta, \beta'$, &c, $c, g, \gamma, \gamma'$, &c, are referred, be drawn, the coordinates of each of the summits of the polygon, with respect to these axes, will be the projections on these same axes of the parts of the polygon intercepted between the point A and this summit. They may be determined in functions of these angles, of the lengths of the sides of the polygon and of the given forces, the general formulæ which are obtained in this manner, will enable us, in each particular case, to construct directly all the summits of the polygon, or one or more of these points, but it is simpler to determine them, one after another, in the manner pointed out above (*a*).

287 When the given forces satisfy the conditions expressed by equations (a), and when the polygon has been made to assume the figure suitable to the equilibrium, the common intensity of the two equal and opposite forces which draw each of the sides in the direction of its production, is the *tension* which this string experiences, it is therefore of consequence, to be

able in practice to compute this tension, and to be assured, by experiment, that it is not greater than that which a string of the same diameter and material can bear without breaking.

Now, from what precedes it appears, that this tension will vary from one side to another of the polygon; the tension of the side MM′ will be equal to the resultant of the forces H and P, or to that of the forces P′, P″, P‴, &c. . . K; the tension of the side M′M″ will be equal to the resultant of the forces H, P, P′, or to that of the forces P″, P‴, . . . K, and so on. It will therefore be easy, in each particular case, to determine the tensions, which all the sides of a polygon in equilibrio experience, when the magnitudes and directions of the forces H, P, P′, P″, . . . K, are given.

If A and B, the extreme points of the polygon, are fixed, the forces H and K will represent both the tensions of the strings that terminate at these points, and also the pressures which these points sustain. In this case, the values of H and K, and of the angles $a$, $b$, $c$, $e$, $f$, $g$, which determine the directions of the two extreme sides of the polygon, will be no longer given; but we shall have eight equations, by means of which these eight unknown quantities can be determined; these will be, equations (a), equations (No. 6), namely,

$$\cos^2 a + \cos^2 b + \cos^2 c = 1, \quad \cos^2 e + \cos^2 f + \cos^2 g = 1,$$

and three equations resulting from the consideration that the position of the two fixed points A and B is given. These are obtained by computing the values of the three coordinates of one of these points referred to axes passing through the other point, that is to say, the projections of the entire polygon on these three axes, and then putting them equal to the given values of these same coordinates.

In general, the determination of these eight unknown quantities is extremely complicated; but after that the funicular polygon has assumed of itself the figure suitable to the equilibrium of the forces, which are applied at its summits,

the tensions of these different sides may be obtained without any difficulty; and this is sufficient in practice. Thus, if the force P applied to the point M be resolved into two others acting in the directions of the productions of the sides AM and MM′, the components, which are given immediately by the rule of the parallellogram of forces, will be the tensions of these two sides. That which will act along the production of AM, must be equal to the force acting in the direction of this first side, when the point A is free, and when it is fixed, it will express the pressure exercised on this point. In like manner, the components of the force P′ in the directions of the productions of MM′ and M′M″ will express the tension of MM′, already known by the decomposition of P, and that of the adjacent side M′M″; and so on

288 As the strings which constitute the different sides of a funicular polygon are always a little extensible, the length of each of them is increased by a small quantity, in consequence of the tension which it experiences in the state of equilibrium, and when this tension is known, the corresponding increase of length may be calculated.

In fact, it appears from experiment, that when the tension of a homogeneous thread of a constant thickness is inconsiderable, compared with the force necessary to break it, its increase of length is proportional to its length and to the tension to which it is subjected; and in different threads, it varies with the thickness and the material of which the thread consists This being so, if to a fixed point a thread is attached of the same thickness and material as the string AM, and if at its inferior extremity there be suspended a given weight $\Pi$ that is very great, relatively to that of the thread, and if $l$ and $l.(1+\omega)$ be its lengths before and after the suspension of the weight $\Pi$, this quantity $\omega$ will be a very small fraction, independent of $l$, and proportional to $\Pi$, (the weight of the thread being neglected), so that if, in another experiment, the three quantities $l$, $\omega$, $\Pi$ become $l'$, $\omega'$, $\Pi'$, we shall have

$$\omega' = \frac{\omega\Pi'}{\Pi},$$

whatever may be $l$ and $l'$. Now, it is evident that a thread attached to a fixed point, and drawn at its other extremity by a force acting in the direction of its production, is in the same condition as if it was drawn in the directions of its two productions by this same force. If therefore, T be the tension of the string AM, and if its length be increased in the ratio of $1 + \tau$ to unity, we shall have

$$\tau = \frac{\omega T}{\Pi},$$

by means of which, this increase of length can be determined; and in the same way the value of $\tau$ for all the other sides of the polygon can be obtained.

289. Whether the extreme points A and B of the polygon be fixed or free, if one or more of the knots M, M′, M″, &c. are replaced by rings, this circumstance will give rise to new conditions of equilibrium. If, for example, M″ is a moveable ring which can slide along the chord M′M″M‴, it is evident that in this motion the sum of the distances M′M″ and M″M‴ of the point M″, from the points M′ and M‴, will remain constant. Now, if the equilibrium obtains, this state will not be deranged by fixing these two last points; but then the point M″ will be in the same circumstances as if it was constrained to exist on an ellipsoid of revolution, of which M′ and M‴ are the two foci, and whose major axis is equal to the given length of the string M′M″M‴, therefore this point cannot be in equilibrio (No. 36), unless the force P″ which is applied to it is perpendicular to this surface; hence, by a known property of the ellipse it follows, that the direction of this force must divide the angle between the two radii vectores M′M″ and M″M‴ into two equal parts.

When, therefore, in performing the construction indicated in No. 286, we come to a moveable ring such as M″, if it is found

that, after we have taken the resultant of the two forces acting in the directions M'' M' and M''C'', of which the side M''M''' is the production, the angles C''M''M' and C''M''M''' are not equal, it follows that the equilibrium does not subsist In general, it is necessary that the direction of the string M''C'' attached to the moveable ring, should not be given beforehand, in order that we may be enabled, by a suitable determination of it, to satisfy the condition of the equality of the angles M'M''C'' and M'''M''C''

It is evident that when there is an equilibrium, the tensions of the two sides adjacent to a moveable ring will be equal to each other, this follows from the circumstance that these two sides make equal angles with the direction of the force applied to this ring, and that their tensions are equal to this force resolved in their respective directions, but, indeed, this equality of tension may be regarded as self-evident, inasmuch as the two sides, in the directions of which the ring can slide, constitute parts of the same string, which necessarily experiences an equal tension throughout its entire extent.

290 What has been stated relatively to a ring constrained to slide along an inextensible and perfectly flexible thread, may be applied to all the points of a system of material points in equilibrio In whatever manner these points are connected together, it is evident that this equilibrium will not be deranged, by fixing all the points of the system except one. Now, if the connexion of this point with the others is such that it may besides describe a surface, or only a curved line about these fixed points, it is evident that the moveable point will be in the same condition as if the surface or curved line actually existed, consequently, the direction of the force which is applied to it must be normal to this surface or this line

Hence it follows, that in any system of material points in equilibrio, the force applied to each of these points is perpendicular to the surface or line on which this point would be

constrained to exist, if, for an instant, all the other points to which it is connected, were considered as fixed points

When this condition relative to the direction of the forces, and to the connexion of the parts of the system, is not satisfied, we may be certain that the equilibrium does not exist, but on the other hand it is not of itself sufficient to secure the equilibrium of the system

291 If all the forces which act on a funicular polygon suspended at the two fixed points A and B, are given weights, it follows, from the construction of No 286, that this polygon must exist altogether in the vertical plane passing through these two points, this is indeed evident of itself, for there is no reason why it should deviate from this plane to one side rather than to the other Hence, if the perpendicular to this plane be assumed for the axis to which $c, g, \gamma', \gamma''$, &c are referred, all these angles will be right, and the third equation (a) will disappear, the two others will become

$$\left.\begin{aligned} \mathrm{H}\cos a + \mathrm{K}\cos e &= 0, \\ \mathrm{H}\cos b + \mathrm{K}\cos f + \mathrm{II} &= 0, \end{aligned}\right\} \qquad (b)$$

in which the angles $a, e, \alpha, \alpha'$, &c refer to an horizontal axis, and the angles $b, f, \beta, \beta'$, &c to an axis drawn in the direction of gravity, II denotes the sum of the weights P, P′, P″, &c which are applied to the polygon.

The equilibrium of this polygon will not be deranged, by making its form invariable, consequently, the force II must be equal and directly opposed to the resultant of the forces H and K From equations (b), we know already that it is equal and contrary to this resultant, it must therefore pass through the point O (fig 72), where the productions of the extreme strings AM and BN intersect, hence this point may be taken for the common point of application of the two forces H and K Thus, in the state of equilibrium, II the resultant of the vertical forces P, P′, P″, &c, will be directed along the vertical OD, and, consequently, we shall have (No. 29), (b)

$$H \quad \Pi \quad \sin BOD \quad \sin AOB,$$

$$K \quad \Pi \quad . \sin AOD \quad \sin AOB,$$

by means of which, the tensions of the extreme strings, or the pressures H and K on the two fixed points A and B, will be known, when the angles AOD and BOD are measured

292 The same remark is applicable to the tensions of strings which support a given weight, as has been already made relatively to the pressures experienced by the points of support of a horizontal plane, on which a weight is placed, (No. 270).

Let us suppose that the three strings attached to the fixed points A, B, C (fig. 73) are reunited at the point M, and that at this point a weight P is suspended, which acts in the direction of the vertical MD. On the production of this line, let a point D′ be assumed, and let a parallelloplped be constructed, of which MD′ is the diagonal, and whose three adjacent sides are MA′, MB′, MC′, taken on the directions of the three strings If the force P be represented by the line MD′, its components in the directions of the lines MA′, MB′, MC′, will be represented by these lines respectively, and they will express the tensions of the three strings MA, MB, MC, or the pressures on the three fixed points A, B, C, which will, in this case, be completely determined. But, when the number of strings which terminate at the point M is four, or a greater number, the force P may be resolved in their directions, in an infinite variety of different ways, so that their tensions and the pressures of the fixed points, will be no longer determined, and one or more of them may be either cypher or any arbitrary magnitude whatever Now, this indetermination really obtains in the abstract question, when the extensibility of the strings is not taken into account, but it no longer exists, when regard is had to this property of the material of which the string consists, then all the strings are lengthened by some small quantities, they may be ever so small, they depend on their num-

ber and on their relative positions, and by measuring the small increase of length for each of them, the tension of each string, or the pressure of each fixed point, which actually has place, may be determined

Thus, if we suppose that the string AM, for example, is lengthened in the ratio of $1 + \delta$ to unity, and if, besides, we know that a string of the same material and diameter is lengthened in the ratio of $1 + \omega$ to unity, when being suspended vertically from a fixed point, the weight P is attached to its inferior extremity, it follows from No 288 that the tension experienced by this string, or the pressure which the point A sustains, is equal to the product $\frac{\delta}{\omega}$ P.

If we denote by $\omega'$ and $\delta'$, $\omega''$ and $\delta''$, &c, what the fractions $\omega$ and $\delta$ become relatively to the strings MB, MC, &c, and by $\gamma, \gamma', \gamma''$, &c. the acute angles which the strings MA, MB, MC, &c., make with the vertical MD′, we must have

$$\frac{\delta}{\omega}\cos\gamma + \frac{\delta'}{\omega'}\cos\gamma' + \frac{\delta''}{\omega''}\cos\gamma'' + \&c, = 1,$$

in order that the sum of the vertical components of all the tensions may be equal to the weight P If the same strings are projected on a horizontal plane drawn through the point M, and if $\eta, \eta', \eta''$, &c, denote the angles which the projections of MA, MB, MC, &c make with a line MO arbitrarily traced in this plane, we shall also have(*d*)

$$\frac{\delta}{\omega}\sin\gamma\sin\eta + \frac{\delta'}{\omega'}\sin\gamma'\sin\eta' + \frac{\delta''}{\omega''}\sin\gamma''\sin\eta'' + \&c. = 0,$$

$$\frac{\delta}{\omega}\sin\gamma\cos\eta + \frac{\delta'}{\omega'}\sin\gamma'\cos\eta' + \frac{\delta''}{\omega''}\sin\gamma''\cos\eta'' + \&c. = 0;$$

which indicate that the resultant of all the tensions is a vertical force

When there are only three strings, these three equations are sufficient to enable us to determine the relations which exist between their tensions and the weight P, or the values of

$\frac{\delta}{\omega}, \frac{\delta'}{\omega'}, \frac{\delta''}{\omega''}$, by means of the angles which these three strings make with the vertical MD′, and of the angles comprised between the planes of this line and their directions When there are only two, their directions and this vertical exist in the same plane, this reduces the two last equations to one

## II *Equilibrium of a flexible Thread*

293. The case which we propose to consider first, is that of a homogeneous thread, which we suppose to be acted on by gravity, and to have a constant diameter If it be perfectly flexible, and attached at its extremities A and C (fig 74) to two fixed points, the figure that it assumes in the state of equilibrium is termed the *catenary*, all its points evidently exist in the vertical plane which passes through the two fixed points A and C, for there is no reason why it should deviate to the one side rather than to the other of this plane.

Let there be drawn through a point O situated in this plane, the two rectangular axes O$x$ and O$y$, which will be those of the positive coordinates, let O$x$ be the horizontal axis drawn on the same side as the point A, and O$y$ the vertical axis drawn through B, the lowest point of the curve, in a direction the opposite to that of gravity Let $x$ and $y$ be the coordinates OP and PM referred to these two axes, of M any point of the catenary, $s$ the arc BM, measured from the point B, and terminating at this point, and let $x'$, $y'$, $s'$ denote what $x$, $y$, $s$ become, relatively to another point of this curve which is so situated that $s' > s$.

If $p$ be the weight of the unit of length of the thread, when it exists in a horizontal plane, $p(s' - s)$ will, in this state, be the weight of any length $s' - s$ of this curve, since it is assumed to be homogeneous and of a constant diameter. If it be suspended at the two fixed points A and C, its different parts will be unequally elongated, on account of their re-

spective tensions, and, at the same time, since their masses do not undergo any change, either their densities, or their thicknesses must diminish, consequently, the weight of this length $s' - s$ will be no longer the same as before, however, if the material of which the thread consists is very little extensible, so that the small dilatations of its parts may be neglected, $p(s' - s)$ may still be considered as the weight of the arc MM' of the catenary

Moreover, let T, T', be the unknown forces that act at its extremities M and M', which arise from this, that these points are connected with the parts CM and AM'. By joining these forces to the weight $p(s' - s)$, we may consider MM' as entirely free, consequently, if $\alpha$ and $\beta$ denote the angles which the direction of the force T makes with the productions of $x$ and $y$, the coordinates of its point of application, and if $\alpha'$ and $\beta'$ denote corresponding angles relatively to the force T', we shall have

$$\left.\begin{aligned} \mathrm{T}\cos\alpha + \mathrm{T}'\cos\alpha' &= 0, \\ \mathrm{T}\cos\beta + \mathrm{T}'\cos\beta' &= p(s' - s), \\ \mathrm{T}(x\cos\beta - y\cos\alpha) + \mathrm{T}'(x'\cos\beta' - y'\cos\alpha') &= p(s' - s)x_1, \end{aligned}\right\} (a)$$

for the equilibrium of these three forces existing in the same plane (No 262), $x_1$ being the horizontal abscissa of the centre of gravity of the arc MM'. These equations will obtain, whatever be the length of this arc, if it be infinitely small, then the infinitely small quantities of the second order may be neglected in these equations, but the quantities of the first order must be retained, this, however, will not prevent us from considering the force T as acting in the direction of MH, which is a tangent to the curve at the extremity M, and the force T' as acting in the direction of M'H', the tangent at the other extremity M'

In order to be satisfied of this, let there be taken on MM' a point $m$, such that the arc M$m$ may be an infinitely small quantity of the second order, this will enable us to neglect the weight

of this part of the catenary The equilibrium will not be deranged by fixing the point $m$, but as the thread is supposed to be perfectly flexible, there is nothing to prevent the force T from causing the arc M$m$ to turn about $m$, if it did not act in the direction of its production MH In the same way it may be shewn, that the force T′ must act in the direction of M′H′.

In consequence of this, we shall have

$$\cos a = -\frac{dx}{ds}, \quad \cos\beta = -\frac{dy}{ds},$$

$$\cos a' = \frac{dx'}{ds'}, \quad \cos\beta' = \frac{dy'}{ds'},$$

and by neglecting infinitely small quantities of the second order, these last values will become

$$\cos a' = \frac{dx}{ds} + d.\frac{dx}{ds}, \quad \cos\beta' = \frac{dy}{ds} + d\ \frac{dy}{ds}$$

It may be likewise shewn, that T′$=$ T $+ d$T In fact, the quantity T is a function of the coordinates of any point M to which it refers, which, consequently, becomes at the point M′ T$+d$T, at this point, it expresses the force which acts in the direction of MH$_1$, the production of H′M′, on AM′ the upper part of the catenary. But, if $m'$ be a point of the curve, the distance of which from M′ is an infinitely small quantity of the second order, the force which acts at $m'$ on the part A$m'$, will be the same, in magnitude and direction, as that which acts at M′ on AM′, consequently, the part M′$m'$ of the catenary is drawn in opposite directions, along M′H′ and $m'$H$_1$, by the forces T′ and T $+ d$T, which must be equal in order that M′$m'$ may continue in equilibrio

This being established, if these different values are substituted in the two first equations (a), they will become, by making $s' - s = ds(e)$,

$$d.\text{T}\frac{dx}{ds} = 0, \quad d.\text{T}\frac{dy}{ds} = pds \qquad \text{(b)}$$

As to the third, it will assume the form

$$d\,\text{T}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)=pxds,$$

in which $x$ is substituted in place of $x_1$ in the second member, which substitution we are permitted to make, as infinitely small quantities of the second order are neglected. Now, this equation being the same thing as

$$xd\,\text{T}\frac{dy}{ds}-yd\,\text{T}\frac{dx}{ds}=pxds,$$

it is evidently a consequence of the two others. In point of fact, the problem cannot depend on more than two equations, inasmuch as there are only two unknown quantities to be determined in functions of $x$, namely, $y$ and T. The first makes known the equation of the curve, the second determines the tension in any point M, that is to say, the magnitude of the equal forces which draw the element M$m$ along its two productions.

294. The integral of the first equation (b) is

$$\text{T}\frac{dx}{ds}=c,$$

in which $c$ denotes a constant arbitrary. At the point B, we have $\frac{dx}{ds}=1$, and $\text{T}=c$; if, therefore, the tension in this lowest point be denoted by the weight of a certain length, such as $h$, of the thread, we shall have $c=ph$, and, in any point whatever,

$$\text{T}=ph\frac{ds}{dx}.$$

The second equation (b) will therefore become

$$hd.\frac{dy}{dx}=ds,$$

from which we deduce

$$s=h\frac{dy}{dx},$$

there is no constant arbitrary, as we have, at the same time, $s = 0$ and $\frac{dy}{dx} = 0$, at the point B By means of these equations, the arc $s$ and the tension T will be immediately known, when the ordinate $y$ shall have been determined in a function of $x$

By substituting in the preceding equation, in the place of $ds$, its value

$$ds = dx\sqrt{1 + \frac{dy^2}{dx^2}},$$

we obtain

$$dx = \frac{hd.\frac{dy}{dx}}{\sqrt{1 + \frac{dy^2}{dx^2}}}.$$

By integrating this expression and observing that at the point B, we have $x = 0$, $\frac{dy}{dx} = 0$, there results

$$x = h\log\left(\frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}}\right),$$

and, consequently,

$$\frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}} = e^{\frac{x}{h}},$$

$e$ denoting, as usual, the base of the Naperian system of logarithms. If this equation be multiplied by

$$\left(\sqrt{1 + \frac{dy^2}{dx^2}} - \frac{dy}{dx}\right)e^{-\frac{x}{h}},$$

there results

$$e^{-\frac{x}{h}} = \sqrt{1 + \frac{dy^2}{dx^2}} - \frac{dy}{dx},$$

consequently, we shall have($f$)

$$ds = \tfrac{1}{2}\left(e^{\frac{x}{h}} + e^{-\frac{x}{h}}\right)dx, \quad dy = \tfrac{1}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right)dx,$$

hence we deduce

$$s = \frac{h}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right), \quad y = \frac{h}{2}\left(e^{\frac{x}{h}} + e^{-\frac{x}{h}}\right), \qquad \text{(c)}$$

in which we suppose that $s = 0$ and $x = 0$, at the point B, and that O the origin of the coordinates, is taken at a distance below this point equal to $h$, so that we have $y = h$ when $x = 0$

From equations (c), we obtain $s = h\frac{dy}{dx}$ as before The second is the equation of the catenary in its simplest form, it shews that this curve is symmetrical on each side of its lowest point

The preceding value of T will become

$$T = ph\frac{ds}{dx} = py,$$

from which it appears, that the tension in any point M is expressed by the weight of a length of the thread, equal to MP, the perpendicular let fall from this point on the horizontal line, passing through the point O Therefore, the tension is least at the point B, where it is equal to $ph$, as has been already supposed

295 It only remains to determine the constant $h$ which occurs in these formulæ The expression for $y$ will then make known the figure of the catenary, but, in order to know its position in the vertical plane passing through A and C, it is necessary also to determine the distance of the axis O$y$ from one of these fixed points

For this purpose, let there be drawn through the point A a horizontal line cutting the axis O$y$ in a point Q, and through the point C, a vertical line meeting AQ in in the point D. The position of the point C, with respect to the point A being known, the distances AD and DC will be given. If these lines be denoted by $a$ and $b$, and if $k$ denotes the distance AQ, we shall have

$$AD = a, \quad DC = b, \quad AQ = k, \quad OB = h,$$

$a$ and $b$ are given, and $k$ and $h$ are the two unknown quantities which it is proposed to determine

Denoting the distance QD by $k'$, the given length of the curve ABC by $l$, its parts AB and BC by $g$ and $g'$, and the sagitta BQ by $f$, we shall have

$$k + k' = a, \quad g + g' = l,$$

in which $k'$ and $g'$ are considered to be positive or negative, according as the point C belongs to the production of AB or to AB itself. The coordinates of the points A and C will be $h + f$ and $h + f - b$, in which $b$ is to be considered as positive or negative, according as C falls below or above the horizontal line, drawn through the point A

If, in equations (c) we first make,

$$x = k, \quad s = g, \quad y = h + f,$$

and then

$$x = -k', \quad s = -g', \quad y = h + f - b,$$

there will result

$$g = \frac{h}{2}\left(e^{\frac{k}{h}} - e^{-\frac{k}{h}}\right), \quad h + f = \frac{h}{2}\left(e^{\frac{k}{h}} + e^{-\frac{k}{h}}\right),$$

$$g' = \frac{h}{2}\left(e^{\frac{k'}{h}} - e^{-\frac{k'}{h}}\right), \quad h + f - b = \frac{h}{2}\left(e^{\frac{k'}{h}} + e^{-\frac{k'}{h}}\right),$$

from which we obtain

$$l = \frac{h}{2}\left(e^{\frac{k}{h}} - e^{-\frac{k}{h}} + e^{\frac{k'}{h}} - e^{-\frac{k'}{h}}\right),$$

$$b = \frac{h}{2}\left(e^{\frac{k}{h}} + e^{-\frac{k}{h}} - e^{\frac{k'}{h}} - e^{-\frac{k'}{h}}\right)$$

Hence and because $k + k' = a$, we obtain $(g)$

$$l^2 - b^2 = h^2\left(e^{\frac{a}{h}} + e^{-\frac{a}{h}} - 2\right),$$

and, consequently,

$$\frac{1}{2a}\left(e^{a} - e^{-a}\right) = n, \qquad \text{(d)}$$

in which we make, in order to abridge,

$$\frac{a}{2h}=\alpha, \quad \sqrt{\frac{l^2-b^2}{a^2}}=n,$$

As $n$ consists entirely of given quantities, equation (d) will make known the value of $\alpha$, and, consequently, that of $h$. In general, this equation can be resolved by trials, and the numerical value of $\alpha$ can be deduced from that of $n$ as accurately as we please If $n$ differs little from unity, the value of $\alpha$ will be very small, and, in this case, if the exponentials be developed, we shall obtain, by neglecting the fourth power of $\alpha$, $\alpha^2 = 6(n-1)$ ($h$)

If in like manner, we make

$$\frac{k-k'}{h}=2\beta,$$

we shall have

$$k=\tfrac{1}{2}a+h\beta, \quad k'=\tfrac{1}{2}a-h\beta,$$

and the preceding value of $b$ will become($i$)

$$b=\frac{h}{2}\left(e^{\frac{a}{2h}}-e^{-\frac{a}{2h}}\right)\left(e^{\beta}-e^{-\beta}\right), \tag{e}$$

by means of which, the value of $\beta$, and, consequently, the quantities $k$ and $k'$ will be known, when that of $h$ is determined. The sign of $k'$ will determine on what side of o$y$, the point C is situated

The simplest case will be that in which the fixed points A and C are situated on the same horizontal line. In this case we have $b=0$, and equation (e) will give $\beta=0$, and, consequently, $k=k'=\frac{1}{2}a$, as we know it ought to be. At the same time, we shall have

$$h+f=\frac{h}{2}\left(e^{\frac{a}{2h}}+e^{-\frac{a}{2h}}\right),$$

by means of which, when the value of $h$ shall have been calculated, the tensions at the points A and C, or the pressures which these fixed points will have to sustain, can be deter-

mined. In the general case, these extreme tensions can be deduced from the values of $y$ corresponding to $x = k$ and $x = -k$.

296. Among all the curves of the same length, which terminate at the given points A and C, the catenary is that whose centre of gravity is the lowest.

In fact, if through the point A (fig. 75) an horizontal axis A$y'$ be drawn, and a vertical axis A$x'$ in the direction of gravity, and if $x'$ and $y'$ be the coordinates of any point M, referred to these axes, we shall have

$$lx_1 = \int_0^b x' \sqrt{1 + \frac{dy'^2}{dx'^2}} \cdot dx',$$

in which $x_1$ denotes the distance of the centre of gravity of AMC any curve whatever, from the axis A$y'$, $b$ the value of $x'$ which refers to the point C, and $l$ the given length of this curve, the value of which we know is

$$l = \int_0^b \sqrt{1 + \frac{dy'^2}{dx'^2}}\, dx'.$$

Now by formula (e) of No. 201, the differential equation of the curve, for which, among all curves of the same length, the first integral is a maximum, is

$$dy' = \frac{c'dx'}{\sqrt{(x'+c)^2 - c'^2}},$$

$c$ and $c'$ being two constant arbitraries. The integration of this gives, as $x'$ and $y'$ are equal to cypher at the same time ($k$)

$$y' = c' \log\left(\frac{x' + c + \sqrt{(x'+c)^2 - c'^2}}{c + \sqrt{c^2 - c'^2}}\right),$$

and, consequently,

$$x' + c\sqrt{(x'+c)^2 - c'^2} = \gamma e^{\frac{y'}{c'}},$$

in which, for the sake of abridging, we make

$$c + \sqrt{c^2 - c'^2} = \gamma.$$

Hence it is easy to perceive that

$$x' + c - \sqrt{(x'+c)^2 - c'^2} = \gamma' e^{-\frac{y'}{c'}},$$

in which, in the same manner, we suppose

$$c - \sqrt{c^2 - c'^2} = \gamma'.$$

Consequently,

$$x' + c = \tfrac{1}{2}\gamma e^{\frac{y'}{c'}} + \tfrac{1}{2}\gamma' e^{-\frac{y'}{c'}} \tag{f}$$

will be the equation of the curve which possesses the required property At the point c, we shall have

$$b + c = \tfrac{1}{2}\gamma e^{\frac{a}{c'}} + \tfrac{1}{2}\gamma' e^{-\frac{a}{c'}},$$

in which $a$ is the given distance of this point from the axis A$x$, so that we have at the same time $x' = b$ and $y' = a$ By means of this particular equation, and of $l$ the length of the curve, the two constant quantities $c$ and $c'$ can be determined. Now, in order to make equation (f) to coincide with that of the catenary, let $\varepsilon$ denote an indeterminate constant, and let the ordinates $x'$ and $y'$ be changed into two others, such that

$$x' + c = -y, \quad y' = \varepsilon - x,$$

so that these new coordinates $x$ and $y$ must be drawn in a contrary direction from that of $x'$ and $y'$, and also be referred to another origin By this change, equation (f) will become

$$y = -\tfrac{1}{2}\gamma e^{\frac{\varepsilon}{c'}} e^{-\frac{x}{c'}} - \gamma' e^{-\frac{\varepsilon}{c'}} e^{\frac{x}{c'}}.$$

$\varepsilon$ can be determined by putting

$$\gamma e^{\frac{\varepsilon}{c'}} = \gamma' e^{-\frac{\varepsilon}{c'}},$$

and denoting their common value by $-h$, so that we may have

$$\gamma e^{\frac{\varepsilon}{c'}} = -h, \quad \gamma' e^{-\frac{\varepsilon}{c'}} = -h.$$

As $\gamma\gamma' = c'^2(l)$, there will result $h = c'$, and the preceding equation of the curve will become

$$y = \tfrac{1}{2}h\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right),$$

which evidently coincides with the second equation (c) that has been found for the catenary.

297 If the vertical force which acts on each element of the thread suspended at the points A and C, (No. 74), in place of being proportional to the element $ds$, was proportional to its horizontal projection $dx$, the second equation (b) will become

$$d\,.\,\mathrm{T}\frac{dy}{ds} = p\,dx,$$

$p$ being a given constant, that denotes the weight of a prism, the altitude of which is the linear unit From the first equation (b) which does not undergo any change, we shall always have

$$\mathrm{T} = ph\frac{ds}{dx},$$

in which $h$ denotes a line of unknown length, and $ph$ a weight equivalent to the tension at B, the lowest point of the curve. From hence there will result

$$hd\,.\frac{dy}{dx} = dx,$$

and, consequently,

$$h\frac{dy}{dx} = x,\quad 2hy = x^2,$$

the origin of the coordinates $x$ and $y$ being placed at the point B In this case, the curve is evidently a parabola, the summit of which is at the lowest point, and the expression for the tension at any point whatever will be $(m)$

$$\mathrm{T} = p\sqrt{h^2 + x^2}.$$

By employing the notations of No 295, we shall have, at the points A and C,

$$2hf = k^2, \quad 2h(f-b) = k'^2,$$

and because $k + k' = a$, there will result

$$2hb = a(k - k'),$$

by means of which $f$, $k$, $k'$, will be known, when $h$ shall have been determined, and the value of this last can be deduced from knowing that of $l$, the length of the thread In fact, we shall have

$$\frac{ds}{dx} = \frac{1}{h}\sqrt{h^2 + x^2}, \quad hl = \int_{-k'}^{k} \sqrt{h^2 + x^2}\,dx,$$

which, by integrating, gives

$$2hl = h^2 \log \frac{\sqrt{h^2 + k^2} + k}{\sqrt{h^2 + k'^2} - k'} + k\sqrt{h^2 + k^2} + k'\sqrt{h^2 + k'^2}$$

If, for greater simplicity, we suppose that the two points A and C exist in the same horizontal line, we shall have

$$b = 0, \quad k = k' = \tfrac{1}{2}a,$$

and the preceding equation will be reduced to ($n$)

$$hl = h^2 \log \frac{k + \sqrt{h^2 + k^2}}{h} + k\sqrt{h^2 + k^2},$$

from which, when the numerical values of $l$ and $k$ are given, an approximate value of $h$ may be deduced by trial This unknown $h$ can be determined with greater facility, when $l$, the length of the curve, differs little from $a$ its projection, for then the value of $h$ will be very great relatively to that of $a$. In this case, we shall have in very convergent series

$$\sqrt{h^2 + k^2} = h + \tfrac{1}{2}\frac{k^2}{h} - \tfrac{1}{8}\frac{k^4}{h^3} + \&c.$$

$$\log \frac{k + \sqrt{h^2 + k^2}}{h} = \frac{k}{h} - \tfrac{1}{6}\frac{k^3}{h^3} + \&c.$$

By means of these values, the preceding equation becomes (*o*) q p.

$$h^2(l-2k)=\tfrac{1}{3}h^3,$$

and, consequently,

$$h=\frac{a\sqrt{2a}}{4\sqrt{3(l-a)}}.$$

This example has been selected, because it may be usefully applied in the construction of suspension bridges, in which it is important to be able to calculate the tension of the chain of suspension, and also the pressure on its points of support

298 If all the points of the thread are supposed to be solicited by any forces whatever, the figure which it will assume in consequence of the action of these forces, will be, in general, a curve of double curvature, the equations of equilibrum of each of its points will be three in number, and if the thread be perfectly flexible, these equations can be obtained in the manner which has been explained in detail in No. 293. In this way, we obtain

$$\left.\begin{aligned} d.\mathrm{T}\frac{dx}{ds}+\mathrm{X}\varepsilon ds&=0,\\ d\,\mathrm{T}\frac{dy}{ds}+\mathrm{Y}\varepsilon ds&=0,\\ d\,\mathrm{T}\frac{dz}{ds}+\mathrm{Z}\varepsilon ds&=0,\end{aligned}\right\}\quad(1)$$

$x$, $y$, $z$ being the rectangular coordinates of M any point whatever of the curve, $ds$ the differential element of its length, $\varepsilon$ the product of the density of the thread and of the section made by a perpendicular to the thread at the point M, so that $\varepsilon ds$ is the element of the mass of the thread, T the tension at this same point, or the force, of unknown magnitude, which draws this element $\varepsilon ds$ in the direction of each of its productions, X, Y, Z the forces referred to the unit of mass, parallel to the axes of $x$, $y$, $z$, the coordinates of the point M, which will be given functions of these three coordinates

In virtue of the tension T, the element $ds$ will experience an extension, and the quantity $\varepsilon$ a diminution, such, however, that the mass $\varepsilon ds$ remains the same; consequently, if $ds'$ and $\varepsilon'$ denote what these quantities were in the natural state of the thread, we shall have

$$\varepsilon ds = \varepsilon' ds',$$

and if the extension be proportional to the force which produces it, (No. 288), we shall, at the same time, have

$$ds = (1 + \omega T)\, ds',$$

$\omega$ being a very small coefficient, depending on the material and thickness of the thread at the point M When the thread is homogeneous, and of a uniform thickness throughout its entire length, $\varepsilon'$ and $\omega$ will be constant quantities, but, in general, these two quantities may be considered as given functions of the arc $s'$, reckoned from a determinate point of the thread, and terminating at the point M.

299. If the thread, whatever be its nature, is only acted on by gravity, and suspended vertically at a fixed point A, the two last equations (1) will disappear, and the third will be reduced to

$$dT + g\varepsilon dx = 0,$$

the axis of the ordinate $x$ being supposed to be vertical, and to act in the direction of $g$ which denotes the force of gravity. If the origin of the ordinates $x$ be placed at A, and if Q denotes the value of T when $x = 0$, that is to say, the load which it will have to support, we shall have for M any point whatever,

$$T = Q - g \int \varepsilon dx,$$

in which the integral vanishes at the same time as $x$

If to B, the lower extremity of the thread, a weight P is attached, and if $l$ denotes the length of AB, it is evident that the tension at the point B will be equal to P; consequently,

we shall have at the same time, $x = l$, and $\text{T} = \text{P}$, by means of which we obtain

$$\text{Q} = \text{P} + g\int_0^l \varepsilon dx,$$

and, consequently,

$$\text{T} = \text{P} + g\int_0^l \varepsilon dx - g\int \varepsilon dx$$

But as the second and third terms of this formula are evidently the weight of the entire thread and of its part AM, it follows that the tension at any point such as M, is equal to the weight of the part BM, increased by the weight P, which indeed is evident of itself.

The law of the extensibility of the thread throughout its entire length, depends on the nature of the material of which it is composed and on its thickness If, for example, it was homogeneous, and of the same thickness throughout, the coefficient $\omega$ would be constant, and if we denote the length of the part AM before the thread is stretched by $x'$, which length becomes $x$ by the effect of the tension, and if, in consequence, $dx'$ and $dx$ be substituted for $ds'$ and $ds$, in equation (2), we shall have

$$dx = (1 + \omega\text{T})\, dx'.$$

Likewise, if $l'$ be the entire length of thread before it is stretched, and $p$ its entire weight, the weight of the part BM will be $\frac{p(l' - x')}{l'}$, and the value of the tension at the point M will be

$$\text{T} = \text{P} + \frac{p\,(l' - x')}{l'}$$

By substituting this expression for T in the preceding equation, and then integrating, we shall obtain, because $x' = 0$, $x = 0$, at the point A, $(p)$

$$x - x' = \omega\text{P}x' + \frac{\omega p\,(2l'x' - x'^2)}{2\,l'},$$

which is, therefore, the quantity by which the length of the part AM is increased. The entire increase of length will be obtained by making $x' = l'$, $x' = l$, this gives

$$l - l' = \omega l' (\mathrm{P} + \tfrac{1}{2}p),$$

from which it appears, that if, in calculating the increase of length of the thread, its own weight is taken into account, half of this weight must be added to that which is attached to its inferior extremity.

300. In the general case, if, after having multiplied equations (1), by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, respectively, they be then added together, there will result ($q$)

$$d\mathrm{T} + \varepsilon (\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = 0, \qquad (3)$$

because

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1,$$

$$\frac{dx}{ds} d \frac{dx}{ds} + \frac{dy}{ds} d . \frac{dy}{ds} + \frac{dz}{ds} d \frac{dz}{ds} = 0$$

If the thread be homogeneous and of a uniform thickness, and if the small dilatation of its parts be neglected, the quantity $\varepsilon$ will be constant, moreover, the formula $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ is, in general, an exact differential of a function of the three variables $x$, $y$, $z$, considered as independent, therefore, by making

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = - d . \phi (x, y, z),$$

we shall have

$$d\mathrm{T} = \varepsilon d . \phi (x, y, z),$$

and, consequently,

$$\mathrm{T} = \varepsilon \phi (x, y, z),$$

in which, the constant arbitrary that is introduced by the integration, is supposed to be comprised in the function $\phi$. This constant will disappear in the expression for the difference of the values of T relative to two points of the thread, it follows

therefore, that it is not necessary to determine the figure which the thread assumes in the state of equilibrium, in order to know the increment of the tension from one point to another, so that the tension will be known throughout the entire length of the curve, if it be known in any one determinate point of it. The curve which the thread assumes will be determined by two of the three equations (1), or by any two combinations of these three equations, in which the preceding value of T should be substituted, so that in general it will be necessary to integrate the system of two differential equations of the second order, in order to have the equation of this curve.

Its radius of curvature at M, any point whatever, will be obtained by means of the following differential formula, which does not surpass the first order, and in which it is only assumed that the direction of the tangent at this point is known.

Equations (1) may be replaced by the following:

$$\frac{dx}{ds}d.\mathrm{T}\frac{dy}{ds} - \frac{dy}{ds}d.\mathrm{T}\frac{dx}{ds} = \varepsilon(\mathrm{X}dy - \mathrm{Y}dx),$$

$$\frac{dz}{ds}d.\mathrm{T}\frac{dx}{ds} - \frac{dx}{ds}d.\mathrm{T}\frac{dz}{ds} = \varepsilon(\mathrm{Z}dx - \mathrm{X}dz),$$

$$\frac{dy}{ds}d.\mathrm{T}\frac{dz}{ds} - \frac{dz}{ds}d.\mathrm{T}\frac{dy}{ds} = \varepsilon(\mathrm{Y}dz - \mathrm{Z}dy),$$

which, by performing the differentiations, and assuming the arc for the independent variable, are the same as

$$\left.\begin{aligned} dxd^2y - dyd^2x &= (\mathrm{X}dy - \mathrm{Y}dx)\frac{\varepsilon ds^2}{\mathrm{T}}, \\ dzd^2x - dxd^2z &= (\mathrm{Z}dx - \mathrm{X}dz)\frac{\varepsilon ds^2}{\mathrm{T}}, \\ dyd^2z - dzd^2y &= (\mathrm{Y}dz - \mathrm{Z}dy)\frac{\varepsilon ds^2}{\mathrm{T}}. \end{aligned}\right\} \quad (4)$$

Now, if the radius of curvature at the point M be denoted by $\rho$, we have (No. 18)

$$\rho = \frac{ds^3}{(dxd^2y - dyd^2x)^2 + (dzd^2x - dxd^2z)^2 + (dyd^2z - dzd^2y)^2};$$

which by means of the preceding equations, and by substituting the value T, becomes

$$\rho = \frac{\phi\,(x, y, z)\, ds}{(\mathrm{X}dy - \mathrm{Y}dx)^2 + (\mathrm{Z}dx - \mathrm{X}dz)^2 + (\mathrm{Y}dz - \mathrm{Z}dy)^2}. \quad (5)$$

In the case of the catenary,

$$\mathrm{X} = 0, \quad \mathrm{Y} = -g, \quad \mathrm{Z} = 0, \quad \phi = gy,$$

the axes and origin of the coordinates being assumed to be the same as in equations (c) of No 294 We shall therefore have

$$\rho = y\frac{ds}{dx},$$

which it is easy to verify, by means of these equations

301 Let these formulæ be applied to the case of a string stretched on the surface of a solid body, and, for greater simplicity, let it be assumed that it is not subjected to the action of any given force, so that the only force which acts on its different points is the unknown resistance of the solid on which it is stretched

Let $\mathrm{N}ds$ be the magnitude of this force applied to $\varepsilon ds$ the element of the thread at any one of its points M, its three components will be $\mathrm{X}\varepsilon ds$, $\mathrm{Y}\varepsilon ds$, $\mathrm{Z}\varepsilon ds$, its direction will be normal to the surface of the solid, and directed from without inwards. The pressure on the part of the solid corresponding to $ds$, will be equal and contrary to this force $\mathrm{N}ds$, so that N will express the measure of the pressure referred to the unit of length If $\lambda$, $\mu$, $\nu$, denote the angles which the exterior part of the normal at M makes with lines drawn through this point, parallel to the axes of $x$, $y$, $z$, we shall have

$$\varepsilon\mathrm{X} = \mathrm{N}\cos\lambda, \quad \varepsilon\mathrm{Y} = \mathrm{N}\cos\mu, \quad \varepsilon\mathrm{Z} = \mathrm{N}\cos\nu$$

Moreover, if $\mathrm{L} = 0$, is the equation of the surface of the solid, and if, for the sake of abridging, we make

$$\mathrm{V} = \left(\frac{d\mathrm{L}^2}{dx^2} + \frac{d\mathrm{L}^2}{dy^2} + \frac{d\mathrm{L}^2}{dz^2}\right)^{-\frac{1}{2}},$$

we shall likewise have (No 21)

$$\cos\lambda = \mathrm{V}\frac{d\mathrm{L}}{dx}, \quad \cos\mu = \mathrm{V}\frac{d\mathrm{L}}{dy}, \quad \cos\nu = \mathrm{V}\frac{d\mathrm{L}}{dz},$$

a suitable sign being given to V This being so, we shall have($s$)

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = \mathrm{NV}d\mathrm{L} = 0;$$

in consequence of which, the value of $d\mathrm{T}$ furnished by equation (3) will be cypher. Hence, whatever be the form of the solid body, the tension will be the same throughout the entire length of the thread. Let its value be supposed to be given, and to be represented by $k$, then if the thread is attached at one of its extremities to a point of the body, and if, at its other extremity, there be suspended vertically a weight such as $p$, that is considerable relatively to that of the thread, which may, therefore, be neglected, this weight will be the tension $k$ and the pressure which the fixed point will experience If the thread is free at its two extremities, and if considerable weights are suspended from them, they will express the extreme tensions, consequently, they must be equal, and each of them will be the tension $k$ Finally, if the two extremities are supposed fixed, its tension $k$ will be deduced from its extension, which will be constant throughout its entire length

302 If $\lambda'$, $\mu'$, $\nu'$, denote the angles which the perpendicular to the osculatory plane at the point M, makes with parallels to the axes of $x$, $y$, $z$, then the radius of curvature at this point being $\rho$, we shall have (No 19)

$$\frac{dxd^2y - dyd^2x}{ds^3} = \rho\cos\nu',$$

$$\frac{dzd^2x - dxd^2z}{ds^3} = \rho\cos\mu',$$

$$\frac{dyd^2z - dzd^2y}{ds^3} = \rho\cos\lambda'$$

If, therefore, equations (4) be multiplied by $\cos\nu$, $\cos\mu$, $\cos\lambda$, respectively, and then added together, there will result, by taking into account the values of X, Y, Z, in the case which we are considering($t$),

$$\cos\nu\cos\nu' + \cos\mu\cos\mu' + \cos\lambda\cos\lambda' = 0,$$

consequently, the normals to the surface of the solid body and to the osculating plane of the curve formed by the thread at each point M, are perpendicular the one to the other, this we have proved to be the characteristic property of the line, the length of which is a *minimum* or *maximum* on a given surface (No 161). It follows, therefore, that a thread stretched on a solid body, traces, in general, the shortest distance from one point to another on the surface Strictly speaking, it is possible that this distance may be, on the contrary, a *maximum* Thus, for example, two given points on a sphere are the common extremities to two arcs of great circles, of which one is the shortest distance between these points, and the other the plane curve which is the longest, now, it is evident, that the equilibrium of the stretched thread will be rigorously possible on these two arcs of the circle, since if it be placed on one of them, there is no reason why it should deviate from it to the one side rather than to the other, but on the small arc the equilibrium will be *stable*, and on the great only instantaneous, so that, *physically*, it cannot subsist except by means of the friction of the thread against the solid body

Likewise, if the values of $\epsilon X$, $\epsilon Y$, $\epsilon Z$, of the preceding number, be substituted in formula (5), we shall have

$$\mathrm{N}\left[\left(\frac{dy}{ds}\cos\lambda - \frac{dx}{ds}\cos\mu\right)^2 + \left(\frac{dx}{ds}\cos\nu - \frac{dz}{ds}\cos\lambda\right)^2 + \left(\frac{dz}{ds}\cos\mu - \frac{dy}{ds}\cos\nu\right)^2\right] = \frac{k}{\rho},$$

because $\epsilon\phi(x, y, z) = k$ We have at the same time

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1,$$

$$\cos\lambda^2 + \cos\mu^2 + \cos\nu^2 = 1,$$

and, as the normal to the surface of the body, and the tangent to the curve of the thread, at each point M, are perpendicular the one to the other, we have also

$$\frac{dx}{ds}\cos\lambda + \frac{dy}{ds}\cos\mu + \frac{dz}{ds}\cos\nu = 0,$$

but, it is easy to shew by means of these three last equations, that the coefficient of N in the preceding expression is unity, hence we have simply($u$)

$$N = \frac{k}{\rho},$$

from which it is evident, that the pressure, referred to the unit of length, exerted by a thread stretched on the surface of a solid body, is, at each point of the curve, equal to the tension divided by the radius of curvature of the thread, that is to say, by the radius of the section which is normal to the surface, and to the tangent to the curve which the thread assumes

303 These results will be modified by the friction of the thread against the surface of the body on which it is stretched In order to show how this force should be taken into account in the equilibrium of a flexible thread, we proceed to consider the equilibrium of a cord ABMCD (fig 76), whose part BMC is applied to the throat of a fixed pulley, and which is drawn in the directions of BA and CD, the productions of this part, by given forces Let the pulley and the line AB be supposed to be vertical, and let the force acting in the direction of BA be represented by the weight $k$, and that in the direction of CD by F, it is evident that $k$ and F denote the tensions at the points B and C, in the directions of the tangents BA and CD If, in like manner, in order to simplify the question, the pulley is supposed to be circular, its radius to be denoted by $c$, and the centre

O, to be the origin of the coordinates, the axis of $z$ will be perpendicular to the pulley, the axis of $y$ vertical and directed upwards, and the axis of $x$ horizontal and passing through the point B. Finally, let C be the origin of the arc $s$, terminating at M, any point of the cord, so that CM $= s$.

This being premised, if there is no friction, it is necessary that in the case of equilibrium we should have $k =$ F, but, in consequence of the friction, the equilibrium may subsist as long as the difference between these two forces $k$ and F does not pass a certain limit. Let us suppose, therefore, that the equilibrium is on the point of giving way in the direction of the weight $k$, this implies that $k >$ F. At this instant, the friction of the cord against the pulley, which has place at any point M, will act in the direction of the part MH of the tangent at this point. If its intensity be denoted by $\mu$, and if, as before, the normal resistance which has place at the point M in the direction of M′O, the production of MO, be denoted by N, then $\mu ds$ and N$ds$ will be the tangential and normal forces which act on $\varepsilon ds$, the element of the cord terminating at the point M, and $\mu$ and N represent these forces, referred to the unit of length. If through this point M, M$x'$ and M$y'$ be drawn parallel to the axes O$x$ and O$y$, we shall have

$$\cos x'\text{MH} = \frac{-y}{c}, \quad \cos y'\text{MH} = \frac{x}{c},$$

$$\cos x'\text{MO}' = \frac{x}{c}, \quad \cos y'\text{MO}' = \frac{y}{c},$$

hence we infer,

$$\varepsilon\text{X} = \frac{\text{N}x}{c} - \frac{\mu y}{c}, \quad \varepsilon\text{Y} = \frac{\text{N}y}{c} + \frac{\mu x}{c};$$

for the values of $\varepsilon$X and $\varepsilon$Y, which should be substituted in equations (1). The force $\varepsilon$Z will be evidently equal to cypher, the third equation (1) will disappear, and the two first will become

$$d\ \mathrm{T}\frac{dx}{ds}+\frac{\mathrm{N}xds}{c}-\frac{\mu yds}{c}=0,$$

$$d.\mathrm{T}\frac{dy}{ds}+\frac{\mathrm{N}yds}{c}+\frac{\mu xds}{c}=0$$

As the point M belongs to the circumference of the pulley, we have

$$x^2+y^2=c^2, \quad xdx+ydy=0,$$

by means of which the two preceding equations may be changed into the following($v$)

$$\left.\begin{aligned} xd\ \mathrm{T}\frac{dx}{ds}+yd\ \mathrm{T}\frac{dy}{ds}+\mathrm{N}cds=0, \\ \frac{dx}{ds}d\ \mathrm{T}\frac{dx}{ds}+\frac{dy}{ds}d.\mathrm{T}\frac{dy}{ds}-\frac{\mu}{c}(ydx-xdy)=0 \end{aligned}\right\} \quad (6)$$

But $\frac{1}{2}(ydx-xdy)$ is the differential of the sector described by the radius OM reckoning from a fixed line (No 156), OC for example This sector being circular and corresponding to the arc $s$, its value will be $\frac{1}{2}cs$, hence we have

$$ydx-xdy=cds$$

Moreover, we have likewise

$$x\frac{dx}{ds}+y\frac{dy}{ds}=0, \quad xd.\frac{dx}{ds}+yd\ \frac{dy}{ds}=-ds,$$

$$\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}=1, \quad \frac{dx}{ds}d\ \frac{dx}{ds}+\frac{dy}{ds}d.\frac{dy}{ds}=0,$$

this reduces equations (6) to($x$),

$$\mathrm{T}=c\mathrm{N}, \quad d\mathrm{T}=\mu ds,$$

hence we obtain

$$cd\mathrm{N}=\mu ds$$

As the pressure at the point M, on the throat of the pulley, is equal and contrary to the force N, if the friction be proportional to the pressure (No 269), we shall have

$$\mu=f\mathrm{N},$$

$f$ being a coefficient which will depend on the nature of the surfaces in contact. Consequently, we shall have

$$c\,d\text{N} = f\text{N}ds,$$

and, by integrating,

$$\text{N} = \text{A}e^{\frac{fs}{c}},$$

A being a constant arbitrary, and $e$ the base of the Naperian system of logarithms. We shall also have at the same time,

$$\text{T} = \text{A}ce^{\frac{fs}{c}}, \quad \mu = \text{A}fe^{\frac{fs}{c}}.$$

At the point C, $s = 0$ and $\text{T} = \text{F}$, hence we have

$$\text{A} = \frac{\text{F}}{c},$$

and denoting the length of the arc CMB by $l$, we shall have $s = l$ and $\text{T} = k$, at its other extremity B. Therefore, we shall have finally at M, any point whatever,

$$\text{N} = \frac{\text{F}}{c}e^{\frac{fs}{c}}, \quad \text{T} = \text{F}e^{\frac{fs}{c}}, \quad \mu = \frac{f\text{F}}{c}e^{\frac{fs}{c}},$$

and moreover, the equation of equilibrium will be

$$k = \text{F}e^{\frac{fl}{c}}.$$

Denoting the total friction through the entire length of CMB by $\text{F}'$, we shall have

$$\text{F}' = \int_0^l \mu ds = \text{F}\,(e^{\frac{fl}{c}} - 1),$$

and the equation of equilibrium may be written thus,

$$k = \text{F} + \text{F}'.$$

By assuming

$$e^{\frac{fl}{c}} - 1 = f',$$

we shall have

$$\text{F}' = f'\text{F}, \quad f' = \frac{k}{\text{F}} - 1,$$

from which it appears, that the total friction F′ is equal to the least of the two forces $k$ and F, multiplied by a coefficient $f'$, which varies not only with the quantity $f$, but also with $l$, the extent of the surface in contact, and $c$ the radius of the pulley. The difference of the forces $k$ and F, at the instant the equilibrium gives way, will make known the value of F′, and their ratio, diminished by unity, will be the value of the coefficient $f'$, from which it is easy to deduce that of $f$. When F is a weight as well as $k$, we should, for greater accuracy, include as belonging to these weights, those of BA and CD, the vertical parts of the string.

304. It is easy, by means of the three equations (1), to shew that the six general equations of equilibrium (No. 261) obtain in the case of a perfectly flexible thread.

For this purpose, let K and K′ be the two extremities of the thread, and $l$ its length, also let the origin of the arc $s$ be fixed at the point K. By integrating the first members of equations (1) from the point K to the point K′, we shall have

$$\left(\text{T}\frac{dx}{ds}\right) - \left[\text{T}\frac{dx}{ds}\right] + \int_0^l \text{X}\varepsilon ds = 0,$$

$$\left(\text{T}\frac{dy}{ds}\right) - \left[\text{T}\frac{dy}{ds}\right] + \int_0^l \text{Y}\varepsilon ds = 0,$$

$$\left(\text{T}\frac{dz}{ds}\right) - \left[\text{T}\frac{dz}{ds}\right] + \int_0^l \text{Z}\varepsilon ds = 0,$$

the quantities comprised within the crotchets refer to the point K, and those which are contained within the parentheses, to the point K′. Besides the forces X, Y, Z, which act through the entire length of the thread, particular forces given in magnitude and direction, may be supposed to be applied at its two extremities, let $k$ be that which acts at the point K, and $\alpha$, $\beta$, $\gamma$, the angles which its direction makes with lines drawn through this point, parallel to the axes of $x$, $y$, $z$, and let $k'$, $\alpha'$, $\beta'$, $\gamma'$ be the corresponding quantities relatively to the point K′. These forces $k$ and $k'$ will be the extreme tensions, in magni-

tude and direction, and as their directions must coincide with parts of the tangents to the curve, which the rod assumes at K and K', we shall have

$$\left.\begin{array}{l}\left[\mathrm{T}\dfrac{dx}{ds}\right]=-k\cos\alpha,\left[\mathrm{T}\dfrac{dy}{ds}\right]=-k\cos\beta,\left[\mathrm{T}\dfrac{dz}{ds}\right]=-k\cos\gamma,\\ \left(\mathrm{T}\dfrac{dx}{ds}\right)=k'\cos\alpha',\left(\mathrm{T}\dfrac{dy}{ds}\right)=k'\cos\beta',\left(\mathrm{T}'\dfrac{dz}{ds}\right)=k'\cos\gamma',\end{array}\right\}(7)$$

consequently, the preceding equations will become

$$\left.\begin{array}{l}k\cos\alpha+k'\cos\alpha'+\int_0^l \mathrm{X}\varepsilon ds=0,\\ k\cos\beta+k'\cos\beta'+\int_0^l \mathrm{Y}\varepsilon ds=0,\\ k\cos\gamma+k'\cos\gamma'+\int_0^l \mathrm{Z}\varepsilon ds=0,\end{array}\right\}\quad(8)$$

and it is evident, that they express the conditions of equilibrium contained in the three first equations (1) of No. 261. As the following equations

$$xd\,\mathrm{T}\frac{dy}{ds}-yd\,\mathrm{T}\frac{dx}{ds}=d\,\mathrm{T}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right),$$

$$zd\,.\,\mathrm{T}\frac{dx}{ds}-xd\,\mathrm{T}\frac{dz}{ds}=d\,\mathrm{T}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right),$$

$$yd\,\mathrm{T}\frac{dz}{ds}-zd\,\mathrm{T}\frac{dy}{ds}=d\,\mathrm{T}\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right),$$

are evidently identical, by equations (1) of No. 298 we shall have

$$d\,.\,\mathrm{T}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)+(x\mathrm{Y}-y\mathrm{X})\,\varepsilon ds=0,$$

$$d\,\mathrm{T}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right)+(z\mathrm{X}-x\mathrm{Z})\,\varepsilon ds=0,$$

$$d\,\mathrm{T}\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right)+(y\mathrm{Z}-z\mathrm{Y})\,\varepsilon ds=0$$

Hence, if the first members of these respective equations be

integrated from the point K to the point K′, and if $a$, $b$, $c$ denote the values of $x$, $y$, $z$, relative to K, and $a'$, $b'$, $c'$, those which refer to K′, we shall obtain by means of equations (7),

$$\left.\begin{aligned} &k(a\cos\beta-b\cos\alpha)+k'(a'\cos\beta'-b'\cos\alpha')+\int_0^l(x\mathrm{Y}-y\mathrm{X})\varepsilon ds=0,\\ &k(c\cos\alpha-a\cos\gamma)+k'(c'\cos\alpha'-a'\cos\gamma')+\int_0^l(z\mathrm{X}-x\mathrm{Z})\varepsilon ds=0,\\ &k(b\cos\gamma-c\cos\beta)+k'(b'\cos\gamma'-c'\cos\beta')+\int_0^l(y\mathrm{Z}-z\mathrm{Y})\varepsilon ds=0,\end{aligned}\right\}\ (9)$$

these express the conditions of equilibrium relative to the moments of the given forces, which are contained in the three last equations (1) of No 261

305 Equations (8) and (9) will, in general, enable us to determine the coordinates $a$, $b$, $c$, $a'$, $b'$, $c'$, of the two extreme points K and K′, nevertheless, there are cases in which some of these quantities must remain indeterminate If, for example, the given forces which act on the thread are the force of gravity and other forces that are independent of the coordinates of their points of application, it is evident that the absolute position of the thread in space cannot be determined, hence the three coordinates of one of the points K and K′ may be arbitrarily selected Equations (9) will determine the three coordinates of the other point, and, in order that the equilibrium may be possible, it is necessary that the given forces should satisfy equations (8) If one of the points K and K′ be fixed, the first, for instance, equations (8) and (9) will still obtain, provided that the force $k$ is considered to be unknown, in magnitude and direction, and as representing the pressure which the point K will have to sustain In this case, the values of $a$, $b$, $c$ will be given, equations (9) will determine those of $a'$, $b'$, $c'$, and equations (8) will make known the three components of the force $k$ When the two points K and K′ are fixed and given in position, their coordinates are determined, and equations (8) and (9) will enable us to determine $k$ and $k'$, the pressures exerted on K and K′, in magnitude and direction.

In all cases, whether the coordinates of K and K′ are given, or whether they have been deduced from equations (8) and (9), the curve formed by the thread must pass through these two points; this will enable us to determine the four constant arbitraries which the complete integrals of these two differential equations of the second order contain. With respect to the constant arbitrary which the function $\phi$ of No 300 contains, its value can be obtained from the given length of the thread, that is to say, from the equation

$$\int_a^{a'} \sqrt{1+\frac{dy^2}{dx^2}+\frac{dz^2}{dx^2}}\, dx = l,$$

in which $y$ and $z$ are regarded as functions of $x$. By this means, the problem will be completely solved

## III. *Equilibrium of an Elastic Rod.*

306 By this term we understand a straight or curved rod, the figure of which cannot be changed, without applying one or more forces to it, while at the same time it resumes its natural form when these forces cease to act, on the contrary, a perfectly *flexible* thread retains, without the aid of any force, the curvature which it has been made to assume, and is elastic only in the direction of its length. In order that a rod may be elastic with respect to its flexion, the material of which it is composed should be very little extensible and contractible, but this is not solely sufficient, it is likewise necessary, that the dimensions of its thickness, although very small relatively to its length, should nevertheless be of a determinate magnitude, for, of whatever material the rod consists, its thickness may be always so diminished, as that it cannot have any sensible tendency to resume the figure from which it has been deflected, by which means it may be reduced to the state of a perfectly flexible thread

When an elastic rod is deflected from its natural form by given forces, each of the longitudinal filaments of which it consists may experience three different effects, each part (the length of which may be as small as we please) may be contracted or dilated, its natural curvature may be either increased or diminished, and this part may have been twisted on itself The tendency of each part to resume its natural state, depends on the mutual attractions and repulsions which obtain between the molecules of all bodies, and which extend only to insensible distances It is the province of mathematical physics to compute the total effect of all the forces which result from them, and which must constitute an equilibrium with the given forces. For information on this subject, the reader is referred to a memoir by the author on the *Equilibrium and Motion of Elastic Bodies*, which is inserted in the *Mémoires de l'Académie des Sciences*, tome viii In the present treatise, the equations of the equilibrium of an elastic rod will be formed, by setting out from such secondary principles as are generally admitted

By an *elastic plate* is understood a rectangular parallelopiped of small thickness, which is bent in the direction of its length, so that it is always comprised between two cylindrical surfaces, whose heights are equal to its breadth. This dimension may be of any magnitude whatever, by dividing it by planes very near to each other, and perpendicular to its direction, the plate will be distributed into rectangular elastic rods. James Bernoulli was the first who determined the figure of an elastic plate in equilibrio, from considerations that we proceed to develope, and by means of which we shall then be enabled to solve the problem completely, in the case of any elastic rod whatever

307 Let us consider an elastic plate made fast at one of its extremities, that is to say, fixed in such a manner that one of the two small rectangles which terminate it perpendicularly to its length, cannot be moved in any direction. If it be supposed

to be bent in the direction of its length by means of a force applied to its other extremity, and which is the only one that acts on the plate, in order that the plane may assume a cylindrical form, as has been stated, it must be terminated at its free extremity by an inflexible rectangle, to the middle of which the given force is applied in a plane perpendicular to the breadth of the plate. All the longitudinal sections, that is, those which are perpendicular to this breadth, will be equal, that which contains the direction of the given force is represented by fig 77, and the curves AMB and A′M′B′ are sections of the two cylindrical surfaces of the plate, which its two faces form in its natural state All the points which, in this state, appertain to the common perpendicular to these two faces, are still supposed, after the plate has been bent, to exist on the common normal to the two cylindrical surfaces, which, in fact, agrees with what has been observed to take place in its change of figure Hence it follows, that if MM′ is normal to the curve AMB, it will be so likewise to A′M′B′, and will contain all those points of the plate, which, previously to the bending of the plate, existed on a common perpendicular to its two faces, it appears also, that if the plate, in its natural state, is divided into longitudinal filaments, and if the curve CND represents what one of those filaments becomes after the change of figure, it will intersect the normal MM′ at right angles in N.

If $m$ be a point of the curve AMB, infinitely near to M, and if $mnm'$ be drawn normal to the three lines AMB, CND, A′M′B′, and intersecting them in $m, n, m'$, respectively, the productions of MNM′ and $m\,n\,m'$ will meet in a point O, which will be the common centre of curvature of these three curves Let $\rho$ denote the radius of curvature of the mean filament, that is, of the filament which is equally distant from AMB and A′M′B′, $\sigma$ the part of this filament contained between the two normals MNM′ and $mnm'$, $u$ the distance of any filament whatever, such as CND, from the mean filament, and $\sigma'$ the length

of $Nn$, $u$ is to be considered as positive or negative, according as CND exists, relatively to the mean filament, nearer to AMB, the convexity of the plate, or on the side of its concavity A′M′B′, NO the radius of curvature of CND will be equal to $\rho + u$, and the infinitely small lengths $\sigma'$ and $\sigma$ will be to each other as $\rho + u$ to $\rho$, so that we shall have

$$\sigma' = \sigma + \frac{u\sigma}{\rho}.$$

In consequence of the bending of the plate, the longitudinal filaments undergo small contractions or dilatations, which will destroy the equality that previously existed between $\sigma$ and $\sigma'$. If their primitive magnitude be denoted by $\gamma$, we shall have

$$\sigma = \gamma\,(1 + \delta), \quad \sigma' = \gamma\,(1 + \delta'),$$

in which $\delta$ and $\delta'$ are very small fractions, positive or negative, according as the mean filament and the filament CND are lengthened or contracted The fraction $\frac{u}{\rho}$ is likewise supposed to be very small, hence if the product of $\delta$ and $\frac{u}{\rho}$ be neglected, we shall have

$$\delta' = \delta + \frac{u}{\rho},$$

which shews that, when the length of the mean filament is not changed, the filaments situated on the side of its convexity are all lengthened, and the filaments situated on the side of the concavity are all contracted, and each proportionably to their respective distances from the mean filament ($a$).

This being established, let the form of each of the two parts of the plate, which correspond to AMM′A′ and B$mm'$B′, be rendered invariable, and, for the sake of abridging, let them be denominated by H and K. The part H will be immoveable, the part K will be drawn towards H, or repelled from it, by the tendency of the intermediate part M$mm'$M′ to resume its natural

state, and become again a section of a constant thickness such as $\gamma$. The filament N$n$ of this slice will tend to contract or dilate itself according as it shall have been lengthened or shortened, that is to say, according as the quantity $\delta'$ will be positive or negative. Consequently, the part K will be drawn in the first case, and pushed in the second case, by a force applied to the point $n$, but as this force arises from the action of N$n$, we may suppose it to be proportional to the quantity $\delta$ and normal to $mnm'$, as if this filament N$n$ was detached. By adopting this hypothesis, the force in question referred to the unit of surface, may be represented by $a\delta'$, and, consequently, the normal force exercised on the transversal element of the surface K, which corresponds to the point $n$, may be represented by $a\delta'\lambda du$, $a$ being a constant depending on the material of which the plate consists, $\lambda$ its breadth, and $\lambda du$ the area of this element Hence denoting the thickness of the plate by $2\varepsilon$, and representing the entire force which draws or pushes K, according as it is positive or negative, by T, we shall have

$$T = a\lambda \int_{-\varepsilon}^{\varepsilon} \delta' du,$$

and, by substituting for $\delta'$ its value,

$$T = 2a\lambda\varepsilon\delta.$$

Moreover, if $\mu$ be the moment of the forces normal to the surface of K, taken with respect to a transversal axis equally distant from the two faces of the plate, we shall likewise have

$$\mu = a\lambda \int_{-\varepsilon}^{\varepsilon} \delta' u du,$$

and, consequently ($b$),

$$\mu = \frac{2a\lambda\varepsilon^3}{3\rho}$$

Hence it appears, 1st, that the force T, which tends to contract or dilate any slice of the plate, is proportional to the

positive or negative extension of the mean filament, and independent of its curvature, 2ndly, that its moment $\mu$ is, on the contrary, independent of this extension, and in the inverse ratio of its radius of curvature, 3rdly, that, the material and breadth of the plate remaining the same, the value of T is proportional to its thickness, and that of $\mu$, to the cube of this dimension.

When the length of the mean filament does not undergo any change, we have $\delta = 0$, $T = 0$, the parallel forces which draw or push K are reduced to two, equal and contrary, but not directly opposed, the moment of which, relative to the transversal axis perpendicular to these forces, is always equal to $\mu$. This quantity $\mu$ is what is termed the *moment of elasticity*, in each point, it is proportional to the curvature of the plate, or to the angle of contact of its mean filament ($c$).

308 It is easy now to obtain the equations of the equilibrium of this plate In the first place, if T′ be what the force T becomes at the point M, it is evident that the infinitely small slice which corresponds to $Mmm'M'$, will be drawn or pushed on one side by this force T′, and on the other by a force equal and contrary to T, and since by hypothesis, no given force is supposed to act on this slice, we must have $T = T'$. Hence it appears that the force T is constant throughout the entire length of the plate, and, consequently, it is equal to the given force which acts at its free extremity, resolved in the direction of this length. In like manner, the dilatation $\delta$ will be constant, proportional to this force, and positive or negative according as this force tends to lengthen or contract the longitudinal filaments. It will not influence the figure of the plate, but by measuring it, it will enable us to determine the value of the constant $a$, with reference to the material of which the plate consists If we suppose that $w$ denotes a weight equivalent to the force which draws the plate in the direction of its length, and that $\omega$ represents the area of each transversal section of the plate, we shall have

$$\omega = 2\lambda\varepsilon, \quad T = w = a\omega\delta, \quad a = \frac{w}{\omega\delta}.$$

In order to determine the figure of the plate, let there be drawn through the point A, in the plane of the mean filament, two rectangular axes A$x$ and A$y$, of which let the first be a tangent to the curve AMB, and drawn in the direction of the plate in its natural state, and let the second be directed from the side of its concavity Let $x$ and $y$ be the coordinates of any point of the mean filament, referred to these two axes, $a$ and $b$ those of its free extremity, which is supposed to be the point of application of the given force that retains the plate in equilibrio, P and Q the components of this force in the direction of the productions of $a$ and $b$. Let there be drawn through the point indicated by $x$ and $y$, an axis perpendicular to the plane of the figure to which the moment denoted by $\mu$ refers, and let there be made a section perpendicular to the mean filament In order that the part of the plate contained between this section and its free extremity may be in equilibrio, it is necessary that the sum of the moment $\mu$, and the moments of P and Q with respect to the same axis, should be equal to zero, regard being had to the direction in which the forces, of which $\mu$ is the moment, and the forces P and Q, tend to make this part of the plate to turn, in this manner we shall have

$$\mu + P(b-y) - Q(a-x) = 0.$$

By assuming the abscissa $x$ for the independent variable, and observing that the plate is convex towards the axis A$x$, we shall have

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \div \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}},$$

in which the radical is considered to be positive. If, therefore, this value be substituted in that of $\mu$, and the resulting expression, in the preceding equation, and if, in order to abridge, we make

$$\tfrac{2}{3}\,a\lambda c^3 = \beta,$$

we shall have

$$\beta\,\frac{d^2y}{dx^2} = [\text{Q}(a-x) - \text{P}\,(b-y)]\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}, \qquad (1)$$

for the equation of the curve formed by the elastic plate in equilibrio.

Its integral will contain two constant arbitraries, which are determined by the conditions $y = \varepsilon$ and $\frac{dy}{dx} = 0$, when $x = 0$, or, if we prefer it, $y = 0$ and $\frac{dy}{dx} = 0$, for this value of $x$, in consequence of the extreme smallness of $\varepsilon$. If then $x$ be made equal to $a$, and $y$ equal to $b$ in this integral, an equation between $a$ and $b$ will be obtained, and this combined with that, which is furnished by the given length of the plate, gives the two equations that are necessary to determine the unknown coordinates $a$ and $b$, and the *elastic curve*, properly so called, will be completely determined.

309. If the plate, instead of being fixed, is entirely free at its extremity A, then in order to maintain it in equilibrio, it is necessary to apply to this extremity a force, the components of which are equal and contrary to P and Q; by taking the corresponding extremity of the mean filament for its point of application, it is, moreover, necessary that the resultant of P and Q should pass through this point, in order that this should take place, we must have

$$\text{Q}a = \text{P}\,(b - \varepsilon)$$

This equation will be sufficient, when the plate is retained by a fixed axis, passing through this extremity of the mean filament, and drawn in the direction of its breadth. If it is merely placed on a plane perpendicular to its length, which does not prevent it from turning about the edge of one of its two faces, it is necessary that the friction of this edge against the plane, or some other force, should prevent the plate from

sliding When the plate is not firmly fixed, the direction of its plane which is a tangent to the curve at the point A, will not be known, if, however, the origin of the coordinates $x$ and $y$, be placed at this point, we shall still have $y = \varepsilon$, or $y = 0$, when $x = 0$, but we can no longer take the axis of $x$ on the tangent at A, since its direction is not known *a priori*. This axis will then be the given direction of the force P, and in the determination of the constant arbitraries, the equation $\frac{dy}{dx} = 0$, when $x = 0$, must be replaced by the preceding equation relative to the moments of the forces P and Q, which can be reduced to $\text{Q}a = \text{P}b$

310. If P be supposed equal to cypher, the plate will be bent by a force Q perpendicular to its primitive direction, which is, for example, the case of a horizontal plate, one end of which is fixed, a given weight Q being attached to the other

If in this case we make

$$\beta = c^2 \text{Q},$$

$c$ being a line, the given length of which is generally very great, unless the weight Q is also very considerable, equation (1) will become

$$c^2 \frac{d^2y}{dx^2} - \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = a - x, \tag{2}$$

and by integrating it so that we may $\frac{dy}{dx} = 0$ when $x = 0$, we shall have

$$2c^2 \frac{dy}{dx} - \sqrt{1 + \frac{dy^2}{dx^2}} = 2ax - x^2$$

Hence we obtain

$$dy = \frac{(2ax - x^2)\, dx}{\sqrt{4c^4 - (2ax - x^2)^2}},$$

$$ds = \frac{2\, c^2 dx}{\sqrt{4c^4 - (2ax - x^2)^2}},$$

$ds$ being the element of the curve. These formulæ may be exactly integrated by means of elliptic functions, but in consequence of the magnitude of $c$, we have $s = x$ very nearly, and the value of $dy$ may be reduced to

$$dy = \frac{1}{2c^2}(2ax - x^2)\,dx,$$

hence we obtain

$$6\,c^2y = 3\,ax^2 - x^3,$$

for the equation of the curve.

If the plate deviates very little from the horizontal direction, the abscissa $a$ may be taken for its length, and the ordinate $b$ will express its greatest deviation Because

$$3\,\text{Q}c^2 = a\omega\varepsilon^2,$$

if, as before, we make $2\varepsilon\lambda = \omega$, we shall have

$$a\omega\varepsilon^2 b = a^3\text{Q},$$

when $x = a$ and $y = b$. Hence it follows, that the nature of the plate remaining the same, the quantity $b$, by which it will be deflected, will be proportional to the weight Q and to the cube of the length $a$, and in the inverse ratio of the square of its thickness $\varepsilon$ and of the area of $\omega$ its transversal section.

If for $a\omega$, its value $\frac{w}{\delta}$, given in No. 308, be substituted, and if $h$ denotes the total lengthening $a\delta$ of the plate, produced by a weight $w$, we shall have

$$b = \frac{ha^2\text{Q}}{\varepsilon^2 w}.$$

If $w$ be supposed equal to Q, it follows from this that when the same weight Q, applied to the free extremity of an elastic plate, acts successively in the direction of its length and perpendicularly to its length, the extension $h$ and flexion $b$, which

are supposed to be very small in respect to the length $a$, will be as the squares of the thickness and of this length($e$).

311 Whatever be the nature of the forces P and Q, a first integral of equation (1) may be always obtained, by reducing it to the form of equation (2), by the transformation of co-ordinates We shall restrict ourselves to the consideration of the case, in which the plate being pressed against a plane, and not fixed in it, deviates little from its natural form This will be, for example, the case of a spring, the lower extremity of which is laid on a horizontal plane, its upper extremity being at the same time loaded with a given weight. Let us suppose that in bending under this load, the spring deviates very little from the vertical AB, and that throughout its entire length, the tangent to the curve, which it forms in its state of equilibrium, makes a very small angle with this right line Figure 78 represents the different forms which it can assume in this state

Let there be taken for the axes of $x$ and $y$, the vertical A$x$ drawn in a direction opposite to that of gravity, and the horizontal line A$y$. The quantity $\frac{dy}{dx}$ being very small by hypothesis, its square may be neglected in equation (1), we shall likewise have $Q = 0$, since the force which acts at the extremity B is vertical, in virtue of the equation $Qa = Pb$ of No 309, it follows that $b = 0$, and as the weight P acts in the direction from B towards A, the sign of this force must be changed in equation (1), in which it is supposed to act in the contrary direction. This equation will thus become simply

$$c^2 \frac{d^2y}{dx^2} = -\pi^2 y,$$

by making, in order to abridge,

$$\beta = \tfrac{1}{3} a\omega\varepsilon^2 = \frac{c^2}{\pi^2} P.$$

In this equation $\omega$ denotes the area of the section of the

spring perpendicular to its length, $\varepsilon$ its semi-thickness, in the direction in which it is bent, and $a$ a quantity depending on the material of which it is composed These three quantities are supposed to be constant, and, consequently, $c$ is a line of a constant and given length.

Since $y = 0$, when $x = 0$, we obtain from this equation

$$y = k \sin \frac{\pi x}{c}, \quad \frac{dy}{dx} = \frac{\pi k}{c} \cos \frac{\pi x}{c}$$

$k$ being a constant arbitrary, which must be either cypher, or very small relatively to $c$.

When $k = 0$, the spring will remain straight, and its length AB will be a little diminished by the weight P. When this coefficient $k$ does not vanish, the spring will bend, at the point B, we shall have $x = a$ and $y = b = 0$, therefore, $i$ denoting any whole number, it is necessary we should have($f$)

$$a = ic$$

for the value of $a$ or AB Naming $l$ the length of the spring, we shall likewise have

$$b = \int_0^a \sqrt{1 + \frac{dy^2}{dx^2}}\, dx = \int_0^a \sqrt{1 + \frac{\pi^2 k^2}{c^2} \cos^2 \frac{\pi x}{c}}\, dx;$$

which, by neglecting the fourth power of $\frac{k}{c}$, and substituting its value for $a$, becomes($g$)

$$l = ic\left(1 + \frac{\pi^2 k^2}{4c^2}\right),$$

from which we obtain

$$k = \frac{2c}{\pi}\sqrt{\frac{l}{ic} - 1}. \tag{3}$$

Thus then, the coefficient $k$ will either vanish, or be expressed by this formula.

312. The following remarkable consequences follow from this result.

1st As long as $l$ is less than $c$, formula (3) will be imaginary for every value of the whole number $i$, the coefficient $k$ cannot then be different from cypher, and the spring will not be *bent* by the weight P.

2ndly. When by increasing the length of the spring, or by diminishing the quantity $c$, (which last is effected by increasing the weight P,) $l$ is made to surpass $c$, the value of $k$, which is different from cypher, and which corresponds to $i=1$, will be real, and the spring can be bent by this weight. If $f$ denotes a very small fraction, and if we make

$$l=c\left(1+\frac{\pi^2 f^2}{4}\right),$$

we shall have

$$i=1, \quad a=c, \quad k=fa,$$

and the equation of the curve of the spring will be consequently

$$y=fa\sin\frac{\pi x}{a},$$

from which it is evident, that it does not intersect the curve between the two points A and B

3rdly The ratio $\frac{l}{c}$ still continuing to increase, if it surpasses 2, the value of $k$ which corresponds to $i=2$ will be real, and the spring can assume a figure different from the preceding. For if $f'$ denotes a very small fraction, and if we make

$$l=2c\,(1+\pi^2 f'^2),$$

we shall have

$$i=2, \quad a=2c, \quad k=f'a,$$

hence there will result,

$$y=f'a\sin\frac{2\pi x}{a},$$

from which it is evident, that in this case, the curve will intersect

the vertical at the middle point of AB, which corresponds to $x = \frac{1}{2}a$

4thly By continuing in this manner, it may be proved, that if $l$ surpasses $ic$ by a small quantity, and if, $\phi$ denoting a very small fraction, we have

$$l = ic\left(1 + \frac{i^2\pi^2\phi^2}{4}\right),$$

we can assume

$$a = ic, \quad k = \phi a,$$

by means of which we obtain

$$y = \phi a \sin\frac{i\pi x}{a},$$

which is the equation of a curve that will intersect AB in the number $i + 1$ of equidistant points, A and B being reckoned in this number($h$).

When $l$ surpasses a multiple of $c$, by a quantity which is not very small, the value of $k$, furnished by formula (3), ceases to be very small relatively to $c$, and as the value of $\frac{dy}{dx}$ is then no longer a very small fraction, the figure of the spring cannot be determined by the preceding analysis It should be observed, that in all cases, the rectilineal figure, which corresponds to $k = 0$, is possible, but it is not necessary and stable except in the case of $l \angle c$

313 By the *force* of a spring, (which, for greater clearness, we suppose to be vertical,) is understood the greatest weight which it can support without bending This weight P is determined by the equation $c = l$, which gives($i$)

$$P = \frac{\pi^2 a\omega\varepsilon^2}{3l^2},$$

from which it appears, that, every thing else being the same, the force of a spring is in the inverse ratio of the square of its length It likewise appears, that when the spring is a rectangular parallellopiped, if its adjacent faces be bent, its

force will be proportional to the square of the thickness which is perpendicular to the face that is bent With respect to the absolute magnitude of P, it may be computed by substituting in the preceding formula the value of $a$, which may be deduced either from $h$ the extension of this spring, or from its flexion $l$, produced by a weight $w$, now, it appears from Nos 308 and 310, and because $a\delta = h$ and $a = l$, that these values are

$$a = \frac{wl}{\omega h}, \quad a = \frac{wl^3}{\omega \varepsilon^2 b},$$

consequently, we shall have

$$\mathrm{P} = \frac{\pi^2 w \varepsilon^2}{3lh}, \quad \mathrm{P} = \frac{\pi^2 wl}{3b}.$$

314 The results of No 307 may be easily extended to an elastic rod, when it is supposed to be straight, or of a single curvature in its natural state, and that, when it is bent, it continues to be of single curvature, and does not experience any torsion

In this case, the mean filament will be assumed to be that which passes through the centres of gravity of all sections perpendicular to its length, which may be either constant, or variable, provided that in each point their dimensions are very small, with respect to the radius of curvature of the rod Let $\omega$ be the area of one of these sections, made through any point whatever of the mean filament, if it be divided into elements perpendicular to the plane of this filament, and if $vdu$ be the area of the element that is at the distance $u$ from this same filament, in this expression the variable $u$ may be either positive or negative, and $v$ denotes a given function of $u$. Likewise, if $k$ and $-k'$ denote the extreme values of $u$, we shall have

$$\int_{-k'}^{k} vdu = \omega, \quad \int_{-k'}^{k} vudu = 0$$

the second equation obtains because the origin of the variable $u$ is at the centre of gravity of $\omega$

Let $\sigma$, $\sigma'$, $\delta$, $\delta'$, $\rho$ denote the same quantities as in No 307, and let $\gamma$, $\gamma'$, $r$, be what $\sigma$, $\sigma'$, $\rho$, were in the natural state of the elastic rod; we shall have for the two states of this rod,

$$\gamma' = \gamma + \frac{u\gamma}{r}, \quad \sigma' = \sigma + \frac{u\sigma}{\rho},$$

and, for the passage from the one to the other,

$$\sigma = \gamma(1 + \delta), \quad \sigma' = \gamma'(1 + \delta').$$

Hence by neglecting the products $\frac{\delta u}{r}$ and $\frac{\delta u}{\rho}$, we shall obtain ($k$)

$$\delta' = \delta + u\left(\frac{1}{\rho} + \frac{1}{r}\right),$$

a value that coincides with that of the number cited, in the case in which the rod is naturally straight, when we have

$$r = \infty$$

Moreover, let T be the sum of the forces perpendicular to $\omega$, which draw or push one of the two parts of the rod that are separated by this normal section. Naming $\mu$ the moment of these forces with respect to the axis passing through the centre of gravity of $\omega$, and perpendicular to the plane of the mean filament, we shall have by the hypothesis of No. 307,

$$\mathrm{T} = a\int_{-k'}^{k} \delta' v\,du, \quad \mu = a\int_{-k'}^{k} \delta' vu\,du,$$

$a$ being a quantity dependent on the material of the rod, which is supposed to be constant through the extent of each section $\omega$, but which may vary from one point to another of the mean filament By substituting for $\delta'$ its preceding value, and making, in order to abridge($l$)

$$\int_{-k'}^{k} vu^2 du = \tfrac{1}{3}\omega q^2,$$

there will result

$$\mathrm{T} = a\omega\delta, \quad \mu = \frac{a\omega q^2}{3}\left(\frac{1}{\rho} - \frac{1}{r}\right).$$

When the elastic rod is a curve of double curvature in its natural state, or after its change of figure, the expression for the force T will be still the same; moreover, the mean filament being always that which passes through the centres of gravity of all the normal sections, and $r$, $\rho$, denoting the radii of curvature in the same point, before and after this change, this value of $\mu$ may be taken to express the moment of elasticity with respect to an axis passing through this point, and perpendicular to the osculating plane of the mean filament; but it will be necessary, besides, to take into account the torsion of the rod, as will be done immediately.

315 It appears from a comparison of this value of $\mu$ with that given in 307, that the second differential equation of the plane curve formed by the mean filament of an elastic rod which does not experience any torsion, differs from that which refers to the elastic plate properly so called, only in this, that it will contain $\frac{1}{\rho} - \frac{1}{r}$ in place of $\frac{1}{\rho}$, and the quantity $q$ instead of the semi-thickness. If the rod is homogeneous, and if, in its natural state it is either a prism or a cylinder of a small diameter, the three quantities $a, \omega, c$, will be constant, and we shall have $r = \infty$. Hence it appears, that the flexion of a rod which is naturally straight, produced by a weight perpendicular to its direction, and the force of the spring, may be deduced from the values of $b$ and P found in Nos. 310 and 313, by substituting $q$ in place of . By means of this substitution, we shall have, ($l$ being the length of this rod,)

$$b = \frac{l^3 Q}{a\omega q^2}, \quad P = \frac{\pi^2 a\omega q^2}{3l^2},$$

or, what comes to the same thing,

$$b = \frac{\pi^2 l Q}{3P}, \quad P = \frac{\pi^2 a}{l^2}\int_{-k}^{k} vu^2 du$$

In the case of two different rods, having the same length,

the flexions produced by the same weight will be in the inverse ratio of the forces of the spring, so that we will be enabled to compare together the magnitudes of these forces, in the different hypotheses made respecting the contour or outline of the normal section

Let, for example, the normal section be an isoceles triangle, and let it be proposed to bend the rod so that the face corresponding to the base of this triangle might become a cylindrical surface, either convex or concave Let $a$ and $c$ denote the base and height of this triangle. When the surface is that of a convex cylinder, towards which the positive values of $u$ (No 307) are directed, we shall have

$$k=\tfrac{1}{3}c, \quad k'=\tfrac{2}{3}c, \quad v=\frac{a}{c}(\tfrac{2}{3}c+u),$$

and there will result from this

$$\text{P}=\frac{\pi^2 aac^3}{36l^3}$$

In the case of the concavity, we shall have

$$k=\tfrac{2}{3}c, \quad k'=\tfrac{1}{3}c, \quad v=\frac{a}{c}(\tfrac{1}{3}c+u),$$

hence we obtain $(m)$

$$\text{P}=\frac{\pi^2 aac^3}{12l^2},$$

which shews, that in this second case, the force of the spring is triple of what it is in the first.

If the normal section is a square represented by $f^2$, and if it is proposed to bend the spring in such a manner that two of its opposite faces may become cylindrical surfaces, we shall have

$$k=k'=\tfrac{1}{2}f, \quad v=f, \quad \text{P}=\frac{\pi^2 af^4}{12l^2}$$

If it is a circle, the radius of which is $k$, we shall have

$$k'=k, \quad v=2\sqrt{k^2-u^2}, \quad \text{P}=\frac{\pi^3 ak^4}{4l^2},$$

and, if the area of the normal section be supposed to be equal

in this and the preceding case, so that we may have $f^2 = \pi k^2$, it is evident that the force of the spring in the first case, is to the force of the spring in the second, in the ratio of $\pi$ to 3(*n*)

Let the cylindrical spring be supposed to be a hollow tube, of which the radii of the interior and exterior concentrical surfaces are $g$ and $g'$ In order to obtain the force of the spring, $g$ and $g'$ ought to be successively substituted in place of $k$ in the last value of P, and then the results should be taken, the one from the other, this gives

$$\text{P} = \frac{\pi^3 a\,(g'^2 + g^2)\,(g'^2 - g^2)}{4l^2}.$$

If the area $\pi\,(g'^2 - g^2)$ of the normal section is equal to $\pi k^2$, we shall have(*o*)

$$\text{P} = \frac{\pi^3 a k^2\,(k^2 + 2g^2)}{4l^2},$$

hence it follows, that the volume, the length and the material being the same, the force of a hollow spring is greater than that of a full spring in the ratio of $1 + \frac{2g^2}{k^2}$ to unity, $2g$ being the interior diameter, and $\pi k^2$ the area of the normal section

316. Let us now proceed to form the equations of equilibrium of any elastic rod whatsoever, all whose points are solicited by given forces

Let A and B be the two extremities of the mean filament, $x, y, z$ the three rectangular coordinates of any point M of this curve, $s$ the arc AM, $\omega$ the normal section of the rod made through the point M, $\gamma$ its density at this point, and consequently, $\gamma\omega ds$ the mass of the infinitely slender slice of the rod If X, Y, Z, be the forces referred to the unit of mass, parallel to the axes of $x$, $y$, $z$, then $\text{X}\gamma\omega ds$, $\text{Y}\gamma\omega ds$, $\text{Z}\gamma\omega ds$, will be the given forces which act on this mass The sum of these forces resolved in the direction of the tangent to the mean filament at the point M, and tending to increase the arc $s$, will be

$$\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)\gamma\omega ds$$

Likewise, if T denotes the force arising from the action of one part of the rod on the adjacent part, applied to one of the faces of the slice $\gamma\omega ds$, perpendicular to $\omega$, and tending to increase or diminish the arc $s$, according as it is positive or negative, the other face of $\gamma\omega ds$ will be drawn or pushed in a contrary direction by a force equal to $T + dT$, consequently, in order that this slice may be in equilibrio, the force $dT$ must be equal and contrary to the given tangential force, hence we must have($p$)

$$dT + \gamma\omega\,(Xdx + Ydy + Zdz) = 0, \qquad \text{(a)}$$

which agrees with equations (3) of No 300

As the material of which the rod consists is very little extensible, $\gamma$ and $\omega$ in this equation (a) may be assumed to be the density, and the normal section of the rod at the point M, in its natural state If these two quantities are constant, and if the formula comprised between the parentheses is an exact differential, the value of T will be had by an immediate integration, and, since $T = c\omega\delta$ (No 307), it is easy to infer the positive or negative dilatation of the element $ds$, which will be lengthened in the ratio of $1 + \delta$ to unity, but this will not make known the dilatation of the normal section $\omega$, nor the change of density of the rod at the point M Now, by what has been established in the memoir cited at the commencement of this paragraph, it appears that the lengthening or shortening of $ds$ is always accompanied by a diminution or increase of $\omega$, but such, that the volume $\omega ds$ will increase and diminish with $ds$, and the density $\gamma$ will vary in the inverse sense It follows from this, that when a prismatic or cylindrical homogeneous rod is fixed at one end, and drawn at its other extremity by a force acting in the direction of the production of its length, it will experience at the same time an extension and increase of volume, proportional to this force, which, in point of fact has

been confirmed by experiment On the other hand, if this rod is placed vertically on a horizontal plane, and loaded with a weight at its upper part, which does not cause it to bend, it will be shortened, and, at the same time, its volume will be diminished in proportion to the magnitude of this weight

317 If on AM the arc of the mean filament, a point $m$ infinitely near to M be taken, and through it a normal section be made, and if the part of the rod comprised between this section and the extremity A, be rendered immoveable, and the form of the part comprised between the other end B, and the section made through M, be merely rendered invariable, the conditions of the equilibrium of this second part, which we shall denote by K, may be determined in the following manner.

In virtue of the torsion of the rod, the points of the slice comprised between the two normal sections made through M and $m$, will be solicited by forces which will tend to untwist its different longitudinal filaments, and will act in planes perpendicular to M$m$, that is to say, to the tangent to the mean filament at M These forces will tend to make K turn about this line, in a direction contrary to that of its torsion. Let $\tau$ be their moment with respect to this line, it will be what is meant by *the moment of torsion* of the rod, corresponding to the point M If through this point lines be drawn parallel to the axes of $x, y, z$, then since the axis of this moment makes with these lines, angles whose cosines are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, we shall have (No. 281)

$$-\tau\frac{dx}{ds}, \quad -\tau\frac{dy}{ds}, \quad -\tau\frac{dz}{ds},$$

for the moments relatively to these three parallels, of the forces which act on K in the direction of the torsion

Let $\mu$ denote the moment of the elasticity with respect to the point M, that is to say, the moment of the forces of which T is the sum, relatively to an axis drawn through this point,

and perpendicular to the osculating plane of the mean filament, we shall have (No. 314), ($r$ and $\rho$ being respectively the radii of curvature at this same point, in the natural state of the plate, and after its form has been changed, and $\beta$ denoting a positive quantity depending on the material and normal section at the point M,)

$$\mu = \beta\left(\frac{1}{\rho} - \frac{1}{r}\right),$$

and if $f, g, h$, be the angles which the axis of this moment makes with parallels to the axes of $x, y, z$, drawn through the point M, the moment of elasticity relative to these lines will be

$$\mu \cos f, \quad \mu \cos g, \quad \mu \cos h.$$

Let M$'$ be any point whatever of the arc MB, $x', y', z'$, its three coordinates, $s'$ the arc AM$'$, and $\gamma', \omega'$, X$'$, Y$'$, Z$'$, what $\gamma, \omega$, X, Y, Z, become relatively to the point M$'$. If we denote the entire length of the mean filament by $l$, and make

$$\int_s^l [\mathrm{Y}'(x' - x) - \mathrm{X}'(y' - y)]\, \gamma'\omega' ds' = \mathrm{Z}_1,$$

$$\int_s^l [\mathrm{X}'(z' - z) - \mathrm{Z}'(x' - x)]\, \gamma'\omega' ds' = \mathrm{Y}_1,$$

$$\int_s^l [\mathrm{Z}'(y' - y) - \mathrm{Y}'(z' - z)]\, \gamma'\omega' ds' = \mathrm{X}_1,$$

these three quantities will be the moments of the given forces which act on K, with respect to the axes drawn through the point M, in the directions of the axes of $x, y, z$.

Finally, if any particular forces be supposed to act at the free extremity of K, and if P, Q, R, denote the sums of their components parallel to the axes of $x, y, z$, and $a', b', c'$, the coordinates of the point of application of their resultant, their moments relatively to the same axes as $\mathrm{Z}_1, \mathrm{Y}_1, \mathrm{X}_1$, will be

$$\mathrm{Q}(a' - x) - \mathrm{P}(b' - y),$$
$$\mathrm{P}(c' - z) - \mathrm{R}(a' - x),$$
$$\mathrm{R}(b' - y) - \mathrm{Q}(c' - z),$$

and if $a, b, c$, be the coordinates of the extremity B of the mean filament, these moments may be replaced by

$$\begin{aligned} &Q(a-x) - P(b-y) + R', \\ &P(c-z) - R(a-x) + Q', \\ &R(b-y) - Q(c-z) + P', \end{aligned}$$

in which, for the sake of abridging, we make

$$\begin{aligned} &Q(a'-a) - P(b'-b) = R', \\ &P(c'-c) - R(a'-a) = Q', \\ &R(b'-b) - Q(c'-c) = P'. \end{aligned}$$

In general, the coordinates $a', b', c'$, will be distinct from $a, b, c$, for the extreme forces P, Q, R, will not be immediately applied to the elastic rod, but will act at the extremities of the arm of a lever. Whether these forces have, or have not, an unique resultant, the quantities P′, Q′, R′, will be their moments with respect to axes drawn through the point B parallel to those of $x, y, z$, therefore, if at this point, we assume

$$\frac{dx}{ds} = \cos\alpha', \quad \frac{dy}{ds} = \cos\beta', \quad \frac{dz}{ds} = \cos\gamma',$$

and if we make

$$P'\cos\alpha' + Q'\cos\beta' + R'\cos\gamma' = L,$$

this quantity L will express the moment of the extreme forces with respect to the tangent at the point B (No 281), hence it appears, that L will be the moment of the extreme torsion, or the value of $\tau$ relative to this same point

This being established, in order to the equilibrium of the part K of the elastic rod, it is necessary that the sum of the moments with respect to each axis, of all the forces which act on its different slices and at its extremities, be equal to cypher, in consequence of which the three following equations result

$$\left.\begin{array}{l}\mu\cos f - \tau\dfrac{dx}{ds} + X_1 + P' + R(b-y) - Q(c-z) = 0,\\ \mu\cos g - \tau\dfrac{dy}{ds} + Y_1 + Q' + P(c-z) - R(a-x) = 0,\\ \mu\cos h - \tau\dfrac{dz}{ds} + Z_1 + R' + Q(a-x) - P(b-y) = 0\end{array}\right\} \quad \text{(b)}$$

318 By the formulæ of No. 19, we have

$$\cos f = \frac{dy\,d^2z - dz\,d^2y}{\lambda ds^3},$$

$$\cos g = \frac{dz\,d^2x - dx\,d^2z}{\lambda ds^3},$$

$$\cos h = \frac{dx\,d^2y - dy\,d^2x}{\lambda ds^3};$$

in which $\lambda ds^3$ represents the square root of the sum of the squares of the three numerators. Hence we obtain

$$d\ \mu\cos f = dy\,d.\frac{\mu d^2z}{\lambda ds^3} - dz\,d\ \frac{\mu d^2y}{\lambda ds^3},$$

$$d.\mu\cos g = dz\,d\ \frac{\mu d^2x}{\lambda ds^3} - dx\,d\ \frac{\mu d^2x}{\lambda ds^3},$$

$$d\ \mu\cos h = dx\,d.\frac{\mu d^2y}{\lambda ds^3} - dy\,d\ \frac{\mu d^2x}{\lambda ds^3},$$

and, consequently,

$$\frac{dx}{ds}d.\mu\cos f + \frac{dy}{ds}d.\mu\cos g + \frac{dz}{ds}d\ \mu\cos h = 0,$$

moreover, since

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1,$$

$$\frac{dx}{ds}.d.\frac{dx}{ds} + \frac{dy}{ds}.d.\frac{dy}{ds} + \frac{dz}{ds}.d.\frac{dz}{ds} = 0,$$

we shall obtain, by multiplying the differentials of equations (b) by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, respectively, then adding them together and reducing($p$)

$$d\tau = \frac{dx}{ds}\,d\text{X}_1 + \frac{dy}{ds}\,d\text{Y}_1 + \frac{dz}{ds}\,d\text{Z}_1,$$

but because the quantities subjected to integration in the expressions of $\text{X}_1$, $\text{Y}_1$, $\text{Z}_1$, vanish at the limit $s' = s$, in order to obtain the values of $d\text{X}_1$, $d\text{Y}_1$, $d\text{Z}_1$, it is sufficient (No 14) to differentiate under the signs $\int$ with respect to $x, y, z$, hence we have simply

$$d\text{X}_1 = dz\int_s^l \text{Y}'\gamma'\omega' ds' - dy\int_s^l \text{Z}'\gamma'\omega' ds',$$

$$d\text{Y}_1 = dx\int_s^l \text{Z}'\gamma'\omega' ds' - dz\int_s^l \text{X}'\gamma'\omega' ds',$$

$$d\text{Z}_1 = dy\int_s^l \text{X}'\gamma'\omega' ds' - dx\int_s^l \text{Y}'\gamma'\omega' ds',$$

and if these values be substituted in the preceding equation, it will become $d\tau = 0$

Thus it appears, that the moment of the torsion is constant throughout the entire length of an elastic rod in equilibrio, consequently, its value will be the same for all its other points as at the two extremities of the rod, and it is easy to show that at the point B, we have $\tau = \text{L}$, as has been stated above In fact, at this point, we have $x = a$, $y = b$, $z = c$, the integrals $\text{X}_1$, $\text{Y}_1$, $\text{Z}_1$, vanish, and equations (b) become($g$)

$$\tau \cos\alpha' = \mu\cos f + \text{P}',$$
$$\tau \cos\beta' = \mu\cos g + \text{Q}',$$
$$\tau \cos\gamma' = \mu\cos h + \text{R}'.$$

As the normal to the osculating plane of the mean filament, and the tangent to this curve, are perpendicular the one to the other, we have at this same point B,

$$\cos\alpha'.\cos f + \cos\beta'.\cos g + \cos\gamma'\ \cos h = 0,$$

hence, if the preceding equations be multiplied by $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, respectively, and then added together, the quantity $\mu$ will disappear, and we shall have

$$\tau.(\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma') = \tau$$
$$= \text{P}'\cos\alpha' + \text{Q}'\cos\beta' + \text{R}'\cos\gamma' = \text{L}.$$

It is only the *moment* of the torsion which we are enabled to determine by the equations of equilibrium, with respect to the torsion itself, it varies along the rod, when the material of which it consists, or the normal section varies from one point to another. When the rod is homogeneous, and the normal section constant, the difference of the angles of torsion is the same at the extremities of any two parts of the rod of *equal* lengths, and proportional to their lengths, when they are *different*. For greater clearness, let us suppose that a prismatic or cylindrical homogeneous rod, is firmly fixed at one extremity, and that at its other extremity, two equal forces, parallel and opposite are applied, this rod will remain straight but it will be twisted on itself proportionably to its length, and to the moment of these two forces with respect to its mean filament, which moment will be the value of the quantity L M Poisson proved in the memoir already cited (No 306), that if the normal section of this rod be a circle, the quantity of torsion will be proportional (every thing else being the same) to the fourth power of its diameter, which accords with experiment.

319 Two of equations (b), or any two combinations of these equations, will enable us, when its value is substituted for $\mu$, and L is put for $\tau$, to determine the figure of the rod in equilibrio If in its natural state it is straight, and if all the forces which are applied to it, exist in the same plane, the three equations (b) will be reduced to one which will be that of the plane curve formed by the mean filament

If the plane of these forces be taken for that of the axes of $x$ and $y$, we shall have

$$z = 0, \quad \cos f = 0, \quad \cos g = 0,$$
$$c = 0, \quad c' = 0, \quad \text{R} = 0, \quad \cos\gamma' = 0,$$

hence there will result

$$\text{X}_1=0,\quad \text{Y}_1=0,\quad \text{P}'=0,\quad \text{Q}'=0,\quad \tau=\text{L}=0,$$

and the two first equations (b) will vanish. Because $r=\infty$, the value of $\mu$ will be $\frac{\beta}{\rho}$, in like manner we shall have $\cos h = \pm 1$; but as we should take into account the direction of the action of T on the part K of the rod (No 314), it is easy to perceive that we must assume $\cos h=-1$ in the third equation (b), which will consequently become

$$\left.\begin{aligned}&\int_s^l [\text{Y}'(x'-x)-\text{X}'(y'-y)]\,\gamma'\omega' ds'\\ &+\text{R}'+\text{Q}(a-x)-\text{P}(b-y)=\frac{\beta}{\rho},\end{aligned}\right\}\quad (c)$$

and we may remark, that if we retain the notations of No. 314, the value of the coefficient $\beta$ will be

$$\beta=a\int_{-k'}^{k} vu^2 du$$

When the forces X and Y vanish, this equation (c) will coincide with equation (1) of No. 308, for then the forces P and Q in equation (c) act at the very extremity of the rod, which renders their moment R′ equal to cypher In all cases, the integrals that occur in this equation (c), can be made to disappear by successive differentiations, by means of which it will become a differential equation of the fourth order

The figure of the rod being determined by equation (c), is necessary moreover that the given forces which are applied to it should satisfy the conditions of equilibrium of No 261, which are reduced to three, because all these forces are comprised in the same plan. Therefore, if we denote by D and E the sums of the particular forces which act at the extremity A of the rod, parallel to the axes of $x$ and $y$, so that D, E, F′, may be, with respect to this point, what P, Q, R′, are relatively

to the other extremity B, the three equations in question will be

$$\left.\begin{aligned}&\mathrm{D}+\mathrm{P}+\int_0^l \mathrm{X}'\gamma'\omega' ds'=0,\\ &\mathrm{E}+\mathrm{Q}+\int_0^l \mathrm{Y}'\gamma'\omega' ds'=0,\\ &\mathrm{F}'+\mathrm{R}'+\mathrm{Q}(a-x)-\mathrm{P}(b-y)\\ &\qquad+\int_0^l[\mathrm{Y}'(x'-x)-\mathrm{X}'(y'-y)]\gamma'\omega' ds'=0,\end{aligned}\right\}\quad\text{(d)}$$

in which the coordinates of the point A should be substituted for $x$ and $y$.

When the two extremities of the rod are entirely free, the extreme forces and their moments will be given If the rod is firmly fixed at the extremity A, the forces D and E, and also their moment F′, will be undetermined, but the values of $x$, $y$, $\frac{dy}{dx}$, relative to this point A, will be known If the rod is merely retained by the fixed point A, the forces D and E will be still undetermined; their resultant will be equal and contrary to the pressure at this point of support, of which it will express the resistance, and we shall have for their moment $\mathrm{F}'=0$, the values of $x$ and $y$ will then be known, but not that of $\frac{dy}{dx}$. The same remarks are applicable to the point B.

320 Let us suppose, for example, that the rod is homogeneous, and in its natural state either of a prismatic or cylindrical form, the three quantities $\gamma$, $\omega$, $\beta$ will then be constant Moreover, let us suppose that it is only subject to the action of forces perpendicular to its length, which cause it to deviate very little from its primitive position, and let the mean filament in this position be taken for the axis of $x$, we shall then have

$$\mathrm{D}=0,\quad \mathrm{X}=0,\quad \mathrm{P}=0,$$

in consequence of which the first equation (d) will disappear. If the square of $\frac{dy}{dx}$ be neglected, we shall also have

$$ds = dx, \quad \frac{1}{\rho} = \frac{d^2y}{dx^2};$$

and equation (c) will be reduced to

$$\beta \frac{d^2y}{dx^2} = \text{R}' + \text{Q}(a - x) + \gamma\omega \int_s^l \text{Y}'(x' - x)ds' \qquad \text{(e)}$$

By differentiating this we obtain

$$\beta \frac{d^3y}{dx^3} = -\text{Q} - \gamma\omega \int_s^l \text{Y}'ds'$$

Likewise, by No. 14, we have

$$d\int_s^l \text{Y}'ds' = -\text{Y}ds,$$

therefore by differentiating a second time, and substituting $dx$ for $ds$, we shall have

$$\beta \frac{d^4y}{dx^4} = \gamma\omega\text{Y}. \qquad \text{(f)}$$

The four constant arbitraries which the complete integral of this last equation will contain, can be determined by means of the conditions relative to the two extremities of the rod, and by observing that the value of $y$ deduced from this equation must satisfy the two preceding for all values of $x$. Now, as equation (f) results from the two others by differentiation, it will be sufficient for this purpose, that this value of $y$ should satisfy these two for a particular value of $x$, hence it will be sufficient if we have

$$\beta \frac{d^2y}{dx^2} = \text{R}', \quad \beta \frac{d^3y}{dx^3} = -\text{Q} \qquad \text{(g)}$$

for $x = a$, conditions which result from equation (e) and its first differential, by ascribing to $x$ this particular value. If we

ascribe to $x$ the value relative to the point A, we shall have, in consequence of equations (d),

$$\beta \frac{d^2y}{dx^2} = -\text{F}', \quad \beta \frac{d^3y}{dx^3} = \text{E}, \tag{h}$$

but these equations do not express new conditions distinct from those contained in equations (d) and (g), so that they may, if we please, be replaced by the system of equations (g) and (h)

321 These formulæ include the case of a heavy rod. The point A is then supposed to be fixed, and to be the origin of the coordinates $x$ and $y$, and also the axis of $x$, which represents the natural direction of the rod, is supposed to be horizontal If the direction of the positive $ys$ be that of gravity, which we suppose to be represented by $g$, we shall have $\text{Y} = g$, and the integral of equation (f) will be ($q$)

$$\beta y = \frac{g\gamma\omega}{24} x^4 + \text{c}x^3 + \text{c}'x^2 + \text{c}''x, \tag{1}$$

c, c$'$, c$''$, denoting three constant arbitraries, the fourth being equal to cypher, because at the point A we have $x = 0$ and $y = 0$ If the rod be firmly fixed at this extremity, we must also have $\frac{dy}{dx} = 0$ when $x = 0$, hence there results $\text{c}'' = 0$. If, moreover, we suppose that the weight Q is directly attached to the other extremity B, so that its moment R$'$ may be cypher, then in consequence of equations (g) which belong to this point or to $x = a$, we shall have

$$\tfrac{1}{2} g\gamma\omega a^2 + 6\,\text{c}a + 2\,\text{c}' = 0,$$

$$g\gamma\omega a + 6\,\text{c} = -\text{Q}$$

By means of these equations the values of c and c$'$ may be obtained, and if they be substituted in equation (1), of which the term c$''a$ may be suppressed, there results by naming $q$ the weight of the rod, in which case we have $q = g\gamma\omega a$, ($r$)

$$\beta y = \frac{qx^4}{24a} - \tfrac{1}{6}(Q + q)\,x^3 + \tfrac{1}{2}(Q + \tfrac{1}{2}\,q)\,ax^2,$$

if in this equation, the weight of the rod be neglected, and if $Qc^2$ be substituted in place of $\beta$, it will coincide with that of No. 310.

In the two cases of $Q = 0$, $q = 0$, we have

$$b = \frac{qa^3}{8\beta}, \quad b = \frac{Qa^3}{3\beta},$$

for the expression of the ordinate of the point B, which expresses the entire flexion of the rod Hence it appears, that if we suppose $Q = q$, the flexions produced by a weight Q suspended at the free extremity of a horizontal rod firmly fixed at its other extremity, is to the flexions produced by the same weight distributed uniformly over the entire length of this rod, as 8 to 3

322 If the point B is, like the point A, fixed and situated on the same horizontal line, we must have $y = 0$ when $x = a$, in consequence of which, equation (1) is changed into the following (s)

$$\beta y = \frac{qx}{24a}(x^3 - a^3) + cx\,(x^2 - a^2) + c'x\,(x - a), \qquad (2)$$

$q$ denoting as above the weight of the rod The determination of the two constants $c$ and $c'$ furnish the following cases.

1st. When the rod is firmly fixed at both ends, we must have $\frac{dy}{dx} = 0$, for $x = 0$, and $x = a$, hence there results ($t$)

$$c = -\tfrac{1}{12}\,q, \quad c' = \tfrac{1}{24}\,aq,$$

and equation (2) becomes

$$\beta y = q\,x^2\,\frac{(x - a)^2}{24a}$$

If $f$ denotes the sagitta of the curve formed by this rod,

that is to say, the value of $y$ at its middle point, when $x = \frac{a}{2}$, we shall have

$$f = \frac{q a^3}{16\ 24\ \beta}.$$

2ndly. If the rod is merely retained by the fixed points A and B, the forces E and Q taken in an opposite direction from that in which they act, will express the pressures exerted against these points of support, and their moments F′ and R′ will be cypher, (No 319). In consequence of the first equations (g) and (h), we shall have $\frac{dy^2}{dx^2} = 0$ for $x = 0$ and $x = a$, hence we infer that

$$C = -\tfrac{1}{12} q, \quad C' = 0,$$

and also

$$\beta y = \frac{qx(a-x)(a^2 + ax - x^2)}{24\,a},$$

and the sagitta $f$ will be

$$f = \frac{5 q a^3}{16.24\,\beta},$$

that is to say, quintuple of what it is in the first case. By means of the last equations (g) and (h), we shall also have

$$E = Q = -\tfrac{1}{2} q,$$

which values also obtain in the first case, as is evident of itself.

3rdly. Finally, when the rod is firmly fixed at its extremity A, and merely retained at its other extremity, we have $\frac{dy}{dx} = 0$ for $x = 0$, and $\frac{d^2y}{dx^2} = 0$ for $x = a$, and, consequently,

$$C = -\frac{5q}{48} \quad C = \frac{qa}{16},$$

by means of which, equation (2) becomes

$$\beta y = \frac{qx^2(a-x)(3a-2x)}{48a}.$$

The second equations (g) and (h), at the same time, give

$$Q = -\tfrac{3}{8}q, \quad E = -\tfrac{5}{8}q,$$

which shews that the weight of the rod is unequally distributed between the two points of support, and that the load at the extremity which is firmly fixed, is to that at the other extremity, in the ratio of five to three.

323. The points A and B being always assumed to be fixed and situated on the same horizontal line, and the rod being also homogeneous and prismatic, let us consider the case in which the other points are loaded with weights unequally distributed throughout its entire length.

Let therefore

$$\gamma\omega Y = \frac{q}{a}\phi x,$$

$q$ denoting the entire weight, and $\phi x$ being a given function, that vanishes when $x = 0$ and $x = a$, which implies that

$$\int_0^a \phi x dx = a$$

This function $\phi x$ may be either continuous or discontinuous, that is to say, its analytic expression may change once or several times between the extreme values $x = 0$ and $x = a$, or, in other words, if it be represented by the ordinate of a line of which $x$ is the abscissa, this line may consist of several portions of different curves. If $\delta$ denotes the length of a line ever so short, we may, for example, suppose that $\phi x = 0$ from $x = 0$ to $x = \frac{1}{2}a - \delta$, and from $x = \frac{1}{2}a + \delta$ to $x = a$, so that this function has not values different from zero except in a very small extent, such as $\delta$, on each side of $x = \frac{1}{2}a$. This case will be that of $q$, a weight acting at the middle of the elastic rod, which we propose to examine particularly immediately.

Whatever be the nature of the function $\phi x$, whether it be continuous or discontinuous, provided that it is equal to cypher for $x=0$ and for $x=a$, we shall have from $x=0$ to $x=a$ inclusively,

$$\phi x=\frac{2}{a}\Sigma\left(\int_0^a \sin\frac{n\pi x'}{a}\phi x'dx'\right).\ \sin\frac{n\pi x}{a}, \tag{a}$$

$n$ denoting any entire and positive number, and the characteristic $\Sigma$ indicating a sum which extends to all values of $n$, from $n=1$ to $n=\infty$. We are indebted to Lagrange for this formula, he first announced it in *Les Anciens Mémoires de l'Académie de Turin*, tome III p 261, a demonstration of it is given in No 325 of this section. Substituting this value of $\phi x$, equation (f) becomes

$$\beta\frac{d^4y}{dx^4}=\frac{2q}{a^2}\Sigma\left(\int_0^a \sin\frac{n\pi x'}{a}\phi x'dx'\right)\sin\frac{n\pi x}{a},$$

and by integrating and observing that $y=0$ for $x=0$ and $x=a$, we shall obtain($u$)

$$\beta y=\frac{2qa^2}{\pi^4}\Sigma\frac{1}{n^4}\left(\int_0^a \sin\frac{n\pi x'}{a}\phi x'dx'\right)\sin\frac{n\pi x}{a}$$
$$+x(a-x)\left[cx+c'(a-x)\right]; \tag{b}$$

c and c′ being the constant arbitraries which can be determined as in the three cases of the preceding numbers.

324. Let us examine in detail the case in which the weight $q$ is suspended at the middle of the rod, that is to say, the case in which, as has been stated above, the function $\phi x'$ is cypher for all values of $x'$ that differ, ever so little, from $\frac{1}{2}a$. We can then make $x'=\frac{1}{2}a$ in the factor $\sin\frac{n\pi x'}{a}$, which contains the integral relative to $x'$, this will give($v$)

$$\int_0^a \sin\frac{n\pi x'}{a}\phi x'dx'=\sin\frac{n\pi}{2}\int_0^a \phi x'dx'=a\sin\frac{n\pi}{2},$$

and cause all the terms of the sum $\Sigma$, in which $n$ is an even

number, to disappear. Let $\imath$ denote any number odd or even, then if we make $n = 2\imath - 1$, and if the sum $\Sigma$ includes all values of $\imath$ from $\imath = 1$ to $\imath = \infty$, since $\sin\frac{(2\imath - 1)\pi}{2} = -(-1)^{\imath}$, equation (b) becomes ($x$)

$$\beta y = x(a-x)\,[cx + c'(a-x)]$$
$$-\frac{2qa^3}{\pi^4}\,\Sigma\frac{(-1)^{\imath}}{(2\imath-1)^4}\sin\frac{(2\imath-1)\pi x}{a}.$$

But by a known formula we shall have, as will be shewn in No 327,

$$\Sigma\frac{(-1)^{\imath}}{(2\imath-1)^{4}}\sin(2\imath-1)\,\omega = \frac{\pi\omega^3}{24} - \frac{\pi^3\omega}{32}$$

for all values of $\omega$, from $\omega = 0$ to $\omega = \frac{\pi}{2}$. If therefore $x \angle \frac{a}{2}$, by making $\omega = \frac{\pi x}{a}$, we shall have ($y$)

$$\Sigma\frac{(-1)^{\imath}}{(2\imath-1)^4}\sin\frac{(2\imath-1)\pi x}{a} = \frac{\pi^4}{96a^3}\,(4x^3 - 3a^2x),$$

if, on the contrary, $x > \frac{1}{2}a$, let $\omega = \frac{\pi(a-x)}{a}$, and as

$$\sin\frac{(2\imath-1)\,\pi(a-x)}{a} = \sin\frac{(2\imath-1)\,\pi x}{a},$$

it follows that

$$\Sigma\frac{(-1)^{\imath}}{(2\imath-1)^4}\sin\frac{(2\imath-1)\pi x}{a} = \frac{\pi^4}{96a^3}\,[4(a-x)^3 - 3a^2(a-x)].$$

In this manner we shall obtain one or other of the following equations

$$\left.\begin{aligned}\beta y &= x(a-x)\,[cx + c'(a-x)] - \frac{q}{48}(4x^3 - 3a^2x)\\ \beta y &= x(a-x)[cx + c'(a-x)] - \frac{q}{48}[4(a-x)^3 - 3a^2(a-x)]\end{aligned}\right\}(1)$$

It only remains to determine the constants c and c′ in the three following cases:

1st The condition $\frac{dy}{dx} = 0$, for $x = 0$ and $x = a$, which obtains when the rod is firmly fixed at its two extremities, gives

$$c' = c = -\frac{q}{16}$$

Equations (1) will become

$$\beta y = \frac{q}{48}(3ax^2 - 4x^3),$$

$$\beta y = \frac{q}{48}[3a(a-x)^2 - 4(a-x)^3];$$

we shall have $\frac{dy}{dx} = 0$, at the middle point of the rod, as well as at the extremities, and the sagitta $f$, or the ordinate corresponding to $x = \frac{1}{2}a$, will be

$$f = \frac{qa^3}{4.48\beta},$$

that is to say, double of what it is in the first case of No. 322. In like manner we shall have, in consequence of the second equations (g) and (h),

$$Q = E = -\tfrac{1}{2}q,$$

which we know ought to be the case

2ndly In the case of a rod merely retained at its two extremities, in which case $\frac{d^2y}{dx^2} = 0$ for $x = 0$ and for $x = a$, there results

$$c = 0, \quad c' = 0,$$

and, consequently,

$$\beta y = \frac{q}{48}(3a^2 - 4x^3),$$

$$\beta y = \frac{q}{48}[3a^2(a-x) - 4(a-x^3)]$$

The tangent at the middle point of the curve is horizontal($z$), and the values of $Q$ and $E$ are $-\frac{1}{2}q$, as in the preceding case, but the expression for the sagitta $f$ is

$$f = \frac{qa^3}{48\beta},$$

so that it is quadruple of the preceding, and greater in the ratio of 8 to 5, than that of the second case of No. 322. If a tangent be drawn to the elastic curve, through one or other of the points A and B, and if we denote its inclination by $\alpha$, and by $f'$ the vertical ordinate of the point of this line, which corresponds to the abscissa equal to $\frac{1}{2}a$, we shall have ($a'$)

$$\operatorname{tang}\alpha = \frac{qa^2}{16\beta}, \quad f' = \tfrac{1}{2}a \operatorname{tang}\alpha,$$

hence we infer

$$f' = \tfrac{3}{2}f.$$

In the second case of No. 322, the ratio of $f'$ to $f$ would be $\frac{8}{5}$.

3. Finally, if the rod is firmly fixed at the extremity A, and merely supported at the other extremity B, we shall have $\frac{dy}{dx}=0$ for $x = 0$, and $\frac{d^2y}{dx^2} = 0$ for $x = a$, from which there results

$$c' = -\frac{q}{16}, \quad c = -\frac{q}{32},$$

and equations (1) will become

$$\beta y = \frac{q}{96}(9ax^2 - 11x^3),$$

$$\beta y = \frac{q}{96}(5x^3 - 15ax^2 + 12a^2x - 2a^3).$$

They give when $x = \frac{1}{2}a$, the same value for $y$, namely

$$y = \frac{7qa^3}{8 \cdot 96\,\beta},$$

but this is not the greatest ordinate. We shall likewise have

$$K = \frac{-11q}{16}, \quad Q = \frac{-5q}{16},$$

so that the weight $q$ will be distributed between the points of support A and B in the ratio of 11 to 5.

325. We now proceed to demonstrate the formula of Lagrange, adverted to in No 323

For this purpose, it may be remarked, that if the quantity

$$\frac{1-h^2}{1-2h\cos\theta+h^2},$$

which is a rational fraction with respect to $h$, and in which $\theta$ is a real angle, be developed with respect to the powers of $h$, its expression will be

$$1+2h\cos\theta+2h^2\cos 2\theta+2h^3\cos 3\theta+2h^4\cos 4\theta+\&c\,,$$

which is easily verified, for if this infinite series be multiplied by the denominator of the fraction, i e by $1-2h\cos\theta+h^2$, the result will be its numerator, as will be evident at once from the consideration that

$$2\cos n\theta\ \cos\theta=\cos(n+1)\theta+\cos(n-1)\theta,$$

whatever be the number $n$. If, abstracting from the sign, $h$ be less than unity, this series will be convergent, and the fraction will be rigorously equal to its development continued *ad infinitum*, hence since

$$1-2h\cos\theta+h^2=(1-h)^2+4h\sin^2\tfrac{1}{2}\theta,$$

we shall have, on this hypothesis,

$$\frac{1-h^2}{(1-h)^2+4h\sin^2\frac{1}{2}\theta}=1+2\Sigma h^n\cos n\theta;$$

in which, under the sum $\Sigma$, are included all values of the integral number $n$, from $n=1$ to $n=\infty$. Therefore, whatever be the nature of the function $f\theta$ and of the real constant $a$, we shall have also

$$\int_0^\pi\frac{(1-h^2)f\theta d\theta}{1-h^2+4h\,.\sin^2\frac{1}{2}(\theta-a)}$$
$$=\int_0^\pi f\theta d\theta+2\Sigma h^n\int_0^\pi f\theta\,.\cos n(\theta-a)\,d\theta$$

Let $g$ be an infinitely small positive quantity, this equation

will still subsist if we make $h = 1 - g$, since it subsists for all values of $h$ less than unity. For all finite values of $n$, we shall have

$$h^n = (1 - g)^n = 1\,,$$

for infinite values of this exponent, $h^n$ may differ from unity, but by partially integrating, we obtain

$$\int f\theta \cos n(\theta - a)\,d\theta = \frac{1}{n} f\theta \sin n(\theta - a) - \frac{1}{n}\int \frac{df\theta}{d\theta} \sin n(\theta - a)\,d\theta;$$

So that if $f\theta$ does not become infinite, between the limits $\theta = 0$ and $\theta = \pi$, nor for these limits, the integral

$$\int_0^\pi f\theta \cos n(\theta - a)\,d\theta$$

by which $h^n$ is multiplied, will vanish, when $n = \infty$; hence it follows that we can always replace $h^n$ by unity under the sign $\Sigma$ In the numerator of the fraction comprised under the sign $\int$, we shall have $1 - h^2 = 2g$, $g^2$ being neglected with respect to $2g$, in the second term of the denominator, we can substitute unity for $h$ or $1 - g$; and, by this means, we shall have

$$\left.\begin{aligned} &\tfrac{1}{2}\int_0^\pi f\theta\,d\theta + \Sigma\int_0^\pi f\theta \cos n(\theta - a)\,d\theta \\ &= \int_0^\pi \frac{g f\theta\,d\theta}{g^2 + 4\sin^2\frac{1}{2}(\theta - a)} \end{aligned}\right\} \quad (1)$$

The coefficient of $d\theta$ in this last integral is infinitely small, except for values of $\theta$, that differ from $a$ by infinitely small quantities, which render its denominator infinitely small; consequently this integral is infinitely small or cypher, when the difference $\theta - a$ is a finite quantity, which will be the case throughout the entire integration, when we suppose $a \angle 0$, or $a > \pi$, hence whenever the constant $a$ falls without the limits zero and $\pi$, the following equation will have place

$$\tfrac{1}{2}\int_0^\pi f\theta\,d\theta + \Sigma\int_0^\pi f\theta \cos n(\theta - a)\,d\theta = 0 \qquad (2)$$

If, on the contrary, we have $a > 0$ and $\angle \pi$, there will be values of $\theta$ which differ infinitely little from $a$, therefore by making

$$\theta = a + u, \quad d\theta = du,$$

the integral in question will still vanish for finite values of $u$, but not for infinitely small values of this variable, whether positive or negative, with respect to these, we shall have

$$f\theta = fa, \quad \sin \tfrac{1}{2}(\theta - a) = \tfrac{1}{2} u,$$

consequently, the second member of equation (1) becomes

$$fa \int \frac{g du}{g^2 + u^2},$$

when $a$ falls between zero and $\pi$ But, as this integral vanishes for every value of $u$ which is not infinitely small, it may now be extended, without altering the value, to any values whatever of $u$, whether positive or negative, so that we may take it, if we please, from $u = -\infty$ to $u = \infty$, in which case, we shall have

$$\int_{-\infty}^{\infty} \frac{g du}{g^2 + u^2} = \pi,$$

and, finally,

$$\tfrac{1}{2} \int_0^\pi f\theta d\theta + \Sigma \int_0^\pi f\theta \cos n(\theta - a) d\theta = \pi fa \tag{3}$$

This reasoning will also suit the case in which $a$ coincides with one of the two limits zero or $\pi$, but if $a = 0$, then we can assign none but positive values to $u$, and only negative values, when $a = \pi$, in order that in these two cases, the variable $\theta$ which is made equal to $a + u$, may not pass the limits of the integration In this manner, the integral relative to $u$ will be reduced to the half of its value in the former case, or to $\frac{\pi}{2}$, so that if $\beta$ and $\gamma$ be the values of $fa$ in the case of $a = 0$ and $a = \pi$, there will result

$$\left.\begin{aligned} &\tfrac{1}{2} \int_0^\pi f\theta d\theta + \Sigma \int_0^\pi f\theta \cos n\theta d\theta = \tfrac{1}{2}\pi\beta, \\ &\tfrac{1}{2} \int_0^\pi f\theta d\theta + \Sigma (-1)^n \int_0^\pi f\theta \cos n\theta d\theta = \tfrac{1}{2}\pi\gamma \end{aligned}\right\} \tag{4}$$

Now if we make

$$\theta = \frac{\pi x'}{a}, \quad d\theta = \frac{\pi dx'}{a},$$

and also

$$f\left(\frac{\pi x'}{a}\right) = \phi x';$$

and if, the quantity $x$ being positive and less than the constant $a$, there be substituted $-\frac{\pi x}{a}$ in the place of $\alpha$, in equation (2), and $\frac{\pi x}{a}$ for $\alpha$ in equation (3), we shall obtain by observing, that the limits relative to $x'$ are cypher and $a$,

$$\left.\begin{aligned} &\frac{1}{2a}\int_0^a \phi x' dx' + \frac{1}{a}\,\Sigma \int_0^a \phi x' \cos \frac{n\pi\,(x'+x)}{a}\,dx' = 0, \\ &\frac{1}{2a}\int_0^a \phi x' dx' + \frac{1}{a}\,\Sigma \int_0^a \phi x' \cos \frac{n\pi\,(x'-x)}{a} = \phi x, \end{aligned}\right\} \quad (5)$$

and, by subtracting the first equation from the second, there results

$$\frac{2}{a}\,\Sigma\left(\int_0^a \phi x' \sin \frac{n\pi x'}{a} dx'\right) \sin \frac{n\pi x}{a} = \phi x,$$

which was to be demonstrated $(b')$

*326 This formula represents the values of the function $\phi x$, for all values of the variable $x$, which are positive and less than $a$, and even for $x = 0$ and $x = a$, when $\phi x$ vanishes for these extreme values. It is important to observe, that the series indicated by $\Sigma$, will be always convergent, when continued to a considerable number of terms, for when the values of $n$ are very great, the integral relative to $x'$ will become a very small quantity, which will diminish more and more according as $n$ increases, and when $n = \infty$, it will become cypher, as was already shewn by means of partial integration This remark is necessary, as it shews that we are justified in the application which we propose to make of the preceding formula

The different formulæ, by means of which portions of arbitrary functions, either continuous or discontinuous, may be thus exhibited in series of periodical quantities always converging, can be deduced from equations (5), which have been demonstrated above. We shall restrict ourselves in this chapter to give two of these formulæ, which will be useful hereafter; the reader is referred for more extensive developments on this matter to the author's *Memoirs on the Integral Calculus*, inserted in the *Journal de l'Ecole Polytechnique*, where will be found a complete theory of this kind of transformations.

If equations (5) be added together, and also if the first be subtracted from the second, and if $2l$ be substituted for $a$, then $x + l$ and $x' + l$ for $x$ and $x'$, and afterwards $\phi x$ and $\phi x'$ in place of $\phi\,(x + l)$, $\phi\,(x' + l)$, the limits of the integrals relative to $x'$ become $\pm\, l$, and these equations will be replaced by the following

$$\phi x = \frac{1}{2l}\int_{-l}^{l} \phi x' dx' + \frac{1}{l}\Sigma\left(\int_{-l}^{l} \phi x' \cos\frac{n\pi(x'+l)}{2l}\,dx'\right)\cos\frac{n\pi(x+l)}{2l},$$

$$\phi x = \frac{1}{l}\Sigma\left(\int_{-l}^{l} \phi x' \sin\frac{n\pi\,(x'+l)}{2l}\,dx'\right)\sin\frac{n\pi\,(x+l)}{2l}.$$

Let each sum $\Sigma$ be divided into two others, of which the one may contain even, and the other odd numbers. For this purpose, let $\imath$ be any integer number whatever, then by making $n$ successively $= 2\imath$, and $2\imath - 1$, we shall have

$$\cos\frac{2\imath\pi\,(x+l)}{2l} = (-1)^{\imath}\cos\frac{\imath\pi x}{l},$$

$$\sin\frac{2\imath\pi\,(x+l)}{2l} = (-1)^{\imath}\sin\frac{\imath\pi x}{l},$$

$$\cos\frac{(2\imath-1)\,\pi\,(x+l)}{2l} = (-1)^{\imath}\sin\frac{(2\imath-1)\,\pi x}{2l},$$

$$\sin\frac{(2\imath-1)\,\pi\,(x+l)}{2l} = (-1)^{\imath}\cos\frac{(2\imath-1)\pi x}{2l}.$$

and, in the same manner, we can express the sines and cosines comprised under the signs $\int$, consequently, we shall have

$$\left.\begin{aligned}
\phi x &= \frac{1}{2l}\int_{-l}^{l}\phi x' dx' + \frac{1}{l}\Sigma\left(\int_{-l}^{l}\phi x' \cos\frac{i\pi x'}{l}dx'\right)\cos\frac{i\pi x}{l}, \\
&+ \frac{1}{l}\Sigma\left(\int_{-l}^{l}\phi x' \sin\frac{(2i-1)\pi x'}{2l}dx'\right)\sin\frac{(2i-1)\pi x}{2l}, \\
\phi x &= \frac{1}{l}\Sigma\left(\int_{-l}^{l}\phi x' \sin\frac{i\pi x'}{l}dx'\right)\sin\frac{i\pi x}{l} \\
&+ \frac{1}{l}\Sigma\left(\int_{-l}^{l}\phi x' \cos\frac{(2i-1)\pi x'}{2l}dx'\right)\cos\frac{(2i-1)\pi x}{2l},
\end{aligned}\right\} \quad (6)$$

in which the sums $\Sigma$ extend to all values of $i$, from $i = 1$ to $i = \infty$. These equations will obtain for all values of $x$ that are comprised between the limits $\pm l$.

This being established, if the function $\phi x$ is such that $\phi(-x) = -\phi x$, there will result

$$\int_{-l}^{l}\phi x' dx' = 0, \quad \int_{-l}^{l}\phi x' \cos\frac{i\pi x'}{l}dx' = 0,$$

$$\int_{-l}^{l}\phi x' \cos\frac{(2i-1)\pi x'}{2l}dx' = 0,$$

and, moreover,

$$\int_{-l}^{l}\phi x' \sin\frac{i\pi x'}{l}dx' = 2\int_{0}^{l}\phi x' \sin\frac{i\pi x'}{l}dx',$$

$$\int_{-l}^{l}\phi x' \sin\frac{(2i-1)\pi x'}{2l}dx' = 2\int_{0}^{l}\phi x' \sin\frac{(2i-1)\pi x'}{2l}dx';$$

by means of which the second equation (6) will coincide with formula (a), by changing in it, $a$ into $l$, and the first will be reduced to

$$\phi x = \frac{2}{l}\Sigma\left(\int_{-l}^{l}\phi x' \sin\frac{(2i-1)\pi x'}{2l}dx'\right)\sin\frac{(2i-1)\pi x}{2l}. \quad (7)$$

If, on the contrary, the function $\phi x$ is such, that $\phi(-x) = \phi x$, we shall have

$$\int_{-l}^{l}\phi x' \sin\frac{(2i-1)\pi x'}{2l}dx' = 0, \quad \int_{-l}^{l}\phi x' \sin\frac{i\pi x'}{l}dx' = 0,$$

and the other integrals may, by doubling the results, be extended from $x=0$ to $x=l$ If $l-x$ be substituted for $x$, and $\phi x$ for $\phi(l-x)$, the second equation (6) will coincide with equation (7) The first equation (6) will become

$$\phi x=\frac{1}{l}\int_0^l \phi x' dx' + \frac{2}{l}\Sigma\left(\int_0^l \phi x' \cos\frac{i\pi x'}{l}dx'\right)\cos\frac{i\pi x}{l}. \quad (8)$$

These formulæ (7) and (8) will represent the values of $\phi x$, from $x=0$ to $x=l$, those which can be deduced from them, by differentiating with respect to $x$, will express the values of $\frac{d\phi x}{dx}$, within the same interval Formula (7) supposes that $\phi x=o$ for $x=0$, and $\frac{d\phi x}{dx}=0$ when $x=l$, in formula (8) $\frac{d\phi x}{dx}=0$ for $x=0$ and $x=l$ When these conditions are not satisfied, neither these formulæ nor their differentials have place for the extreme values of $x(c')$

327. Conversely, formulæ of this kind make known the sums of numerous periodical series, which have been obtained by different means Thus, for example, in order to infer from them the sum of the series made use of in No 324, let $-a$ be substituted in place of $a$ in equation (2), and then let it and equation (3) be added together, there will result

$$\int_0^\pi f\theta d\theta + 2\,\Sigma\left(\int_0^\pi f\theta\cos n\theta d\theta\right)\cos na=\pi fa$$

If then we assume $f\theta=\theta$, we shall have

$$\int_0^\pi f\theta\cos n\theta d\theta=\frac{\cos n\pi-1}{n^2},$$

which is equal to cypher for all even numbers, and to $-\frac{2}{(2i-1)^2}$ for $n=2i-1$ Hence the preceding equation becomes

$$\Sigma.\frac{\cos(2i-1)a}{(2i-1)^2}=\frac{\pi}{8}(\pi-2a),$$

in which the sum $\Sigma$ extends to all values of the whole number $i$, from $i=1$ to $i=\infty$.

Multiplying by $da$, and integrating, we obtain

$$\Sigma\frac{\sin(2i-1)a}{(2i-1)^3}=\frac{\pi}{8}(\pi-a)a.$$

It is not necessary to add any constant arbitrary, for the two members of this equation vanish for $a=0$, and for $a=\pi$, so that this equation obtains for all values of $a$, from $a=0$ to $a=\pi$ inclusively, if we make $a=\frac{1}{2}\pi+\omega$, we shall have

$$\sin(2i-1)a=-(-1)^i\cos(2i-1)\omega,$$

and, consequently,

$$\Sigma\frac{(-1)^i\cos(2i-1)\omega}{(2i-1)^3}=\frac{\pi}{8}(\omega^2-\tfrac{1}{4}\pi^2),$$

from $\omega=-\frac{1}{2}\pi$ to $\omega=\frac{1}{2}\pi$. If this be multiplied by $d\omega$ and integrated again, there results

$$\Sigma\frac{(-1)^i\sin(2i-1)\omega}{(2i-1)^4}=\frac{\pi\omega^3}{24}-\frac{\pi^3\omega}{32};$$

which was required to be obtained.

328. If $2a$ be substituted in place of $a$, and then $x+a$, $x'+a$ in place of $x$ and $x'$ in the second equation (5), we shall obtain, by making $\phi(a+x)=\mathrm{F}x$,

$$\mathrm{F}x=\frac{1}{4a}\int_{-a}^{a}\mathrm{F}x'dx'+\frac{1}{2a}\Sigma\int_{-a}^{a}\mathrm{F}x'\cos\frac{n\pi(x'-x)}{2a}dx',$$

for all values of $x$ comprised between $\pm a$

By making

$$\frac{\pi}{2a}=\varepsilon,\quad \frac{nx}{2a}=n\varepsilon=u,$$

this equation may be written as follows,

$$\mathrm{F}x=\frac{\varepsilon}{2\pi}\int_{-a}^{a}\mathrm{F}x'dx'+\frac{1}{\pi}\Sigma\left[\int_{-a}^{a}\mathrm{F}x'\cos u(x'-x)\,dx\right]\varepsilon,$$

in which we suppose that $u$ is a multiple of $\varepsilon$, and that the sum $\Sigma$ extends to all values of $u$, from $u = \varepsilon$ to $u = \infty$. But if the constant $a$ becomes infinite, $\varepsilon$ the difference of the consecutive values of $u$ will become infinitely small, and the sum $\Sigma$ will be changed into an integral which should be taken from $u = \varepsilon$, or $u = 0$, to $u = \infty$. Consequently, making $a = \infty$, $\varepsilon = du$, and substituting the sign $\int$ for $\Sigma$, we shall obtain, by suppressing the first term of the preceding formula,

$$\mathrm{F}x = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} \mathrm{F}x' \cos u\,(x' - x)\,dx'\right] du$$

This important formula, which extends to all real values of the variable $x$, positive or negative, and is applicable like the preceding, from which it has been deduced, to any function $\mathrm{F}x$, whether continuous or discontinuous, was first given by Fourier

## CHAPTER IV

### PRINCIPLE OF VIRTUAL VELOCITIES

329 In the simplest cases of the equilibrium of machines, the power and resistance are reciprocally proportional to the spaces which their points of application would simultaneously describe, if the equilibrium was destroyed But, since in consequence of the connexion between the points of application, the spaces which these points would describe, if they were entirely free, is different from the spaces that they actually describe, in order that this relation may always obtain, the infinitely small spaces which are described in the first instant, should be replaced by their projections on the directions of the forces. This relation has been a long time recognized in the case of simple machines, John Bernouilli afterwards extended it by induction to any system whatever of material points solicited by given forces, and, under the denomination of the *principle of virtual velocities*, it is now become the general principle of equilibrium We propose to demonstrate it here in all its generality, after having first verified it, in the following examples

1st Let (fig 79) A, A′, A″, . . be a series of pulleys contained in the same block, and constituting a fixed system, and B, B′, B″, . another series of pulleys also contained in one block, and constituting a moveable system Let us suppose a thread attached to the inferior extremity of the fixed system, to be successively rolled round all the pulleys by passing alternately from one system to the other. To the free extremity of this thread let a weight P be suspended, which may constitute

an equilibrium with a weight R suspended at the inferior pulley of the moveable system The tension of the thread will be the same throughout its entire length, and equal to the weight P, moreover, if the diameters of the pulleys be very small, relatively to the distance which separates the two systems, the strings which pass alternately from the one to the other will be sensibly parallel and vertical, consequently, the force which sustains the weight R will be equal to the sum of their tensions, or to $n$ times the weight P, $n$ being the number of these strings, hence, in the case of equilibrium, we shall have

$$R = nP$$

Now, if the equilibrium is destroyed, and the weight R ascends or descends by a quantity $a$, all the strings which terminate in the moveable system will be shortened or lengthened by this same quantity. As the entire length of the thread remains the same, the part to which the weight P is attached will be lengthened or shortened by $n$ times this quantity $a$, hence if $\beta$ denotes the space through which the weight P ascends or descends, we shall have $\beta = na$, and, consequently,

$$Ra = P\beta,$$

which contains the principle of virtual velocities already adverted to

2ndly. ABC (fig. 80) represents the *wheel*, and A′B′C′ the intersection of the vertical plane of this wheel and of the surface of the cylinder, in the machine called the axle in the wheel, O is the common centre of these two circumferences, and AOC, A′OC′ are their horizontal diameters. A thread coiled round the wheel is attached to one of its points, another thread attached to one of the points of the cylinder, is in the same manner coiled round its surface. A weight P is suspended from the first thread, and a weight R from the second, these two weights tend to turn the machine in opposite direc-

tions, and are supposed to be in equilibrio. This being agreed on, if to the point C′ there be applied two vertical forces R′ and R″, equal, and acting in opposite directions, the equilibrium will not be disturbed, if, moreover, these forces are respectively equal to R, the force R″, and the weight R, will constitute an equilibrium, for there is no reason why their simultaneous action should cause the machine to turn in one direction rather than in the opposite, there must therefore be likewise an equilibrium between the weight P and the force R′, which act perpendicularly to AOC′ at the extremity of this lever, of which the point O is the fulcrum. Hence, $r$ denoting AO the radius of wheel, and $r'$ OC′ the radius of the cylinder, the equation of equilibrium will be (because $R'=R$)

$$Pr = Rr',$$

Now, if the equilibrium is destroyed, and if the weight R rises or falls by a quantity $a$, while the weight P falls or rises by a quantity $\beta$, it is evident, from the nature of the machine, that we shall have $\beta r' = ar$, hence we infer

$$P\beta = Ra,$$

conformably to the principle which it was proposed to verify.

3rdly. Let a vertical *screw* be loaded at its upper extremity by a weight R, and let a horizontal wheel, having its centre in the axis of this screw, be adapted to its inferior extremity, let then a thread be wrapped on this wheel, and fixed to its circumference by one extremity, while at its other extremity a horizontal force F acting in the direction of a tangent to the wheel, constitutes an equilibrium with the weight R. By placing a fixed vertical pulley in the direction of this tangent, we can give the thread a vertical direction, and thus replace F by P a weight equal to this force, and attached to the free extremity of the vertical part of the thread. Denoting the interval between the threads of the screw by $h$, and by $c$ the cir-

cumference of the wheel, we shall have, by the known condition of equilibrium in this machine,

$$\text{P}c = \text{R}h.$$

The two weights R and P tend to make the screw to turn in opposite directions, if the equilibrium is destroyed, one of the weights will rise while the other falls, and if the weight R rises or falls by a quantity equal to $h$, the interval between the threads of the screw, the weight P will fall or rise through a space equal to $c$ the circumference of the wheel, hence it follows, that, in general, denoting the spaces simultaneously traversed by the two weights R and P, by $\alpha$ and $\beta$, we shall have $\alpha c = \beta h$, and, consequently,

$$\text{P}\beta = \text{R}\alpha,$$

conformably to the principle in question

4thly. Let us consider the case of two weights, P and R, placed on two inclined planes, and connected together by a thread passing over a fixed pulley situated above the two planes, which are supposed to rest against each other. Figure 81 represents a vertical section of this system, AC is the length of the plane on which the weight R is placed, BC that of the plane which supports the weight P, AB is a horizontal line, and CD a vertical line, equal to the common height of the two planes Let

$$\text{AC} = a, \quad \text{BC} = b, \quad \text{CD} = h,$$

the component of R in the direction of CA will be $\text{R}\,\frac{h}{a}$, and that of P in the direction CB will be $\text{P}\,\frac{h}{b}$ In order that there may be an equilibrium, these two components should be equal, consequently, we must have

$$\text{P}a = \text{R}b.$$

If the equilibrium is destroyed, and the weight R slides on

the plane CB by a quantity equal to $\gamma$, the weight P will slide by the same quantity, but in a contrary direction, on the plane AC, denoting by $\alpha$ the vertical height by which the weight R is elevated or depressed, and by $\beta$, that by which the weight P is depressed or elevated, it is easy to perceive, that we shall have

$$\alpha = \frac{\gamma h}{a}, \quad \beta = \frac{\gamma h}{b}$$

and, consequently,

$$\text{P}\beta = \text{R}\alpha,$$

as in the preceding examples, but in this case $\alpha$ and $\beta$ are the vertical projections of the spaces simultaneously described by the weights R and P, while, in the preceding case, $\alpha$ and $\beta$ were those spaces themselves.

330 It appears from No 49, that two forces which constitute an equilibrium, through the intervention of any lever whatever, are in the inverse ratio of the infinitely small spaces, described in the same time by their points of application, projected on the respective directions of these forces.

This relation, which exists in the case of the lever between two forces in equilibrio, is true also when two forces, applied to any other machine, are in equilibrio Thus, if we denote the power and resistance which are in equilibrio, by the intervention of any machine whatever, by P and R, when an infinitely small motion is impressed on this machine, we shall always have

$$\text{P}\beta = \text{R}\alpha,$$

$\beta$ and $\alpha$ being the projections on the directions of these forces, of the spaces which would be simultaneously described by their points of application, it is necessary besides to take notice that one of these projections must be taken in the direction of the corresponding force, and the other on its production, as is the case in the lever

It is only requisite in practice, that the motion impressed

on the machine be very small By measuring the lengths of the projections $\beta$ and $a$, the ratio of the power to the resistance may be immediately obtained, without knowing anything of the particular construction of the machine

331 This mode of expressing the relations between the power and resistance when they are in equilibrio, is not only true in the case of every machine, but it may be also extended to any number of forces whatever in equilibrio Therefore, let M, M′, M″, &c, (fig 82), be a system of material points connected together in any manner whatever, let us suppose that the forces P, P′, P″, &c, act on these points in the directions MA, M′A′, M″A″, &c, if these points be made to undergo infinitely small displacements compatible with the conditions of the system, and if in this way they are transferred to N, N′, N″, &c, and if finally N, N′, N″, &c, be projected on the lines MA, M′A′, M″A″, &c, to $a, a', a''$, &c, so that

$$\text{M}a = p, \quad \text{M}'a' = p', \quad \text{M}''\text{A}'' = p'', \text{ \&c },$$

then these projections $p, p', p''$, &c., being considered as positive or negative, according as they fall on the directions of the corresponding forces, or on their productions, we shall have, when the equilibrium obtains,

$$\text{P}p + \text{P}'p' + \text{P}''p'' + \text{\&c.} = 0,$$

and conversely, there will be an equilibrium, when this equation subsists for all displacements compatible with the conditions of the system

The infinitely small lines MN, M′N′, M″N″, &c, are termed the *virtual velocities* of the points M, M′, M″, &c, because they are the spaces that would be simultaneously described by the points of the system, in the very first instant in which the equilibrium is destroyed.

It should be observed, that the principle of virtual velocities contained in the formula stated above, furnishes merely the *conditions* of equilibrium which may be expressed by

equations, but not those which are relative to the directions of certain forces, and to the extent within which they should meet a fixed plane (No 266) The motions compatible with the conditions of the system, which furnish the equations of equilibrium, are such, that motions directly contrary are equally possible For example, if a material point is placed on a fixed plane, the motion will be possible in every direction taken *in this plane*, and also in the corresponding opposite direction, but perpendicularly to this plane, it can only have place in one sole direction Now, the consideration of the motions performed in the plane, will furnish us with conditions of equilibrium which may be expressed by equations, and the consideration of the perpendicular motion will only determine the direction of the normal force, which ought to be contrary to that of the possible motion In the general statement of the principle of virtual velocities, it is implicitly supposed, that each of the motions compatible with the conditions of the system, and the directly contrary motion, are equally possible, and if the preceding equation be successively applied to these two motions, all the quantities $p, p', p''$, &c., will change their signs, and there will only result but one equation of equilibrium

If the force P is the resultant of several given forces Q, Q′, Q″, &c, and if $q, q', q''$, &c., denote the projections of MN on their directions, we shall have (No 34),

$$Pp = Qq + Q'q' + Q''q'' + \&c.,$$

so that we can replace in the preceding equation, the term P$p$ relative to the force P, by this sum of terms of the same nature, which refer to its components, and we may make the same substitution in case of the forces P, P′, P″, &c, if they are also the resultants of several other forces

It appears from No 39, that in the case of an isolated point, the principle of virtual velocities is a consequence of this last equation, whether the point in question be entirely free, or whether it be constrained to remain on a given curve

or surface. We proceed now to demonstrate this general principle in the case of any system whatever of material points M, M′, M″, &c.

332 Let us suppose that these points are connected together, either by inflexible rods, or by flexible threads, of which the first must be firmly attached to these points, while the second traverse them like moveable rings In this last case, these points or rings are free to slide along the threads that traverse them, which, therefore, must be perfectly flexible It is evident, that, after the given forces P, P′, P″, &c , are applied to the points M, M′, M″, &c , and the equilibrium is established, the threads which connect these points two by two, must each of them experience a particular tension, that is to say, each of these threads will be drawn at its two extremities by equal and opposite forces, acting in the directions of its productions, as has been already stated in the case of the funicular polygon (No 285) The intensity of this force will measure the unknown tension which this thread experiences. Any thread that is not stretched, will contribute nothing to the equilibrium, and, therefore, need not be taken into account

The tension may vary from one thread to another, but in the case of two threads, that are the production, the one of the other, through a ring, the tension is the same, in these two parts of the same thread, which must necessarily experience the same tension throughout its entire length (No 289) Thus, for example, if M is a ring traversed by the thread M′MM″, the tension of MM′ is the same as that of MM″

When several threads cross each other in passing through the same ring, the tension is the same in the two parts of each thread, but it may vary from one thread to another If, therefore, beside the thread M′MM″, the thread M‴MM$^{IV}$ likewise passes through the ring M, the tension will be the same in the two parts MM‴ and MM$^{IV}$ of this last thread, and, in general, it will be different from that of the two parts MM′ and

MM″ of the first thread And, if another thread, such as MM^v should terminate at the same ring M, to which it is firmly attached, its particular tension will, generally speaking, be different from the tensions of the other threads that terminate at the same point M

It is likewise to be observed, that if M′ is a ring as well as M, and if the thread M″MM′, after having traversed the ring M, passes likewise through the ring M′, and terminates at the point M‴, the tension will be the same in the three threads M″M, MM′, M′M‴, for then these three threads constitute only one M″MM′M‴ In general, when a thread is distributed into several parts by moveable rings, the tension is the same in all its parts.

With respect to inflexible rods, when the equilibrium obtains, they are drawn or pushed in the direction of their length, by equal and contrary forces, acting at their extremities The common intensity of these two forces, in the case of each rod, is the measure of the tension or contraction that it undergoes. If there be any rods in the system which are neither stretched or contracted, they do not contribute to the equilibrium, and, being useless, they may be suppressed Hence, in what follows it is assumed, that the rods or threads which connect the different points of the system, are stretched or contracted in the direction of their length by unknown forces

The advantage of the principle of virtual velocities, consists in this, that it furnishes the equation of equilibrium in each particular case, without requiring us to compute these interior forces, but as the demonstration which we propose to give of this principle, is founded on the consideration of these forces whose magnitude is unknown, the following notation may be advantageously employed to represent them.

The tension or contraction of the flexible or inflexible thread which connects any two points M and M′ of the system, will be denoted by $[m, m']$ In this manner $[m, m'']$, $[m', m'']$,

&c., will represent the tensions or contractions of threads which connect M and M″, M′ and M″, &c.

333 We must likewise consider the infinitely small variations which the distances of the points M, M′, M″, &c., taken two by two, undergo, when only one of these points changes its position, and also when they are both displaced simultaneously. Denoting the distance between any two points M and M′, M and M″, M′ and M″, &c., by $(m, m')$, $(m, m'')$, $(m' m'')$, &c., if we employ the characteristic $\delta_1$ to denote the variations of these distances relatively to the displacement of the point M, the characteristic $\delta_1'$ to denote those which obtain when it is the point M′ that is displaced, the characteristic $\delta_1''$ to indicate the variations arising from the displacement of M″, and so on, and finally, if the characteristic $\delta$ be reserved to denote the variation of the distance of the two points, resulting from their mutual displacements, we shall have

$$\delta\,(m, m') = \text{MM}' - \text{NN}',$$
$$\delta_1\,(m, m') = \text{MM}' - \text{NM}',$$
$$\delta_1'\,(m, m') = \text{MM}' - \text{MN}',$$

for M has been supposed to have been transferred from M to N, and M′ from M′ to N′.

It is of consequence to observe, that the entire variation indicated by $\delta$, is equal to the sum of the partial variations indicated by $\delta_1$ and $\delta_1'$, so that for any two points whatever, we have

$$\delta\,(m, m') = \delta_1\,(m, m') + \delta_1'\,(m, m'),$$

this equation obtains, because the displacements of M and M′ are supposed to be infinitely small, and it is only true on this hypothesis. In fact, $(mm')$ is a function of the coordinates of these two points, and when M and M′ are transferred to N and N′, these variables experience infinitely small increments, positive or negative; now, if the powers of these increments higher than the first, be neglected, it is evident that the entire

increment of any function whatever of these coordinates, is equal to the sum of the partial increments arising from the variation of each coordinate separately; consequently, the entire variation of $(m, m')$, indicated by the characteristic $\delta$, must be equal to the sum of its partial variations which refer to $\delta_1$ and $\delta_1'(a)$.

334 What precedes being admitted, let any point M, to which the given force P is applied, be considered. This point is connected with the others by the threads MM′, MM″, &c ; it is, therefore, drawn or pushed in the direction of each of these threads by a force equal to the contraction or tension which this thread experiences; so that besides the given force P, the point M is likewise subject to the action of as many other forces as there are threads terminating at this point. These interior forces being thus taken into account, we may abstract altogether from the consideration of the threads which connect M with the other points of the system, and regard it as an isolated point, which is in equilibrio, in consequence of the action of the force P, and of the forces $[m, m']$, $[m, m'']$, &c If M is a fixed point, no equation of condition will result from them, but if it is entirely free, or if it is only constrained to remain on a given curve or surface, we shall have between these forces, the equation of virtual velocities, which has been already demonstrated to obtain in the case of a material point in equilibrio

In order to form this equation, let a point be taken infinitely near to M, and appertaining to the curve or surface on which M is constrained to exist, if it is not entirely free If $p$, $t$, $t'$, $t''$, &c , be the projections of MN on the directions of the forces P, $[m, m']$, $[m, m'']$, $[m, m''']$, &c.; by No. 39 we shall have

$$Pp + [m, m']\ t + [m, m'']\ t' + [m, m''']\ t'' + \&c = 0.$$

But, as the line MN is by hypothesis infinitely small, it is easy to perceive that its projection on MM′, is q p. the dif-

ference of the two distances MM′ and NM′, for if from the point N (fig. 83) the perpendicular NH be let fall on MM′, the line MH will be this projection, and we shall have

$$\mathrm{MH} = \mathrm{MM}' - \mathrm{HM}'$$

We have also, by neglecting infinitely small quantities of the second and higher orders,

$$\mathrm{HM}' = \sqrt{(\mathrm{NM}')^2 - (\mathrm{NH})^2} = \mathrm{NM}',$$

hence, therefore,

$$\mathrm{MH} = \mathrm{MM}' - \mathrm{NM}'$$

According to the notation explained above, this equation is

$$t = \delta_1 (m, m'),$$

and in the same manner we shall have

$$t' = \delta_1 (m, m''), \quad t'' = \delta_1 (m, m'''), \text{ \&c },$$

consequently, the equation of equilibrium will become

$$\mathrm{P}p + [m, m'] \, \delta_1 (m, m') + [m, m''] \, \delta_1 (m, m'') + [m, m'''] \, \delta_1 (m, m''') + \text{\&c.} = 0$$

In the case of each of the other points M′, M″, M‴, &c, of the system, similar equations may be obtained, these equations will be

$$\mathrm{P}'p' + [m', m] . \delta_1' (m', m) + [m', m''] \, \delta_1' (m', m'') + [m', m'''] \, \delta_1' (m', m''') + \text{\&c.} = 0,$$

$$\mathrm{P}''p'' + [m'', m] \, \delta_1'' (m'', m) + [m'', m'] \, \delta_1'' (m'', m') + [m'', m'''] \, \delta_1'' (m'', m''') + \text{\&c.} = 0,$$

$$\mathrm{P}'''p''' + [m''', m] \, \delta_1''' (m''', m) + [m''', m'] \, \delta_1''' (m'', m') + [m''', m''] \, \delta_1''' (m''', m'') \text{ \&c } = 0,$$

$p'$, $p''$, $p'''$, &c, being the respective virtual velocities of M′, M″, M‴, &c., projected on the directions of the given forces P′, P″, P‴, &c., which act on these material points.

We shall obtain, by adding these equations together, and by observing, that $[m, m']$ and $(m, m')$ are the same thing as $[m', m]$ and $(m', m)$, and that the same is likewise true for the other points $m''$, $m'''$, &c ,

$$\left.\begin{array}{l} \text{P}p + \text{P}'p' + \text{P}''p'' + \text{P}'''p''' + \&c. \\ + [m, m'] \ \delta(m, m') + [m, m''] \ \delta(m, m'') \\ \qquad + [m, m'''] \ \delta(m, m''') + \&c., \\ + [m', m''] \ \delta(m', m'') + [m' m'''] \ \delta(m', m''') \\ + [m'', m'''] . \delta(m'', m''') + \&c , \\ + \&c = 0. \end{array}\right\} \text{(a)}$$

in which the total variation of each distance is substituted in place of the sum of its partial variations

335. In what precedes the displacements MN, M′N′, M″N″, &c , (fig 82), are supposed to be independent of each other , and in equation (a) it is only implied that these points have remained on the given surfaces or curves, on which they are obliged to exist, but, if we suppose, in addition, that, in consequence of these displacements, the points of the system which are connected by a rod, or stretched thread, have preserved the same respective distances, we shall have

$$\delta(m, m') = 0, \quad \delta(m, m'') = 0, \quad \delta(m', m'') = 0, \ \&c ,$$

and equation (a) will be reduced to the following

$$\text{P}p + \text{P}'p' + \text{P}''p'' + \text{P}'''p''' + \&c = 0, \qquad \text{(b)}$$

which is precisely that of the principle of virtual velocities (No 331)

If in the displacements of the points M, M′, M″, &c , those which are rings slide along the threads which traverse them, equation (b) will still obtain, provided that the total length of these threads does not vary Let us, for example, suppose that M is a ring which slides along the thread M′MM″, then we shall have no longer separately $\delta(m, m')$ and $\delta(m, m'') = 0$, but we shall always have

$$\delta(m, m') + \delta(m, m'') = 0,$$

since the entire length of the thread continues constant. But, in this case, as the tensions $[m, m']$ and $[m, m'']$ of the two parts of this thread are equal, the terms which contain these tensions in equation (a) may be written thus,

$$[m, m'] \quad [\delta(m, m') + \delta(m, m'')],$$

and, consequently, they destroy one another

In general, when a flexible thread passes through any number whatever of rings, the equal tensions of its different parts will disappear from equation (a), as often as the entire length of this thread does not vary. We can, therefore, finally infer,

1st That the equation resulting from the principle of virtual velocities obtains in the case of all infinitely small motions which can be impressed on a solid body, whether free or constrained by fixed obstacles, for in all these motions, the respective distances of the points of this body are invariable.

2ndly. That this equation obtains also in the case of all the infinitely small motions which a system of points or rings connected together by flexible threads can acquire, provided that these threads remain straight or stretched When this condition is not satisfied, all the tensions would not disappear in equation (a), and, consequently, equation (b) no longer obtains

336 It is necessary likewise to demonstrate that, conversely, when equation (b) obtains for all the infinitely small motions which can be impressed on a system of points M, M′, M″, &c , the given forces P, P′, P″, &c , are in equilibrio, as has been already stated, (No 331).

If for one instant it be supposed that the equilibrium does not obtain, the points M, M′, M″, &c , or at least some of them, will begin to move, and in the first instant they will describe simultaneously right lines such as MN, M′N′, M″N″, &c., therefore, all these points may be reduced to a state of rest by impressing on them suitable forces, acting along the productions of these lines,

in a direction opposite to that of the motions produced, consequently, if we denote these unknown forces by R, R′, R″, &c., there will be an equilibrium between the forces P, P′, P″, &c, and R, R′, R″, &c, so that if $r$, $r'$, $r''$, &c, denote the virtual velocities projected on the directions of these new forces R, R′, R″, &c., we shall have, by the principle of virtual velocities that has been demonstrated,

$$Pp + P'p' + P''p'' + \&c + Rr + R'r' + R''r'' + \&c = 0,$$

or simply

$$Rr + R'r' + R''r'' + \&c. = 0, \qquad (c)$$

in virtue of equation (b), which is supposed to have place

As equation (c) obtains for all infinitely small motions compatible with the conditions of the system of points M, M′, M″, &c, we may select for their virtual velocities, MN, M′N′, M″N″, &c, the spaces actually described in the same instant; but as these lines are laid off on the productions of the directions of R, R′, R″, &c, it follows that all the projections $r$, $r'$, $r''$, &c. will be negative, (No 331), and, abstracting from the sign, equal to these very lines MN, M′N′, M″N″, &c. In this case, then, all the terms of equation (c) have the same sign, and consequently, their sum cannot vanish, unless that each term in particular is equal to cypher, hence we shall have

$$R \cdot MN = 0, \quad R' \cdot M'N' = 0, \quad R'' \cdot M''N'' = 0, \ \&c$$

Now, if the product R . MN = 0, we must have either R = 0, or MN = 0; in both which cases, it follows that the point M cannot move, it is the same with respect to all the other points; consequently the entire system is in equilibrio, which it was proposed to demonstrate.

337 In the case of fluids, it will be shewn hereafter, by means of their fundamental property, that the principle of virtual velocities obtains likewise in the equilibrium of a system of forces, the actions of which are transmitted by the intervention of a fluid contained in a canal or vessel of any form

whatever. So that this principle of equilibrium, being applicable in every case, will have all the required generality; for when material points are detached from each other, the only way in which the action of these forces can be transmitted from one point to another, is either by means of inflexible rods, of stretched threads, or of fluids contained in canals; and moreover, if some of these points are immoveable, others perfectly free, and others constrained to exist on given curves or surfaces, the system of material points thus constituted will be the most general which there will be any occasion to consider. Nevertheless, it will be perhaps not unnecessary to give another demonstration of the same principle, for which we are indebted to Lagrange, and which is grounded on notions that are more elementary than the preceding; in fact, it is founded on the possibility of our being able to replace all the forces applied to any system whatever of material points, by one weight acting in a manner which we now proceed to explain.

338. If a point M (fig. 84) is solicited by a force P acting in the direction of the line MA, we may, in the first place, suppose that this force is applied to the point A, and that it acts by means of a cord MA attached to this point M. We may then substitute for this cord, a thread which is alternately rolled round a fixed and moveable system of pulleys, and is attached by one of its two extremities to one or other of these two systems; that which is fixed being supposed to refer to the point A, and that which is moveable to the point M. If at the free extremity of the thread, a weight K is suspended vertically, its tension throughout its entire length will be equal to K. If the dimensions of the pulleys be supposed to be infinitely small, the tensions of all the parts of this thread, which terminate at the moveable system, will have the same direction, and, denoting their number by $i$, their resultant will be equal to $iK$, which will act at the point M in the direction of

MA, consequently, if $iK = P$, the action of the force P may be replaced by that of the weight K

The same will be the case with respect to the other forces $P'$, $P''$, &c, applied to the points $M'$, $M''$, &c, and acting in the directions $M'A'$, $M''A''$, &c, each of them can be replaced by a weight equal to a submultiple of its intensity, and acting in the manner that has been explained with respect to the force P Moreover, it is easy to perceive that, as is represented in figure 85, we can always make the same thread to pass successively over all the fixed systems at $A$, $A'$, $A''$, &c., and over all the moveable systems attached to the points $M$, $M'$, $M''$, &c Hence if we make

$$i'K = P, \quad i'K = P', \quad i''K = P'', \text{ \&c} \qquad (d)$$

$i$, $i'$, $i''$, &c, being whole numbers, we can, by suspending the weight K at the free extremity of this thread, replace the system of given forces $P$, $P'$, $P''$, &c, by this weight, the action of which will be transmitted to the points $M$, $M'$, $M''$, &c, through the intervention of this thread, and of the fixed and moveable systems Indeed, it is implied in equations (d), that the forces $P$, $P'$, $P''$, &c are commensurable, but this hypothesis is always admissible, because their common measure K may be a weight as small as we please, and even, if necessary, infinitely small(e)

339 Let us now conceive that there is impressed on the points $M$, $M'$, $M''$, &c, a motion which, as also the directly contrary motion, may be consistent with the conditions of the system, after an infinitely small portion of time, let $N$, $N'$, $N''$, &c, be their positions, and let, as before, $p$, $p'$, $p''$, &c, denote the projections of $MN$, $M'N'$, $M'N''$, &c. on the directions of $P$, $P'$, $P''$, &c, or on their productions. The point N being projected at $a$ on the right line MA, each of the cords which issue from A to M will be shortened by a quantity $AM - AN$, which, if infinitely small increments of the second order be neglected, may be considered as equal to $Ma$, on the other hand, this cord will be lengthened by a quantity equal to $Ma$,

if the point $a$ falls on the production of AM; hence it follows, that in consequence of the displacement of M, the weight K will descend in the first case, and ascend in the second, by a quantity equal to the product of M$a$ and $\iota$, which implies, agreeably to what was stated relative to the sign of $p$, (No 331), that, in consequence of this sole displacement, the positive or negative variation of its vertical height will be expressed by $\iota p$ The same will be the case with respect to all the other points M$'$, M$''$, &c, consequently, if $\zeta$ be an infinitely small quantity that represents, according as it is positive or negative, the entire quantity by which the weight K descends or ascends in consequence of the simultaneous displacements of all the points of the system, we shall have

$$\zeta = \iota p + \iota' p' + \iota'' p'' + \&c$$

Now, when the weight K tends to descend, as it is the only force which acts on the system, it is evident that nothing will prevent it from producing the motion in question, if this value of $\zeta$ is positive, and that, if it is negative, nothing will prevent the weight K from producing the direct contrary motion, which is supposed to be equally possible, and for which the sign of $\zeta$ must be changed It is therefore necessary, in order that the equilibrium may obtain, that $\zeta$ should be equal to zero Conversely, as the weight K cannot produce any motion whatever, without descending by an infinitely small quantity in the first instant, it follows that it will not produce any, and that if $\zeta = 0$, the equilibrium will obtain for all the infinitely small displacements of the points M, M$'$, M$''$, &c., which are compatible with the conditions of the system

Now, if K be multiplied by the equation

$$\iota p + \iota' p' + \iota'' p'' + \&c = 0,$$

which is necessary, and suffices to insure the equilibrium, it will be changed, by having regard to equations (d), into equation (b), which is that of the principle of virtual velocities, it was proposed to obtain

340. In this demonstration it is not assumed, as in the former, that the principle has been previously demonstrated in the case of an isolated material point If the system be reduced to one sole point M, to which the forces P, P′, P″, &c, given in magnitude and position, are supposed to be applied, we thould substitute for their simultaneous action that of one sole weight K, as in No 338; and, in the case of the equilibrium of these forces, the principle of virtual velocities may be deduced from this substitution by the same mode of reasoning as in the case just considered Now, this principle will at once furnish the equations of equilibrium of the point M, constrained to remain on a curve or a surface, or which may be entirely free, (No 39) In this last case, if one of the forces be considered as being equal and contrary to the resultant of all the others, the rules of their composition and resolution, and also the theorem of the parallelogram of forces may be deduced from it. By applying this principle to the equilibrium of three parallel forces, one of which is consequently equal and contrary to the resultant of the two others, the rules of the composition and resolution of parallel forces may, in like manner, be deduced

We may also, without difficulty, infer from the general principle of virtual velocities, the equations of equilibrium of a solid body entirely free, which have been already obtained in another manner in No. 260.

In fact, we can, in the first place, suppose that all the points of this body describe right lines mutually equal to each other, and parallel to one of the axes of the coordinates Denoting the length of these lines by $h$, and by $a, a', a''$, &c, the angles which their common direction makes with the directions of the given forces, we shall have

$$p = h\cos a, \quad p' = h\cos a', \quad p'' = h\cos a'', \text{ \&c},$$

for the virtual velocities of M, M′, M″, &c, the points of the solid body, projected on the directions of the forces P, P′, P″,

&c., which are applied to these points, therefore, if these values be substituted in equation (b), we shall obtain, by suppressing the factor $h$, which is common to all the terms, the equation of equilibrium

$$P\cos a + P'\cos a' + P''\cos a'' + \&c. = 0$$

If the motions of the body parallel to the two other axes of coordinates be respectively considered, two other equations of equilibrium similar to this may be obtained in the same manner.

We may also cause the body to turn about one of the axes of the coordinates. In order to obtain the equation which corresponds to this motion, let the coordinates of the points $M, M', M''$, &c., and the angles which the directions of the forces $P, P', P''$, &c., make with those of these coordinates, be denoted by the same letters as in No. 260. If the rotation takes place about the axis of $z$, each of these points will describe an arc of a circle parallel to the plane of the axes of $x$ and $y$, the radius of which will be the perpendicular let fall from this point on this axis. Moreover, as the body is supposed to be solid, and therefore all its points to be firmly connected together, by the nature of the solid body, the angle described by this perpendicular will be the same for all its points. If, therefore, it be supposed to be infinitely small, and denoted by $\omega$, and if $r, r', r''$, &c. denote the distances of the points $M, M', M''$, &c., from the axis of $z$, their respective virtual velocities will be $r\omega, r'\omega', r''\omega''$, &c., and if $\delta, \delta', \delta''$, &c., denote the angles, either acute or obtuse, which the directions of these velocities make with those of the forces $P, P', P''$, &c., there will result

$$p = r\,\omega\cos\delta, \quad p' = r'\omega\cos\delta', \quad p'' = r''\omega\cos\delta'', \ \&c.,$$

for the expressions of the projections of these same velocities, on the directions of these forces, or on their productions.

Moreover, if $a, b, c$ be the angles comprised between the

direction of the velocity $r\omega$ and the parallels to the axes of $x$, $y$, $z$, drawn through the point M, the same angles relative to the direction of the force P being $\alpha$, $\beta$, $\gamma$, we shall have

$$\cos\delta = \cos a \cos\alpha + \cos b \cos\beta + \cos c \cos\gamma,$$

but because $r\omega$ is in the direction of a tangent at the point M to the circle, the centre of which is in the axis of $z$, it is easy to perceive that

$$\cos b = \pm \frac{x}{r}, \quad \cos a = \mp \frac{y}{r}, \quad \cos c = 0,$$

and, consequently,

$$p = r\omega \cos\delta = \pm\,(x\cos\beta - y\cos\alpha)\,\omega$$

In like manner we shall have

$$p' = \pm\,(x'\cos\beta' - y'\cos\alpha')\,\omega,$$
$$p'' = \pm\,(x''\cos\beta'' - y''\cos\alpha'')\,\omega,$$
$$+\ \&c$$

The signs will depend on the direction of the rotation, we should take the superior or the inferior signs in all these values at the same time, therefore, if they be substituted in equation (1), we shall obtain, by suppressing the factor $\pm\,\omega$, which is common to all the terms,

$$p\,(x\cos\beta - y\cos\alpha) + p'(x'\cos\beta' - y'\cos\alpha') + \&c = 0.$$

This equation of equilibrium is that of the moments with respect to the axis of $z$, about which the motion has place, the equations of the moments, with respect to the axes of the co-ordinates of $x$ and $y$, may be obtained in the same manner, by making the solid body to turn successively about these two right lines

341 Equation (b) may be made to assume another form, which will render it of easier application in particular cases.

For this purpose, let $x$, $y$, $z$ be the coordinates of the point M in its position of equilibrium, $x + \delta x$, $y + \delta y$, $z + \delta z$,

what these become, when this material point is transferred to N a point infinitely near, X, Y, Z, the components of the force P in the directions of the productions of $x, y, z$, estimated positively, these infinitely small quantities $\delta x, \delta y, \delta z$ will be the projections of the virtual velocity MN on the directions of X, Y, Z, and if $p$ denote, as before, its projection on the direction of P, we shall have (No 331)

$$\mathrm{P}p = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z$$

If the corresponding quantities which refer to the points M′, M″, &c., be denoted by the same letters with accents, we shall also have

$$\mathrm{P}'p' = \mathrm{X}'\delta x' + \mathrm{Y}'\delta y' + \mathrm{Z}'\delta z',$$
$$\mathrm{P}''p'' = \mathrm{X}''\delta x'' + \mathrm{Y}''\delta y'' + \mathrm{Z}''\delta z'',$$
&c

By adding these equations to the preceding, we can write

$$\mathrm{P}p + \mathrm{P}'p' + \mathrm{P}''p'' + \&c = \Sigma(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z):$$

the sum $\Sigma$ being supposed to extend to M, M′, M″, &c, all the points of the system, and it, consequently, consists of a number of similar parts, equal to that of these points By this means, equation (b) will assume the form

$$\Sigma(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) = 0, \tag{e}$$

which it was proposed to give it

Now, whatever be the connexion between the points of the system, it may be always expressed by one or more equations between their coordinates Therefore if L, L′, L″, &c, be given functions of $x, y, z, x', y', z', x''$, &c, or of a part of these coordinates, and if we suppose that these equations are

$$\mathrm{L} = 0, \quad \mathrm{L}' = 0, \quad \mathrm{L}'' = 0, \ \&c, \tag{f}$$

then as the simultaneous displacements of all the points of the system must be compatible with the conditions to which it is subjected, it is necessary that the coordinates $x, y, z, x'\ y'$, &c.,

of M, M', M'', &c , and $x+\delta x$, $y+\delta y$, $z+\delta z$, $x'+\delta x'$, &c , the coordinates of N, N', N'', &c , should successively satisfy these equations, consequently, if infinitely small quantities of the second order be neglected, we shall have

$$\left.\begin{aligned}
&\frac{d\text{L}}{dx}\delta x+\frac{d\text{L}}{dy}\delta y+\frac{d\text{L}}{dz}\delta z+\frac{d\text{L}}{dx'}\delta x'+\&\text{c.}=0,\\
&\frac{d\text{L}'}{dx}\delta x+\frac{d\text{L}'}{dy}\delta y+\frac{d\text{L}'}{dz}\delta z+\frac{d\text{L}'}{dx'}\delta x'+\&\text{c}\;=0,\\
&\frac{d\text{L}''}{dx}\delta x+\frac{d\text{L}''}{dy}\delta y+\frac{d\text{L}''}{dz}\delta z+\frac{d\text{L}''}{dx'}\delta x'+\&\text{c}\;=0,\\
&\&\text{c}
\end{aligned}\right\}\quad(\text{g})$$

If the direction of the displacements of all the points of the system be changed simultaneously, the signs of $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c , will be all changed at the same time, and these equations will be still satisfied, so that the infinitely small motions to which they refer, and the motions directly contrary, are equally compatible with the given conditions, as is implicitly supposed in the general statement of the principle of virtual velocities, (No. 331).

This being established, by means of these equations (g), there can, in each case, be eliminated from equation (e) a number of the quantities $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c , equal to that of equations (f) , those of these quantities which will afterwards remain in the first member of equation (e), will be independent of each other , consequently their coefficients may be put separately equal to cipher, this will furnish all the equations of equilibrium of the system, the number of which will be equal to three times that of the material points M, M', M'', &c , minus the number of equations (f). When the positions of these points, that is to say, the values of their coordinates $x$, $y$, $z$, $x'$, &c , are given, it is necessary that the components of the forces P, P', P'', &c., should satisfy these equations of equilibrium; when, on the contrary, these forces being given in magnitude and direction, the positions of the points of the

system are unknown, these same equations, combined with equations (f), will enable us to determine all their coordinates

342. As equations (e) and (g) are linear with respect to $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c, the elimination of a part of these quantities may be effected by the usual method, i e by adding these equations together, after having multiplied equations (g) by indeterminate factors, and then, making in this sum, the coefficients of the quantities $\delta x$, $\delta y$, $dz$, $\delta x'$, &c, which it is proposed to eliminate, equal to cipher As the coefficients of the remaining quantities must be likewise equal to cipher, it follows that the coefficients of all the quantities $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c, should be indiscriminately put equal to cipher, in the sum in question, hence there will result a number of equations equal to that of the coordinates, between which there will remain, in each case, indeterminate factors to be eliminated, in order to obtain the equations of equilibrium of the system.

If we denote the factors by which equations (g) should be multiplied, by $\lambda$, $\lambda'$, $\lambda''$, &c., we shall have, by what precedes

$$\left.\begin{aligned}
\text{X} + \lambda \frac{d\text{L}}{dx} + \lambda' \frac{d\text{L}'}{dx} + \lambda'' \frac{d\text{L}''}{dx} + \&\text{c} &= 0, \\
\text{Y} + \lambda \frac{d\text{L}}{dy} + \lambda' \frac{d\text{L}'}{dy} + \lambda'' \frac{d\text{L}''}{dy} + \&\text{c} &= 0, \\
\text{Z} + \lambda \frac{d\text{L}}{dz} + \lambda' \frac{d\text{L}'}{dz} + \lambda'' \frac{d\text{L}''}{dz} + \&\text{c} &= 0,
\end{aligned}\right\} \quad (\text{p})$$

for the equations arising from the coefficients of $\delta x$, $\delta y$, $\delta z$, in like manner we shall have

$$\left.\begin{aligned}
\text{X}' + \lambda \frac{d\text{L}}{dx'} + \lambda' \frac{d\text{L}'}{dx'} + \lambda'' \frac{d\text{L}''}{dx'} + \&\text{c} &= 0, \\
\text{Y}' + \lambda \frac{d\text{L}}{dy'} + \lambda' \frac{d\text{L}'}{dy'} + \lambda'' \frac{d\text{L}''}{dy'} + \&\text{c} &= 0, \\
\text{Z}' + \lambda \frac{d\text{L}}{dz'} + \lambda' \frac{d\text{L}'}{dz'} + \lambda'' \frac{d\text{L}''}{dz'} + \&\text{c} &= 0,
\end{aligned}\right\} \quad (\text{h}')$$

for those which arise from the coefficients of $\delta x'$, $\delta y'$, $\delta z'$, and so on

Instead of simply eliminating $\lambda, \lambda', \lambda''$, &c, their values may be obtained by means of these equations, and we now proceed to shew how we may deduce from them, in magnitude and direction, the forces arising from the connexion of the points of the system, which act on all these points, and constitute an equilibrium with the given forces P, P', P'', &c. The determination of these unknown forces is an important part of the problem of equilibrium, the general and complete solution of which will be found thus comprised in equations (f), (h), (h'), &c, taken together.

343 If all the points of the system, minus the point M, be rendered fixed, the equilibrium will not be disturbed. In virtue of equation $L = 0$, the point M will be then constrained to move on the surface of which $L = 0$ is the equation, and in which the coordinates $x, y, z$, are the sole variables. Now, if the resistance of this surface be denoted by $\mu$, it will act in the direction of one of the two parts of the normal at M, and we can substitute for this surface, or the equation of condition $L = 0$, this unknown force. In like manner, we can replace $L' = 0$, by a force $\mu_{\prime}$ acting perpendicularly to the surface which belongs to this equation, $L'' = 0$ by a force $\mu_{\prime\prime}$ normal to the corresponding surface, and so on; therefore, if these normal forces $\mu, \mu_{\prime}, \mu_{\prime\prime}$, &c, be joined with the given force P, or its components X, Y, Z, the point M may be regarded as altogether free and isolated. Consequently, if $a, b, c$, denote the angles which the direction of the force $\mu$ makes with lines parallel to the axes of $x, y, z$, drawn through the point M, and if $a_1, b_1, c_1$, be the same angles relative to the force $\mu_1$, and so on, we shall have

$$\begin{aligned}
X + \mu \cos a + \mu_{\prime} \cos a_{\prime} + \mu_{\prime\prime} \cos a_{\prime\prime} + \&c &= 0, \\
Y + \mu \cos b + \mu_{\prime} \cos b_{\prime} + \mu_{\prime\prime} \cos b_{\prime\prime} + \&c &= 0, \\
Z + \mu \cos c + \mu_{\prime} \cos c_{\prime} + \mu_{\prime\prime} \cos c_{\prime\prime} + \&c &= 0,
\end{aligned}$$

for the three equations of equilibrium of the point M. Moreover, if, in order to abridge, we make

$$\nu = \sqrt{\left(\frac{d\text{L}}{dx}\right)^2 + \left(\frac{d\text{L}}{dy}\right)^2 + \left(\frac{d\text{L}}{dz}\right)^2},$$

$$\nu_{,} = \sqrt{\left(\frac{d\text{L}'}{dx}\right)^2 + \left(\frac{d\text{L}'}{dy}\right)^2 + \left(\frac{d\text{L}'}{dz}\right)^2},$$

$$\nu_{,,} = \sqrt{\left(\frac{d\text{L}''}{dx}\right)^2 + \left(\frac{d\text{L}''}{dy}\right)^2 + \left(\frac{d\text{L}''}{dz}\right)^2},$$

$+$ &c

we shall have also, by known formulæ (No 21),

$$\left.\begin{array}{lll}
\cos a = \dfrac{1}{\nu}\dfrac{d\text{L}}{dx}, & \cos b = \dfrac{1}{\nu}\dfrac{d\text{L}}{dy}, & \cos c = \dfrac{1}{\nu}\dfrac{d\text{L}}{dz}, \\
\cos a_{,} = \dfrac{1}{\nu_{,}}\dfrac{d\text{L}'}{dx}, & \cos b_{,} = \dfrac{1}{\nu_{,}}\dfrac{d\text{L}'}{dy}, & \cos c_{,} = \dfrac{1}{\nu_{,}}\dfrac{d\text{L}'}{dz}, \\
\cos a_{,,} = \dfrac{1}{\nu_{,,}}\dfrac{d\text{L}''}{dx}, & \cos b_{,,} = \dfrac{1}{\nu_{,,}}\dfrac{d\text{L}''}{dy}, & \cos c_{,,} = \dfrac{1}{\nu_{,,}}\dfrac{d\text{L}''}{dz}, \\
\&\text{c.},
\end{array}\right\} \quad (1)$$

by means of which, the three equations of equilibrium will be changed into the following

$$\text{X} + \frac{\mu}{\nu}\frac{d\text{L}}{dx} + \frac{\mu_{,}}{\nu_{,}}\frac{d\text{L}'}{dx} + \frac{\mu_{,,}}{\nu_{,,}}\frac{d\text{L}''}{dx} + \&\text{c.} = 0,$$

$$\text{Y} + \frac{\mu}{\nu}\frac{d\text{L}}{dy} + \frac{\mu_{,}}{\nu_{,}}\frac{d\text{L}'}{dy} + \frac{\mu_{,,}}{\nu_{,,}}\frac{d\text{L}''}{dy} + \&\text{c.} = 0,$$

$$\text{Z} + \frac{\mu}{\nu}\frac{d\text{L}}{dy} + \frac{\mu_{,}}{\nu_{,}}\frac{d\text{L}'}{dz} + \frac{\mu_{,,}}{\nu_{,,}}\frac{d\text{L}''}{dz} + \&\text{c.} = 0$$

Now from a comparison of these equations with equations (h), with which they are identical, it appears that

$$\mu = \nu\lambda, \quad \mu_{,} = \nu_{,}\lambda', \quad \mu_{,,} = \nu_{,,}\lambda'', \ \&\text{c}$$

Thus, relatively to the point M, the forces which arise from its connexion with the other points of the system, are expressed by the products $\nu\lambda$, $\nu_{,}\lambda'$, $\nu_{,,}\lambda''$, &c , as these forces must be positive quantities, the radicals $\nu$, $\nu_{,}$, $\nu_{,,}$, &c , must have the

same signs as the quantities $\lambda, \lambda_{/}, \lambda_{//}$, &c., and their directions will be completely determined by equations (i).

If, in the same manner, $\mu', \mu_{/}', \mu_{//}''$, &c., denote the forces arising from the connexion of the system, which act on the point M′, and are normal to the different surfaces on which it is constrained to exist, when all the other points M, M″, M‴, &c., are fixed, we shall find

$$\mu' = \nu'\lambda, \quad \mu_1' = \nu_1'\lambda', \quad \mu_{//} = \nu_{//}'\lambda'', \text{ \&c },$$

in which, in order to abridge, we suppose

$$\nu' = \sqrt{\left(\frac{d\text{L}}{dx'}\right)^2 + \left(\frac{d\text{L}}{dy'}\right)^2 + \left(\frac{d\text{L}}{dz'}\right)^2},$$

$$\nu_{/}' = \sqrt{\left(\frac{d\text{L}'}{dx'}\right)^2 + \left(\frac{d\text{L}'}{dy'}\right)^2 + \left(\frac{d\text{L}'}{dz'}\right)^2},$$

$$\nu_{//}' = \sqrt{\left(\frac{d\text{L}''}{dx'}\right)^2 + \left(\frac{d\text{L}''}{dy'}\right)^2 + \left(\frac{d\text{L}''}{dz}\right)^2},$$

&c

Similar expressions may be obtained for the forces relative to the points M″, M‴, &c

344. It appears from a comparison of the values of $\mu$ an $\mu'$, that

$$\mu\nu' = \mu'\nu,$$

so that they are as the quantities $\nu$ and $\nu'$. When, therefore, two material points M and M′ are connected together, and also, if we please, with any number whatever of other points, by an equation $\text{L} = 0$, there results in the state of equilibrium, the forces $\mu$ and $\mu'$ applied to M and M′, the magnitudes of which are as $\nu$ and $\nu'$, and these forces make with the axes of the coordinates, angles whose cosines are

$$\frac{1}{\nu}\frac{d\text{L}}{dx}, \quad \frac{1}{\nu}\frac{d\text{L}}{dy}, \quad \frac{1}{\nu}\frac{d\text{L}}{dz},$$

for the force $\mu$, and

$$\frac{1}{\nu'}\frac{d\text{L}}{dx'}, \quad \frac{1}{\nu'}\frac{d\text{L}}{dy'}, \quad \frac{1}{\nu'}\frac{d\text{L}}{dz'},$$

for the force $\mu'$. The direction and magnitude of these forces, depend on the sign and magnitude of $\lambda$, which in each case may be deduced from the equations of equilibrium

The consideration of the surfaces on which each of the points of a system are at liberty to move, when all the others are supposed fixed, determines the normal directions of the forces arising from the connexion of these moveables, for each of the equations by which this connexion is expressed (No 290), but we cannot deduce from them any relation between the forces relative to two material points connected together by the same equation, and it is the principle of virtual velocities, or equations ($h$), ($h'$), &c, that have been deduced from it, which enables us to determine this ratio *à priori*, in the case of equilibrium

345 For an application of these formulæ, let us consider the case of the funicular polygon, which has been already discussed in the first section of the preceding chapter, and let us suppose that the material points M, M′, M″, &c, are the successive summits of this polygon

If $l$, $l'$, $l''$, &c, be the given lengths of the sides MM′, M′M″, M″M‴, &c, equations (f) will in this case be

$$\begin{aligned}
\text{L} &= \sqrt{(x-x')^2+(y-y')^2+(z-z)^2} - l = 0,\\
\text{L}' &= \sqrt{(x'-x'')^2+(y'-y'')+(z'-z'')^2} - l' = 0,\\
\text{L}'' &= \sqrt{(x''-x''')^2+(y''-y''')^2+(z''-z''')^2} - l'' = 0,\\
&\text{\&c},
\end{aligned}$$

hence there will result($b$)

$$\frac{d\text{L}}{dx} = -\frac{d\text{L}}{dx'} = \frac{x-x'}{l}, \quad \frac{d\text{L}'}{dx'} = -\frac{d\text{L}'}{dx''} = \frac{x'-x''}{l'}, \text{ \&c},$$

$$\frac{d\text{L}}{dy} = -\frac{d\text{L}}{dy'} = \frac{y-y'}{l}, \quad \frac{d\text{L}'}{dy'} = -\frac{d\text{L}'}{dy''} = \frac{y'-y''}{l'}, \text{ \&c},$$

$$\frac{d\text{L}}{dz} = -\frac{d\text{L}}{dz'} = \frac{z-z'}{l}, \quad \frac{d\text{L}'}{dz'} = -\frac{d\text{L}'}{dz''} = \frac{z'-z''}{l'}, \text{ \&c},$$

and all the other partial differences of L, L′, L″, &c., which occur in the preceding formulæ, will be equal to cipher.

In the case of the two points M and M′, we shall have

$$\nu = \nu' = \pm 1, \quad \mu = \mu' = \pm \lambda,$$

in which the superior or inferior signs should be taken, according as the value of $\lambda$ is positive or negative. From this and the preceding equations, we may infer that the points M and M′, will be solicited by equal and opposite forces, acting either in the direction of the line MM′ or of its productions, and the quantity $\lambda$ will, abstracting from the sign, express the common intensity of these forces. The same will be the case relatively to the points M′ and M″, M″ and M‴, &c., so that in the state of equilibrium, the quantities $\lambda, \lambda', \lambda''$, &c., will express the contractions or tensions of the successive sides MM′, M′M″, M″M‴, &c. Since, by equations (1), we have

$$\cos a = \pm \frac{x - x'}{l}, \quad \cos b = \pm \frac{y - y'}{l}, \quad \cos c = \pm \frac{z - z'}{l},$$

in which the superior or inferior signs should be taken, according as the value of $\lambda$ is positive or negative, it follows, for example, that the force applied to the point M will be directed from M towards M′, and will express a contraction of the side MM′ when this value is negative, and, when this value is positive, this force will act in the opposite direction, and express a tension, and will indicate that the string is stretched. One or other of these cases is possible, if the sides of the polygon are inflexible rods, joined by hinges, and the second case can alone obtain, if the sides are flexible threads.

Equations (h), (h′), (h″), &c. may be written as follows

$$\mathrm{X} = \lambda \left(\frac{x' - x}{l}\right),$$

$$\mathrm{Y} = \lambda \left(\frac{y' - y}{l}\right),$$

$$\mathrm{Z} = \lambda \left(\frac{z' - z}{l}\right),$$

$$X' + \lambda\left(\frac{x'-x}{l}\right) = \lambda'\left(\frac{x''-x'}{l'}\right),$$

$$Y' + \lambda\left(\frac{y'-y}{l}\right) = \lambda'\left(\frac{y''-y'}{l'}\right),$$

$$Z' + \lambda\left(\frac{z'-z}{l}\right) = \lambda'\left(\frac{z''-z'}{l'}\right),$$

$$X'' + \lambda'\left(\frac{x''-x'}{l'}\right) = \lambda''\left(\frac{x'''-x''}{l''}\right),$$

$$Y'' + \lambda'\left(\frac{y''-y'}{l'}\right) = \lambda''\left(\frac{y'''-y''}{l''}\right),$$

$$Z'' + \lambda'\left(\frac{z''-z'}{l'}\right) = \lambda''\left(\frac{z'''-z''}{l''}\right),$$

It appears from a consideration of the three first, that the tension $\lambda$ is the resultant of the forces X, Y, Z $(c)$ By adding them to the three following, we shall obtain

$$X + X' = \lambda'\left(\frac{x''-x'}{l'}\right),$$

$$Y + Y' = \lambda'\left(\frac{y''-y'}{l'}\right),$$

$$Z + Z' = \lambda'\left(\frac{z''-z'}{l'}\right),$$

which shews that the tension $\lambda'$ is the resultant of X′, Y′, Z′, and of the forces X, Y, Z, transferred to the point M, parallel to themselves By proceeding in this manner, we shall obtain, for the tension of any side whatever, the same value as in No 287.

The number of summits M, M′, M″, &c , being denoted by $n$, that of the preceding equations will be $3n$, and that of the tensions $\lambda, \lambda', \lambda''$, &c , will be equal to $n-1 (d)$ Therefore if these quantities be eliminated, there will result $2n+1$ equations of equilibrium, which combined with $l, l', l''$, &c , the $n-1$ given lengths of the sides of the polygon, will be sufficient to determine the $3n$ coordinates of its summits, and,

consequently, its figure of equilibrium. But this mode of determining the figure, &c , is attended with no practical advantage, it is therefore preferable, as has been done in No 286, to trace the sides of the funicular polygon successively, by means of the given magnitudes and directions of the forces that act at its different summits

346 In the case of M, M′, M″, &c , any system whatever of material points, if the given forces which are applied to these points, arise from their mutual attractions or repulsions, or from similar forces which emanate from one or more centres, we shall have

$$\Sigma(\text{X}dx + \text{Y}dy + \text{Z}dz) = d\ \phi(x, y, z, x', y', z', \&c),$$

in which $\phi$ denotes a given function of the coordinates of M, M′, M″, &c , dependent on the law of these forces in a function of the distances

In fact, with respect to forces emanating from fixed centres, this follows from what has been established in No. 158 Moreover, let U express the mutual action of M and M′, which we will suppose to be attractive, let also $u$ be their mutual distance, so that U may be a given function of $u$, it will be expressed by the equation

$$u^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2.$$

The cosines of the angles which MM′ makes with lines drawn through the point M, in the directions of the positive axes of $x$, $y$, $z$, will be

$$\frac{x'-x}{u}, \quad \frac{y'-y}{u}, \quad \frac{z'-z}{u};$$

if these be respectively multiplied by U, we shall obtain the components of this force applied to the point M, and acting in the direction MM′ Those of the same force U, applied to the point M′ in the direction M′M, will be equal and contrary, hence it follows that the part of the sum $\Sigma$ which arises from the action and reaction of M and M′, will be equal to

$$\frac{\mathrm{U}}{u}\left[(x'-x)(dx-dx')+(y'-y)(dy-dy')+(z'-z)(dz-dz')\right]$$

But, by differentiating the value of $u^2$, there results

$$udu=(x'-x)(dx'-dx)+(y'-y)(dy'-dy)+(z'-z)(dz'-dz),$$

from which it is evident that the preceding quantity is equal to $-\mathrm{U}du$, that is to say, to the differential of a function of $u$. Similar expressions may be obtained for the parts of the sum $\Sigma$, which arise from the mutual actions of the other points of the system; consequently, its entire value will consist of terms, all of which will be exact differentials, and this value will be also the differential of a given function of the coordinates of all these points.

In virtue of equation (e), this function, which we will denote by $\phi$, will be a *maximum* or a *minimum*, relatively to the values of the coordinates that belong to a position of equilibrium of the system; and, conversely, if the *maximum* or *minimum* of the function $\phi$ be determined, regard being had to the equations (f) which may exist between the coordinates, the values that will be obtained for these variables, will refer to positions of equilibrium Hence it follows, that when the system of points M, M′, M″, &c. is in motion, in which case their coordinates, and consequently, the quantity $\phi$ are functions of the time, this function $\phi$ will attain its *maximum* or *minimum*, whenever the system passes through a position in which it would remain in equilibrio, if the points of which it is composed had not any acquired velocity.

347 There is an essential difference between the maximum and minimum of the function $\phi$, which, as it is of consequence to take into account, we now proceed to explain

The state of equilibrium of a body or a system is said to be *stable*, when, if the moveables be made to deviate, ever so little, from their positions, they tend to revert to them, by making small oscillations which eventually, the frictions, and the resistances of the media in which they move, render

insensible, or finally extinguish. The equilibrium is instable or *instantaneous*, when the body, or system of bodies, being made to deviate a little from it, recedes more and more from this position, until it finally terminates by being turned upside down If we assume that there is no friction, which does to a certain extent retain the bodies in their positions, this second state of equilibrium is a case purely mathematical, that can never be observed, since the least disturbing force is sufficient to destroy it

This being established, the equations furnished by the principle of virtual velocities, or, what is the same thing, by the condition of the maximum or minimum of the function $\phi$, are common to these two states, but the *maximum* refers to the case of stable, and the *minimum* to the case of instantaneous equilibrium; and this is, in fact, what will be shewn in a subsequent chapter, when the nature of the motion that has place when a system of material points is made to deviate from any state of equilibrium will be considered At present, we shall content ourselves with furnishing examples of these two states of equilibrium in the case of a system of heavy bodies, and with pointing out a remarkable property of its centre of gravity.

348. Let gravity be supposed to be the sole force applied to the points M, M′, M″, &c, which are the centres of gravity of bodies, whose weights are denoted by $\omega, \omega', \omega''$, &c, then as this force is vertical, and acts in the direction of the weight, we shall have

$$Z = \omega, \quad Z' = \omega', \quad Z'' = \omega'', \text{ \&c },$$

the other components will all vanish, and there will result

$$d\phi = \omega dz + \omega' dz' + \omega'' dz'' + \text{\&c}$$

But if $\Pi$ denotes the sum of the weights $\omega, \omega', \omega''$, &c, and $z_1$ the vertical ordinate of their centre of gravity, drawn in the direction of the gravity, we have likewise (No 64),

$$\Pi z_1 = \omega z + \omega' z' + \omega'' z'' + \&c\ ,$$

consequently, there results

$$d\varphi = \Pi dz_1, \quad \phi = c + \Pi z_1,$$

$c$ being an arbitrary constant.

From this it follows—1st, that the ordinate $z_1$ is a quantity which should be either a *maximum* or *minimum*, when the system is in equilibrio, and conversely, 2ndly, that the *maximum* of $z_1$ corresponds to the case of stable equilibrium, and its *minimum* to the case of instantaneous equilibrium

Thus, the condition of equilibrium of any system whatever of heavy bodies, consists in this, that the centre of gravity of the entire system is the lowest or highest possible, the lowest, when the equilibrium is stable, and the highest when it is only instantaneous.

349 It follows from this theorem, that if a heavy chain attached by its extremities to fixed points, is in equilibrio, its centre of gravity will be the lowest possible, this agrees with the result of No 296

If a heavy material point is placed on a curve, and if in several points of this curve, the tangent is horizontal, the vertical ordinate of the material point, estimated in the direction of gravity, will be a *maximum* for those of its points, in which the concavity is turned upwards, and a *minimum* in the case of those points for which the concavity is directly opposite, consequently, the first will be positions of stable, and the last positions of instantaneous equilibrium

If a homogeneous heavy ellipsoid rests on a fixed horizontal plane, its centre of gravity or of figure will be the lowest possible, when the ellipsoid touches the fixed plane in one of the two extremities of the least of its three axes, and in this case the equilibrium will be stable When it touches it in one of the two extremities of the greatest of its three axes, its centre of gravity will be the highest possible, and the equilibrium will be only instantaneous

Finally, if the point of contact is an extremity of its mean axis, the elevation of the centre of gravity will be a *minimum* for one part of the sections of the body, and a *maximum* for the other sections, consequently, the equilibrium will be stable or instable according as the displacements have place in the direction of the first or last sections. As what has been now stated is evident *à priori*, it will enable us to verify the theorem of the preceding number

Let us now suppose that two homogeneous heavy liquids are poured into a vessel, if the surface of separation and that which terminates the upper liquid are respectively horizontal, and if this liquid be the one which has the least density, the centre of gravity of these two liquids will be the lowest possible; for it is easy to perceive, that if one or other of these two surfaces be inclined or curved, the centre of gravity of the system will be always elevated. These two surfaces being always horizontal, if the less dense liquid be below the other, in the same manner, it will appear that the centre of gravity of the system will be the highest possible. Consequently, in order that these fluids may be in equilibrio, it is necessary and sufficient that each of them be terminated by a horizontal plane, but in order that the equilibrium may be stable, it is moreover necessary that the densest liquid should occupy the inferior part of the vessel When the difference of the two densities is inconsiderable, it is possible, by taking proper precaution, to make the denser liquid float on the other, but this instable equilibrium can only maintain itself sufficiently long to be observed, in consequence of the friction of the two liquids against the sides of the vessel.

# NOTES.

## INTRODUCTION

($a$) As there are ninety degrees in the quadrant of the circumference, there must be 90, 60, or 5400 minutes, and 5400.60, or 32400 seconds in it.

($b$) By stating this proportion we have

$$\omega \; 32400'' : n \; \frac{\pi}{2} \cdot \omega = \frac{32400''\, 2n}{\pi},$$

and by substituting the value of $\pi$ given in the text, we shall obtain the expression for $\omega$.

($c$) In fact, if $\lambda, \lambda', \lambda''$, be the projections of a given area $l$ on these planes, and if $\iota, \iota', \iota''$, be the respective inclinations of $l$ on these planes, by what as been just established, we shall have

$\lambda = l\cos\iota, \quad \lambda' = l\cos\iota', \quad \lambda'' = l\cos\iota'', \qquad \lambda^2 + \lambda'^2 + \lambda''^2 = l^2(\cos^2\iota + \cos^2\iota' + \cos^2\iota'') = l^2$, since $\cos^2\iota + \cos^2\iota' + \cos^2\iota'' = 1$ See No 281.

($d$) See on this point *Le Journal de l'Ecole Polytechnique*, 18th number, page 320

($e$) By substituting in the expression for $Fb - Fa$, the values which have been deduced for its second and third terms, we obtain as $\delta^3$, $\delta^4$, &c. are neglected

$$Fb - Fa = \delta\Sigma f(a + \iota\delta) + \tfrac{1}{2}\delta(fb - fa) - \tfrac{1}{4}\delta^2(f'b - f'a) + \tfrac{1}{6}\delta^2\,(f'b - f'a),$$

which is evidently equal to the value of $Fb - Fa$ given in text, and if $f(a + \iota\delta)$ be expressed in a series, there results by obliterating the term $\frac{\delta}{2}fa$, the expression for $\int_a^b fx\,dx$ given in the text

($f$) When one of these equations is taken from the other, their difference, which is equal to cipher, becomes

$$2(x'-x)dx+2(y'-y)dy+2(z'-z)dz+dx^2+dy^2+dz^2=0,$$

but $dx^2, dy^2, dz^2$, being infinitely small quantities of the second order, may be neglected

($g$) By Taylor's theorem, if $u=\text{F}x$, when $x$ receives an increment $dx$, and $u$ becomes $u'$, the value of $u'$ expressed in a series is

$$u'=u+\frac{du}{1}+\frac{d^2u}{1\ 2}+\frac{d^3u}{1\ 2\ 3}+\&c\ ,$$

if $u=\cos\alpha$, when $\alpha$ receives an increment and becomes $\alpha'$, then

$$\cos\alpha'=\cos\alpha+\frac{d\ \cos\alpha}{1}+\frac{d^2\cos\alpha}{1\ 2}+\frac{d^3\cos\alpha}{1\ 2\ 3}+\&c$$

Now if in the value of $\sin^2\delta$ given in the text, we substitute for $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, their values, and omit all terms in the resulting expressions after the third, as involving infinitely small quantities of a higher order than the second, we obtain

$$\sin^2\delta=1-\left[\cos\alpha\left(\cos\alpha+\frac{d\cos\alpha}{1}+\frac{d^2\cos\alpha}{2}\right)-\right.$$
$$\left.\cos\beta\left(\cos\beta+\frac{d\cos\beta}{1}+\frac{d^2\cos\beta}{2}\right)-\cos\gamma\left(\cos\gamma+\frac{d\cos\gamma}{1}+\frac{d^2\cos\gamma}{2}\right)\right]$$

Now since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, and therefore $\cos\alpha d\cos\alpha+\cos\beta d\cos\beta+\cos\gamma d\cos\gamma=0$, the expression between brackets becomes equal to

$$-[1+\tfrac{1}{2}\cos\alpha\, d^2\cos\alpha+\tfrac{1}{2}\cos\beta d^2\cos\beta+\tfrac{1}{2}\cos\gamma d^2\cos\gamma]^2,$$

that is to

$$-1-\cos\alpha d^2\cos\alpha-\cos\beta d^2\cos\beta-\cos\gamma d^2\cos\gamma,$$

as infinitely small quantities of a higher order than the second are neglected; hence it is evident, that the values of $\sin^2 d$ is that given in the text

($h$) By substituting in the expression for $d\cos\alpha$, the values of $ds^2$ and $ds\,d^2s$ we obtain

$$d\ \cos\alpha=\frac{(dx^2+dy^2+dz^2)d^2x-(dxd^2x+dyd^2y+dzd^2z)dx}{ds^3},$$

which is equal to the value of $d\cos\alpha$, as will be evident by obliterating $dx^2d^2x$, and concinnating the other terms.

(*i*) If the squares of $d\cos\alpha$, $d\cos\beta$, $d\cos\gamma$, be taken, and then added together, we obtain

$$(d\cos\alpha)^2+(d\cos\beta)^2+(d\cos\gamma)^2=\frac{dx^2+dy^2}{ds^6}(dy\,d^2x-dx\,d^2y)^2+$$

$$\frac{dx^2+dz^2}{ds^6}(dz\,d^2x-dx\,d^2z)^2+\frac{dy^2+dz^2}{ds^6}(dz\,d^2y-dy\,d^2z)^2+\frac{2dy}{ds^3}\,\frac{dz}{ds^3}$$

$$(dy\,d^2x-dx\,d^2y)\,(dz\,d^2x-dx\,d^2z)+2\frac{dx}{ds^3}\,\frac{dz}{ds^3}\,(dx\,d^2y-dy\,d^2x)$$

$$(dz\,d^2y-dy\,d^2z)+2\frac{dx}{ds^3}\,\frac{dy}{ds^3}(dx\,d^2z-dz\,d^2x)\,(dy\,d^2z-dz\,d^2y)$$

Now if the factors in the three last terms be respectively multiplied together, they will be equal (by concinnating and obliterating the quantities which mutually destroy each other) to

$$\frac{dz^2}{ds^6}\,(dy\,d^2x-dx\,d^2y)^2+\frac{dy^2}{ds^6}\,(dz\,d^2x-dx\,d^2z)^2+\frac{dx^2}{ds^6}\,(dz\,d^2y-dy\,d^2z)^2.$$

hence then we obtain

$$(d\cos\alpha)^2+(d\cos\beta)^2+(d\cos\gamma)^2=$$

$$\left(\frac{dx^2+dy^2+dz^2}{ds^6}\right)[(dy\,d^2x-dx\,d^2y)^2+(dz\,d^2x-dx\,d^2z)^2+(dz\,d^2y-dy\,d^2z)^2],$$

which is evidently the same as the expression for $\delta^2$ given in text

(*k*) By substituting these expressions of A, B, C, it is easy to perceive that the resulting expressions will be identical

(*l*) When the equation

$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0,$$

is differenced with respect to $x$, $y$, $z$, there results

$$(x'-x)\,d^2x+(y'-y)\,d^2y+(z'-z)\,d^2z-dx^2-dy^2-dz^2=$$
$$(x'-x)\,d^2x+(y'-y)\,d^2y+(z'-z)\,d^2z-ds^2=0$$

Now if the equation

$$(x'-x)\,dx+(y'-y)\,dy+(z'-z)\,dz=0$$

be multiplied by $d^2y$, and if from the product, the value of

$$(x'-x)\,d^2x+(y'-y)\,d^2y+(z'-z)\,d^2z=ds^2,$$

multiplied by $dy$, be subtracted, there will result by concinnating

$$(x'-x)(dxd^2y-dyd^2x)=(z'-z)(dyd^2z-dzd^2y)+ds^2dy, \quad (1)$$

and if the same equations be multiplied by $dx$ and $dy$ respectively, and then the second be taken from the first, we obtain in like manner,

$$(y'-y)(dyd^2x-dxd^2y)=(z'-z)(dxd^2z-dzd^2x)+ds^2dx \quad (2)$$

Now if equation (1) be multiplied by $dyd^2z-dzd^2y$, and equation (2) by $dxd^2z-dzd^2x$, there results

$$(x'-x)(dyd^2z-dzd^2y)(dxd^2y-dyd^2x)=(z'-z)(dyd^2z-dzd^2y)+ds^2dy(dyd^2z-dzd^2y),$$

$$(y'-y)(dxd^2z-dzd^2x)(dyd^2x-dxd^2y)=(z'-z)(dxd^2z-dzd^2x)+ds^2dx(dxd^2z-dzd^2x),$$

and if the second be taken from the first, we obtain

$$[(x'-x)(dyd^2z-dzd^2y)+(y'-y)(dxd^2z-dzd^2x)](dxd^2y-dyd^2x)= (z'-z)[(dyd^2z-dzd^2y)^2+(dzd^2x-dxd^2z)]^2+ds[(dyd^2z-dzd^2y)dy - (dzd^2x-dxd^2z)dx],$$

but from the equation of the osculating plane given above, it is evident that

$$[(x'-x)(dyd^2z-dzd^2y)+(y'-y)(dzd^2x-dxd^2z)](dxd^2y-dyd^2x)= -(z'-z)(dxd^2y-dyd^2x)^2,$$

therefore by substituting this last quantity for that to which it is equal, we obtain

$$(z'-z)[(dyd^2z-dzd^2y)^2+(dzd^2x-dxd^2z)^2+(dxd^2y-dyd^2x)^2]= ds^2[dy(dyd^2z-dzd^2y)-dx(dzd^2x-dxd^2z)],$$

and, consequently,

$$z'-z=\frac{ds^2[dy(dyd^2z-dzd^2y)-dx(dzd^2x-dxd^2z)]}{(dxd^2y-dyd^2x)^2+(dzd^2x-dxd^2z)^2+(dyd^2z-dzd^2y)^2},$$

and if for the denominator of this expression its value $\frac{\rho}{ds^6}$ be substituted, the value of $z'-z$ given in the text will be obtained, the expressions for $x'-x$, $y'-y$, may be obtained in the same manner.

(*m*) When the values of $(x'-x)^2$, $(y'-y)^2$, $(z'-z)^2$, are added together there results, after all reductions, a factor of $\frac{\rho^4}{ds^4}$ equal to $\frac{ds^5}{r}$ as is evident from the value of $\delta$ determined in note (*l*)

# BOOK I

## CHAPTER I

(*a*) This is, in fact, a particular case of that axiom which was used by Archimedes in demonstrating some of the fundamental propositions of mechanics; it was Leibnitz however who first announced it as a general principle of philosophy in the following manner "Nothing exists in any state that is not determined by *some reason* to be in that state rather than in another." Hence if, as in the present case, there are two conditions, and no reason to determine a subject to be in one of them rather in the other, we are to conclude that it is in neither. It has been denominated the principle of *sufficient reason*.—See *King de Origine Mali Bishop Law's Notes*, and also *Theorie Analytique des Probabilities, Introduction*, page 2. This principle is frequently assumed in the following treatise.

(*b*) As $\cos\beta = \cos^2\tfrac{1}{2}\beta - \sin^2\tfrac{1}{2}\beta = 2\cos^2\tfrac{1}{2}\beta - 1$,

$$(\phi\tfrac{1}{2}\beta)^2 = 2\cos\beta + 2 = 4\cos^2\tfrac{1}{2}\beta$$

(*c*) For by performing the multiplication we have

$$\text{R}\ \cos \text{RMO} = \text{R}\cos a\cos g + \text{R}\cos b\cos h + \text{R}\cos c\cos k =$$
$$\text{X}\cos g + \text{Y}\cos h + \text{Z}\cos k,$$

and if equations (c) be multiplied by $\cos g$, $\cos h$, $\cos k$, and then added together, there results

$$\text{X}\cos g + \text{Y}\cos h + \text{Z}\cos k =$$
$$\text{P}(\cos g\cos\alpha + \cos h\cos\beta + \cos k\cos\gamma + \&c.) +$$
$$\text{P}'(\cos g\cos\alpha' + \cos h\cos\beta' + \cos k\cos\gamma' + \&c.) +$$
$$\text{P}''(\cos g\cos\alpha'' + \cos h\cos\beta'' + \cos k\cos\gamma'' + \&c.),$$

that is,

$$\text{R}\ \cos\text{RMO} = \text{P}\cos\text{PMO} + \text{P}'\cos\text{P}'\text{MO} + \&c.$$

(*d*) By means of these formulæ (b) we have

$$\text{X} - \frac{d\text{L}}{dx} = \text{Y} - \frac{d\text{L}}{dy} \quad \text{Y}\frac{d\text{L}}{dx} = \text{X}\frac{d\text{L}}{dy}$$

## CHAPTER II.

(*a*) Since Q, Q′, the components of S′ are less than S or S′, and as the effect of Q′ is destroyed, because it is supposed to pass through the prop, the only forces that remain will be S and Q, and since they act in opposite directions, their resultant will be S − Q.

## CHAPTER III

(*a*) By similar triangles we have

$$NG : M'H :: MN : MM' :: P' : P + P' \;\therefore\; (P + P')\,NG = M'H \cdot P,$$

and by adding the identical equation to both members of this, we obtain

$$(P + P')(NG + GK) = P \cdot MQ + P'(M'H + HQ') =$$

$$(\text{as } NG + GK) = NK, \text{ and } MQ = z,\ M'H + HQ' = M'Q' = z',)$$

the expression in the text

(*b*) It is not implied in the proof that these planes are at right angles to each other, and it is easy to demonstrate that if this condition is satisfied with respect to any two planes parallel to the common directions of these forces, it will be so with respect to all others

(*c*) This follows from equations (1) of No 55.

## CHAPTER IV

(*a*) By adding these equations together, we obtain

$$m(m + m' + m'' + \&c)\,x^2 + m'(m + m' + m'' + \&c)\,x'^2 +$$
$$m''(m + m' + m'' + \&c)\,x''\ \&c = mm'(x^2 + x'^2 - 2xx') +$$
$$mm''(x^2 + x''^2 - 2xx'') + m'm''(x'^2 + x''^2 - 2x'x'') + \&c.,$$

that is,

$$(m + m' + m'' + \&c)(mx^2 + m'x'^2 + m''x''^2) =$$
$$M\,(mx^2 + m'x'^2 + m''^2x''^2 + \&c) =$$
$$mm'(x - x')^2 + mm''(x - x'')^2 + m'm''(x' - x'')^2 + \&c$$

## CHAPTER V.

(*a*) By making this substitution, in the first of equations (1), we obtain

$$lx_1 = \int a\,ds + \int s\,ds \cos\alpha,$$

and by integrating between the limits $s_0 = 0$, $s_1 = l$, there results

$$lx_1 = al + \frac{l^2}{2}\cos\alpha, \quad \therefore x_1 = a + \frac{l}{2}\cos\alpha$$

(*b*) $x = a\cos\frac{s}{a}$, $\quad x\,ds = a\cos\frac{s}{a}\,ds$, and $\int x\,ds = a^2\sin\frac{s}{a}$, when this integral is taken between the limits $\frac{l}{2a}$, $-\frac{l}{2a}$, we shall evidently

obtain the value of $lx_1$ given in the text, and as the chord of an arc is equal to twice the sine of half the arc, $c = 2a \sin \frac{l}{2a}$, $lx_1 = ac$ When $l$ is a quadrant, and equal to $\frac{a\pi}{2}$, then $c = a\sqrt{2}$, $x' = \frac{2^{\frac{1}{2}}a}{\pi}$, when $l$ is equal to $a\pi$, then $x_1 = \frac{2a}{\pi}$ and $a = \frac{\pi x_1}{2}$, i e. in this case $a$ = quadrant of the circle whose radius is $x_1$, and when $l = a$, $x_1 = c$

($c$) In all conic sections, the two last equations (2) may be integrated in a finite form, as is immediately evident by substituting for $\sqrt{1 + \frac{dy^2}{dx^2}}$ its value in function of $x$, for example, the two last equations become, in the case of the parabola, in which $y^2 = px$

$$lx_1 = \int_\alpha^\beta x\left((1 + \frac{p}{4x}\right)^{\frac{1}{2}} dx,$$

$$ly^1 = \int_\alpha^\beta px\left(1 + \frac{p}{4x}\right)^{\frac{1}{2}} dx;$$

which equations can be integrated in a finite form, consequently, if the value of $l$ was also given in a finite form, $x_1$ and $y_1$ will be obtained in functions of $\alpha$ and $\beta$, in the case of an ellipse, of which the equation is $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, $ds = \frac{dx\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}$, which by making $a = 1$, $a^2 - b^2 = e^2$, can be reduced to $\frac{dx\sqrt{1 - e^2x^2}}{\sqrt{1 - x^2}}$, and in the hyperbola of which the equation is $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$, $ds = \frac{dx\sqrt{(a^2 + b^2)x^2 - a^4}}{a\sqrt{x^2 - a^2}}$, which by making $a^2 = 1$, $a^2 + b^2 = 1 + b^2 = e^2$, can be reduced to $\int \frac{dx\sqrt{e^2x^2 - 1}}{\sqrt{x^2 - 1}}$, now these values of $ds$, in the case of the ellipse, cannot be obtained in a finite form, though when they are respectively multiplied by $x$ and $y$, the integrals of the resulting expressions can be obtained in a finite form, and therefore the values of $y_1$ and $x_1$

($d$) See the Journal of the Polytechnic School, 18th Number, page 431

(*e*) By differentiating this equation we obtain

$$dp + \frac{(c-q)}{\sqrt{2cq-q^2}}\,dq = \frac{cdq}{\sqrt{2cq-q^2}} \qquad dp = \frac{qdq}{\sqrt{2cq-q^2}}.$$

(*f*) By substituting the value of $dy^2$ in the equation $ds^2 = dy^2 + dx^2$, there results

$$ds^2 = dx^2\left(\frac{a^2 - 2ax + x^2}{ax - x^2}\right) + dx^2,$$

that is,

$$ds^2 = \left(\frac{a^2 - 2ax + x^2 + ax - x^2}{ax - x^2}\right)dx^2,$$

$$= \frac{a}{x}\cdot\frac{(a-x)}{a-x}\,dx^2 \quad \therefore\ ds = \sqrt{\frac{a}{x}}\cdot dx$$

(*g*) In the equation $2x_1\sqrt{x} = \int\sqrt{x}\,dx$, there results by integrating, $2x_1\sqrt{x} = \frac{2}{3}x^{\frac{3}{2}}$ $\therefore x_1 = \frac{x}{3}$, and in the equation $2y_1\sqrt{x} = \int\frac{ydx}{\sqrt{x}}$, by partially integrating we obtain

$$2y_1\sqrt{x} = 2y\sqrt{x} - 2\int\sqrt{x}\,dy,$$

and by substituting for $dy$, we obtain

$$2\int\sqrt{x}\,dy = 2\int\frac{\sqrt{x}\,(a-x)\,dx}{\sqrt{x}\,\sqrt{a-x}} = 2\sqrt{a-x}\,dx,$$

the integral of this last quantity, taken between the limits $a$ and $0$, is $\frac{4}{3}((a-x)^{\frac{3}{2}} - a^{\frac{3}{2}})$ consequently the value of $y_1$ will be that given in the text

(*h*) When a semi-circle revolves about its diameter as in this case, $y_1 = \frac{2a}{\pi}$ (see note (*b*)) $\therefore 2\pi l y_1 = 4\pi a^2$, i.e. the surface of the sphere is four times a great circle of a sphere

(*i*) In the equation $\frac{1}{3}a^2c = \frac{1}{3}ch^2 + \frac{1}{2}(al - ch)\,x_1$, if for $c$ and $h$, their respective values $2a\sin\frac{l}{2a}$ and $a\cos\frac{l}{a}$ be substituted, it becomes

$$\frac{2}{3}a^3\sin\frac{l}{2a} = \frac{2}{3}a^3\sin\frac{l}{2a}\cos^2\frac{l}{2a} + \frac{1}{2}\left(al - 2a^2\sin\frac{l}{2a}\cos\frac{l}{2a}\right)x_1,$$

and as $\cos^2\frac{l}{2a} = 1 - \sin^2\frac{l}{2a}$, $2\sin\frac{l}{2a}\cos\frac{l}{2a} = \sin\frac{l}{a}$, there results

$$\frac{2}{3}a^3\sin\frac{l}{2a} = \frac{2}{3}a^3\sin\frac{l}{2a} - \frac{2}{3}a^3\sin^3\frac{l}{2a} + \frac{1}{2}\left(al - a^2\sin\frac{l}{a}\right)x_1,$$

by obliterating the quantities which are common to both sides of the equation, and dividing by $a$, there results

$$\tfrac{2}{3}a^2\sin^3\frac{l}{2a}=\tfrac{1}{2}\left(l-a\sin\frac{l}{a}\right)x_1$$

when $l=a\pi$, the first member of this equation becomes $\frac{2}{3}a^2$, and the second $\frac{1}{2}\pi a x_1$

(k) By equation (a) of No 73, we have $dy=\dfrac{(a-x)}{\sqrt{ax-x^2}}dx$, and

$xdy=\dfrac{(ax-x^2)}{\sqrt{ax-x^2}}dx=\sqrt{ax-x^2}\,dx$, and $\int x^2dy=\int x\sqrt{ax-x^2}\,dx$

$$=\tfrac{1}{2}a\int\sqrt{ax-x^2}\,dx-\int\left(\frac{a}{2}-x\right)\sqrt{ax-x^2}\,dx,$$

and because $\int\sqrt{ax-x^2}\,dx=\gamma$, we obtain

$$\lambda x_1=\tfrac{1}{2}x^2y-\tfrac{1}{2}\int x^2dy=\tfrac{1}{2}x^2y-\tfrac{1}{4}a\gamma+\tfrac{1}{6}(ax-x^2)^{\frac{3}{2}}$$

(l) Since $\int xydy=\int y\sqrt{ax-x^2}\,dx$, by substituting for $y$ its value, there results

$$\int xydy=\int(ax-x^2)dx+\frac{a}{2}\int z\sqrt{ax-x^2}\,dx=$$

$$\frac{a}{2}x^2-\frac{x^3}{3}+\frac{a}{2}\int z\sqrt{ax-x^2}\,dx$$

(m) $d\gamma=\sqrt{ax-x^2}\,dx=\dfrac{a^2}{4}\dfrac{dx}{\sqrt{ax-x^2}}-\dfrac{\left(\frac{a}{2}-x\right)^2}{\sqrt{ax-x^2}}dx$,

$$\gamma=\frac{a^2}{4}\int\frac{dx}{\sqrt{ax-x^2}}-\left(\frac{a}{2}-x\right)\sqrt{ax-x^2}-\int\sqrt{ax-x^2}\,dx,$$

hence as $\gamma=\int\sqrt{ax-x^2}\,dx$, $z=\int\dfrac{dx}{\sqrt{ax-x^2}}$, there results by substituting $\gamma$ and $z$ for these values, and concinnating

$$2\gamma=\frac{a^2}{4}z-\left(\frac{a}{2}-x\right)\sqrt{ax-x^2},$$

as $dz=\dfrac{dx}{\sqrt{ax-x^2}}$, if the first member of this equation be multiplied by $dz$, and the first term of second member by $dz$, and the second term by its equivalent $\dfrac{dx}{\sqrt{ax-x^2}}$, there results by dividing by two, and integrating $\int\gamma dz=\dfrac{a^2}{16}z^2-\frac{1}{4}(ax-x^2)$

($n$) Now if these values of $\gamma dz$ and $\gamma$ be substituted in equation (6), there results

$$\int z\sqrt{ax-z^2}\,dx = \tfrac{1}{8}a^2z^2 - \frac{z}{2}\left(\tfrac{1}{2}a - x\right)\sqrt{ax-x^2} - \tfrac{1}{16}a^2z^2 + \tfrac{1}{4}(ax - x^2),$$

from which it is easy to obtain equation (7)

($o$) In this case as $dy = \dfrac{-x\,dx}{\sqrt{a^2-x^2}}$, $dy^2 + dx^2 = \dfrac{a^2}{a^2-x^2}dx^2$, and

$$s = 2\pi\int_\alpha^\beta a\,dx,\quad sx_1 = 2\pi\int_\alpha^\beta x\,dx,\qquad s = 2\pi a(\beta-\alpha),\ sx_1 = \pi a(\beta^2-\alpha^2)$$

($p$) By performing the integration, we obtain as $dy = \dfrac{a-x}{\sqrt{ax-x^2}}\,dx$,

$$s = 4\pi y\sqrt{ax} - 4\pi\int\sqrt{ax}\,dy = (-4\pi\sqrt{a}\int\sqrt{a-x}\,dx,$$
$$= 4\pi y\sqrt{ax} + \frac{4\cdot 2}{3}\pi\sqrt{a}(a-x)^{\frac{3}{2}} + c$$

Now as $s = 0$ when $x = 0$, $c$ must be equal to $-\dfrac{4\cdot 2}{3}\pi a^2$, hence the value of $s$ is that given in the text. In like manner we shall have

$$sx_1 = \tfrac{4}{3}\pi\sqrt{a}\,yx^{\frac{3}{2}} - \tfrac{4}{3}\pi\sqrt{a}\int x^{\frac{3}{2}}dy\ (= -\tfrac{4}{3}\pi\sqrt{a}\int x\sqrt{a-x}\,dx)$$
$$\left(\text{as} \int x\sqrt{a-x}\,dx = -\tfrac{2}{3}x(a-x)^{\frac{3}{2}} + \tfrac{2}{3}\cdot\tfrac{2}{5}(a-x)^{\frac{5}{2}}\right),$$
$$sx_1 = \tfrac{4}{3}\pi\sqrt{a}\,yx^{\frac{3}{2}} + \tfrac{4}{3}\cdot\tfrac{2}{3}\pi x\sqrt{a}(a-x)^{\frac{3}{2}} + \tfrac{16}{45}\pi\sqrt{a}(a-x)^{\frac{5}{2}} + c',$$

when $x = 0$, $c' = -\tfrac{4}{9}\cdot\tfrac{4}{5}\pi a^3$, and the value of $sx_1$ is that given in the text

($q$) In this case, the arc of the generating curve whose value is given in the first of equations (8), becomes (as $y$ is changed into $x$) equal to the expression for $s$ given in the text

($r$) The third equation of No 75 is $\lambda y_1 = \frac{1}{2}\int_\alpha^\beta (y^2 - y'^2)dx$, we have $2\pi\lambda y' = \pi\int_\alpha^\beta (y^2 - y'^2)\,dx = v$. This is called the Centrobaric method of determining centres of gravity, it was discovered by Pappus, but from the frequent applications made of it by Guldin, it is more commonly denominated Guldin's method It follows from it, that if several solids are generated by a revolution of surfaces about a common axis, the sum of them all is equal to the circumference described by their common centre of gravity, multiplied into the sum of the generating surfaces

(*s*) In this case we have the following proportion

$$L : 2\pi\lambda y_1 :: l : 2\pi y_1 \quad L = l\lambda$$

(*r*) If the area of the base of the cone $V'$ be denoted by $b$ and the height by $h$, we shall have $V' = \frac{hb}{3}$, in this case $b = \frac{\pi c^2}{4}$, and $h = a - f$ $\quad V' = \frac{1}{4 \cdot 3}\pi c^2(a - f)$

(*s*) As $S$ is equal to $2\pi a(\beta - \alpha)$, (see No 81), in the case of a sphere, $\beta - \alpha =$ the difference between the radius and cosine of the spherical segment, and therefore equal to $f$, the sagitta of this segment. It appears from this expression $2\pi af$, that the entire surface of the sphere is equal to four times the area of a great circle of the sphere, as we know from other considerations should be the case.

(*t*) It appears from No 81, that the distance of the centre of gravity of the surface, generated by the revolution of the arc $A'D'B'$, from the centre, is equal to $\frac{1}{2}(CD' + CE') = \frac{3}{4}CD - \frac{1}{2}D'E' = \frac{3}{4}(a - \frac{1}{2}f)$, for $D'E' = \frac{3}{4}f$

(*u*) If in the equation $\frac{2}{3}\pi a^2 f = \frac{1}{12}\pi c^2(a - f) + V_1$, their values be substituted for $c$ and $f$, there will result

$$\frac{2}{3}\pi a^3\left(1 - \cos\frac{l}{2a}\right) = \frac{1}{12}\pi\, 4a^2\sin^2\frac{l}{2a}\; a\cos\frac{l}{2a} + V_1,$$

$$\frac{2}{3}\pi a^3\left(1 - \cos\frac{l}{2a} - \frac{1}{2}\sin^2\frac{l}{2a}\cos\frac{l}{2a}\right) = V_1,$$

and if these values be substituted in the equation

$$\frac{1}{2}\pi a^2 f\left(a - \frac{1}{2}f\right) = \frac{1}{16}\pi c^2(a - f)^2 + V_1 t_1,$$

we shall have

$$\left(\text{as } a - \frac{f}{2} = \frac{a}{2}\left(1 + \cos\frac{l}{2a}\right),\right)$$

$$\frac{1}{2}\pi a^2\left(1 - \cos\frac{l}{2a}\right)\frac{1}{2}a^2\left(1 + \cos\frac{l}{2a}\right) = \frac{1}{16}\pi\, 4a^2\sin^2\frac{l}{2a}\; a^2\cos^2\frac{l}{2a} + V_1 t_1,$$

that is

$$\frac{1}{4}\pi a^4\sin^2\frac{l}{2a} = \frac{1}{4}\pi a^4\left(\sin^2\frac{l}{2a} - \sin^4\frac{l}{2a}\right) + V_1 t_1,$$

and, if the terms which are common to the two sides of this equation be obliterated, there will result by substituting for $V_1$ its value, the expression for $t_1$ which is given in the text

($x$) Each section of an ellipsoid, made by a plane parallel to one passing through two of the principal axes, is an ellipse, the major and minor axis of which are the ordinates of the ellipsoid corresponding to $x$ the distance of the plane from the centre of the solid, when the cutting plane is parallel to the plane of the axes $b$ and $c$, the axes of the ellipse made by the section are $\frac{b}{a}\sqrt{a^2-x^2}$ and $\frac{c}{a}\sqrt{a^2-x^2}$, for they are evidently the ordinates of the ellipsoid corresponding to the abscissa $x$, and in this case $\text{v}dx = \pi bc\left(dx - \frac{x^2 dx}{a^2}\right)$, the integral of which taken between the limits $\alpha$ and $\beta$, is $\pi bc\left(\beta - \alpha - \frac{\beta^3 - \alpha^3}{3a^2}\right)$ $= \pi bc\ (\beta-\alpha)\left(1 - \frac{\alpha^2 + \alpha\beta + \beta^2}{3a^2}\right)$, because $\beta^3 - \alpha^3 = (\alpha - \beta)\ (\alpha^2 + \alpha\beta + \beta^2)$ In like manner $\text{v}x_1 = \pi bc\int\left(x dx - \frac{x^3 dx}{a^2}\right) =$ (when the integral is taken between the limits $\alpha$ and $\beta$) $\pi bc\left(\frac{\beta^2 - \alpha^2}{2} - \frac{\beta^4 - \alpha^4}{4a^2}\right)$, evidently equal to the value of $\text{v}x_1$ given in the text, from which, by substituting for v its value obtained above, there results the value of $x_1$

($y$) In this value of v, if for $\int y dx$ its value $\int(\sqrt{ax - x^2} + \frac{1}{2}az)dx$ be substituted, we have

$$\text{v} = \pi x y^2 - 2\pi\int(ax - x^2)\,dx - a\pi\int z\sqrt{ax - x^2}\ dx,$$

i e, by integrating and substituting for this last quantity, its value given in No 80,

$$\text{v} = \pi x y^2 - 2\pi\left(\frac{ax^2}{2} - \frac{x^3}{3}\right) - \frac{a\pi a^2 z^2}{16} + \frac{a\pi z}{2}[(\tfrac{1}{2}a - x)\sqrt{ax - x^2} + \tfrac{1}{4}(ax - x^2)]$$

equal when $x = a$ and $y = \frac{\pi a}{2}$,

$$\pi a\frac{\pi^2 a^2}{4} - 2\pi\left(\frac{a^3}{2} - \frac{a^3}{3}\right) - \frac{a^3\pi^3}{16},$$ the value of v when the arc is a semicycloid With respect to the value of $\text{v}x_1$, then as $y = \sqrt{ax - x^2} + \frac{1}{2}az$, we obtain $\text{v}x_1 = \frac{1}{2}\pi x^2 y^2 - \pi\int x(ax - x^2) - \int\frac{1}{2} azx\sqrt{ax - x^2}dx$, in this case we must make $\gamma = \int x\ \sqrt{ax - x^2}dx$ $= \int\sqrt{ax^3 - x^4}dx$, and the value of $\gamma$ may be written as follows

$\gamma = \frac{a'}{4}\int\frac{z\,dz}{\sqrt{az^3-z^4}} - \int\frac{(a_1 z - z^2)^2 dz}{\sqrt{az^3-z^4}}$, the second term can be treated as the corresponding term in Number 80, and the first $=\frac{a^2}{4}\int\frac{z\,dz}{\sqrt{az-z^2}}$, and if we make $\frac{a}{2}-z=x$ we obtain $az-z^2=\frac{a^2}{4}-x^2$ and $z\,dz=\frac{a}{2}dx - x\,dx$, by means of which the value of $\mathrm{v}z_1$ may be obtained

($z$) $\mathrm{M}=\iiint\rho r^2\sin\theta\,dr\,d\theta\,d\psi = \mathrm{A}\iint d\psi\,d\theta\sin\theta = \mathrm{A}(\alpha'-\alpha).\int\sin\theta\,d\theta = \mathrm{A}(\alpha'-\alpha)(\cos\omega-\cos\omega')$, in like manner as $\int\sin\theta\cos\theta\,d\theta=\frac{1}{2}\cos^2\theta$, $=$ (between the limits $\theta=\omega$, and $\theta=\omega'$) $\frac{1}{2}(\cos^2\omega-\cos^2\omega')$, it follows that the value of $\mathrm{M}z_1$ is that given in the text, likewise, as $\int\sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta = -\frac{1}{4}\sin 2\theta+\frac{1}{2}\theta$, $=$ (between the limits $\omega,\omega'$) $-\frac{1}{4}\sin 2\omega'+\frac{1}{4}\sin 2\omega+\frac{1}{2}\omega'-\frac{1}{2}\omega$, it is evident that the value of $\mathrm{M}y_1$ is also that given in the text

($a'$) When M is a complete ring, that is, when $\alpha'=\alpha+2\pi$, it is evident that the factor $\sin\alpha'-\sin\alpha$ in the value of $\mathrm{M}y_1$, and $\cos\alpha-\cos\alpha'$ in the value of $\mathrm{M}x_1$ are respectively cipher, consequently, in this case, $y_1$ and $x_1$ are cipher, and $z_1=\frac{\mathrm{B}}{2\mathrm{M}}(\alpha'-\alpha)(\cos^2\omega-\cos^2\omega')$ becomes by substituting for M its value $=\frac{\mathrm{B}}{2\mathrm{A}}(\cos\omega+\cos\omega')$

($b'$) When $\omega=0$ and $a=0$, then $\mathrm{B}=\frac{1}{4}\rho a'^4$, $\mathrm{A}=\frac{1}{3}\rho a'^3$, consequently, $z_1=\frac{3a'}{8}(1+\cos\omega')$

($c'$) The base of the parallellopided $=r\,d\psi\times r\sin\theta\,d\theta = r^2\sin\theta\,d\theta\,d\psi$, and when $r=1$, the integral of this expression is $-\cos\theta\;\psi$, which taken between the prescribed limits, gives the integral $=2\pi\;2$, in the same way the volume of the sphere is $\frac{r^3}{3}4\pi$

## CHAPTER VI

($a$) The body whose mass is $m$ and volume $v$ being supposed to be condensed into its centre of gravity, $\rho$ the density which is equal to $\frac{m}{v}$ must be that of the centre of gravity, now if the shape of all the molecules be the same we shall have when their number is $n$,

$$v=n\varepsilon^3,\quad \frac{v}{n}=\varepsilon^3$$

(*b*) $u^2 = (\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2$ becomes, by expanding,

$$= \alpha^2 + \beta^2 + \gamma^2 - 2(\alpha x + \beta y + \gamma z) + x^2 + y^2 + z^2,$$

and if we make

$$\alpha^2 + \beta^2 + \gamma^2 = \delta^2 \text{ and } 2(\alpha x + \beta y + \gamma z) - (x^2 + y^2 + z^2) = p,$$

we shall have

$$\frac{1}{u^2} = \frac{1}{\delta^2 - p} \text{ and } \frac{1}{u} = \frac{1}{\delta} + \frac{1}{2}\frac{p}{\delta^3} + \frac{1.3}{2\,4}\frac{p^2}{\delta^5} + \&c.$$

Now if there be substituted for $\delta$ and $p$ their respective values, there results when no powers of $x$, $y$, $z$, higher than the square, are taken into account

$$\frac{1}{u} = \frac{1}{\delta} + \frac{2(\alpha x + \beta y + \gamma z) - (x^2 + y^2 + z^2)}{2\delta^3} + \frac{1\,3}{2\,4}\left(\frac{2(\alpha x + \beta y + \gamma z)}{\delta}\right)^2,$$

which is evidently equal to the value of $\frac{1}{u}$ given in the text; and if this value be substituted in the expression $T = \iiint \frac{dm}{u}$ it is evident, that when the origin of the coordinates is at the centre of gravity, the second term of its value given in the text will vanish. It likewise appears, that when OD or $\delta$ is so great that the value of T is reduced to its first term, then $\frac{dT}{d\alpha} = -\frac{M\alpha}{\delta^3}$ and $A = \frac{\mu M/\alpha}{\delta^3}$.

(*c*) A threefold integration is to be performed relatively to $r$, $\theta$, $\psi$, with respect to $\psi$, as neither this quantity nor any function of it occurs under the sign of integration, the integration relative to $d\psi$ may be effected by writing $2\pi$ without the sign $\int$, for it should be integrated between the limits $0, 2\pi$, and then there only remains the double integration to be performed relative to $\theta$ and $r$. With respect to $\theta$, the radical $\sqrt{\alpha^2 - 2\alpha r \cos\theta + r^2}$, becomes, when $\theta = \pi$, $\sqrt{\alpha^2 + 2\alpha r + r^2}$, and when $\theta = 0$, $\sqrt{\alpha^2 - 2\alpha r + r^2}$, now as the indefinite integral relative to $\theta$ is $\frac{1}{\alpha}\sqrt{\alpha^2 - 2\alpha r \cos\theta + r^2} + \text{const}$ when taken between the limits $\theta = \pi$ and $\theta = 0$, it becomes, when the attracted point is within the stratum,

$$\frac{1}{\alpha}[(r + \alpha) - (r - \alpha)] = 2,$$

consequently in this case the value of T will not depend on $\alpha$, and

$A = -\mu f \frac{dr}{da} = 0$ When the point is without the stratum the value of $\frac{1}{\alpha}\sqrt{\alpha^2 - 2\alpha r\cos\theta + r^2}$, taken between the same limits $\pi$ and 0, is

$$\frac{1}{\alpha}[(\alpha + r) - (\alpha - r)] = \frac{2r}{\alpha},$$

if this value be substituted in the expression for T, there results

$$T = \frac{4\pi\rho}{\alpha}\int_b^a r^2 dr,$$

which, integrated between the limits $a$ and $b$, gives

$$T = \frac{4\pi\rho}{3\alpha}(a^3 - b^3),$$

and as the masses of the spheres, whose radii are $a$ and $b$, are respectively $\frac{4\pi\rho a^3}{3}$, $\frac{4\pi\rho b^3}{3}$, $4\pi\rho\frac{(a^3 - b^3)}{3} = M$ is equal to the difference of the masses of these spheres.

(*d*) $A = \mu f \iiint \frac{a - t}{u^3} dm = \left(\text{as } dm = \rho u^2 du d\omega,\ \cos g = \left(\frac{t - \omega}{u}\right)\right)$ $\mu f \iiint \rho \cos g\, du d\omega$, and when $\rho$ is constant, by integrating with respect to $du$, from $u = 0$ to $u = r$, we obtain $A = \mu f\, \rho \iint r \cos g\, d\omega$

(*e*) By making this substitution we obtain

$$\left(\frac{\alpha + r\cos g}{a}\right)^2 + \left(\frac{\beta + r\cos h}{b}\right)^2 + \left(\frac{\gamma + r\cos k}{c}\right)^2 = 1,$$

this becomes by developing and concinnating

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 2\left(\frac{\alpha\cos g}{a^2} + \frac{\beta\cos h}{b^2} + \frac{\gamma\cos k}{c^2}\right) r +$$

$$r^2\left(\frac{\cos^2 g}{a^2} + \frac{\cos^2 h}{b^2} + \frac{\cos^2 k}{c^2}\right) = 1,$$

which, by substituting $l$, $p$, $q$ for their respective values, becomes the expression in the text

(*f*) Since the parts of the value of $r$ under the radical sign mutually destroy each other, there remains only the quantity $\frac{-q}{p}$ to be taken into account, and if for $q$ its value be substituted, there results the expression in the text

(*g*) If $p$ and its value given above, be multiplied by $a^2b^2c^2$, there will result by substituting for $\cos g$, $\cos h$, $\cos k$, their values, in terms of the polar coordinates, the expression in text ; and if in value of A the expressions for $d\omega$ and $\cos g$ be substituted, we obtain this value of A

(*h*) When $\psi = \frac{\pi}{2}$, $\phi (= \tan\psi)$ is $\infty$, hence when $\frac{d\psi}{p}$ is integrated between 0 and $\frac{\pi}{2}$, the equivalent expression

$$a^2b^2c^2 \frac{d\phi}{(b^2\cos^2\theta + a^2\sin^2\theta)\,c^2 + (c^2\cos^2\theta + a^2\sin^2\theta)\,b^2\phi^2}$$

must be integrated between the limits 0 and $\infty$. It is easy to show that the two expressions are equivalent, for if there be substituted for $d\psi$ and $p$ their respective values, namely,

$$\frac{d\phi}{1+\phi^2} \text{ and } \frac{b^2c^2\cos^2\theta + (c^2\cos^2\psi + b^2\sin^2\psi)\,a^2\sin^2\theta}{a^2b^2c^2},$$

and also for $\cos^2\psi$ its value $\frac{1}{1+\phi^2}$, and $\frac{\phi^2}{1+\phi^2}$ for $\sin^2\psi$, there results the second expression given above.

Now if $(b^2\cos^2\theta + a^2\sin^2\theta)c^2 = m$, and $(c^2\cos^2\theta + a^2\sin^2\theta)b^2 = n$,

$$\text{then } a^2b^2c^2 \int_0^\infty \frac{d\phi}{(b^2\cos^2\theta + a^2\sin^2\theta)\,c^2 + (c^2\cos^2\theta + a^2\sin^2\theta)\,b^2\phi^2} =$$

$$a^2b^2c^2 \int_0^\infty \frac{d\phi}{m + n\phi^2},$$

the integral of this last expression is equal to

$$\frac{a^2b^2c^2}{\sqrt{mn}} \tan^{-1} \phi \sqrt{\frac{n}{m}},$$

which, when taken between the prescribed limits, becomes, by substituting for $m$ and $n$ their values, the integral given in the text

(*i*) That is, if there be taken any two points in the ellipsoid, the force acting on these points, resolved parallel to an axis, is proportional to the ordinates of the respective points

(*k*) Since $b = c$, the part under the radical in the value of A is equal to

$$a^2\cos^2\theta + a^2e^2\cos^2\theta + a^2\sin^2\theta = a^2(1 + e^2\cos^2\theta),$$

and by substituting this value in A, we obtain

$$\text{A} = 4\pi\mu f\rho a\int_0^{\frac{1}{2}\pi}\frac{b^2\cos^2\theta\sin\theta d\theta}{a^2(1+e^2\cos^2\theta)},$$

and as

$$b^2 = a^2(1+e^2) \text{ and } 4\pi\rho = \frac{3m}{a^3(1+e^2)},$$

it is evident, when these values are put for $b^2$ and $4\pi\rho$ in the value of A, that the expression given in the text will be obtained

Now in order to integrate this expression, let $e^2\cos^2\theta = y^2$, then $\frac{\cos^2\theta\sin\theta d\theta}{1+e^2\cos^2\theta} = -\frac{y^2dy}{e^3(1+y^2)}$, of which the integral is

$$-\frac{1}{e}(y - \text{arc tang} = y) = \frac{1}{e^3}(e\cos\theta - \text{arc tang} = e\cos\theta),$$

and this expression, when taken between the limits 0 and $\frac{\pi}{2}$, becomes $\frac{1}{e^3}(e - \text{arc tang} = e)$, hence we obtain at once the value of A

($l$) If we suppose as in the last note, that $e\cos\theta = y$, then we have

$$\frac{3\mu fma'}{a^3(1+e^2)}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\sin\theta d\theta}{\sqrt{1+e^2\sin^2\theta}} = \frac{3\mu fma'}{a^3(1+e^2)}\int_0^{\frac{\pi}{2}}\frac{y^2dy}{e^3\sqrt{1+e^2-y^2}}$$

Now the integral of $\frac{y^2dy}{\sqrt{1+e^2-y^2}}$ is evidently equal to

$$-\frac{y}{2}\sqrt{1+e^2-y^2}+\frac{1+e^2}{2}\text{arc sin} = \frac{y}{\sqrt{1+e^2}}\left(=\frac{1+e^2}{2}\text{arc tang} = \frac{y}{\sqrt{1+e^2-y^2}}\right),$$

and if for $y$ its value $e\cos\theta$ be substituted, and if the integral be then taken between the limits $\theta = 0$, $\theta = \frac{\pi}{2}$, it becomes

$$-\frac{e}{2}+\frac{1+e^2}{2}\text{arc tang} = e, \text{ and } \frac{3\mu fma'}{a^3(1+e^2)}\int_0^{\frac{\pi}{2}}\frac{y^2dy}{e^3\sqrt{1+e^2-y^2}} =$$

$$\frac{3\mu fma'}{a^3e^3(1+e^2)}\left(-\frac{e}{2}+\frac{1+e^2}{2}\text{arc tang} = e\right),$$

which when concinnated is the expression in the text

($m$) By substituting for arc tang $= e$ we obtain

$$\text{A} = \frac{3f\mu ma}{a^3e^3}\left(e - e + \frac{e^3}{3} - \frac{e^5}{5} + \&c\right) = \frac{\mu fma}{a^2}\left(1 - \frac{3e^2}{5} + \&c\right),$$

and

$$v' = \frac{3\mu fma'}{2a\,e}\left(e - \frac{e^3}{3} + \frac{e^5}{5} - \&c - e + e^3 - e^5 + \&c\right),$$

equal, by dividing as before the terms of the series by $e^3$

$$\frac{\mu fma'}{a^3}\left(1 - \frac{6e^2}{5} + \&c\right)$$

(*n*) If in the value of A there be substituted for $x_1$, $y'$, $z'$, $dy'$, $dz'$, their values in terms of $\theta$ and $\psi$, it is evident that when in this expression, $\pi - \theta$ is put for $\theta$, it will coincide with the first term, so that if the second be integrated between 0 and $\frac{\pi}{2}$, the result is the same as if the first was integrated between $\frac{\pi}{2}$ and $\pi$

(*o*) $\alpha^2 = a_1^2\cos^2 p$, $\beta^2 = b_1^2\sin^2 p\cos^2 q = a_1^2\sin^2 p\cos^2 q + h\sin^2 p\cos^2 q$,
$\gamma^2 = c_1^2\sin^2 p\sin^2 q = a_1^2\sin^2 p\sin^2 q + k\sin^2 p\sin^2 q$,
$a^2\cos^2\theta + b^2\sin^2\theta\cos^2\psi + c^2\sin^2\theta\sin^2\psi$,
= (by substituting for $b^2$ and $c^2$ their respective values, namely, $a^2 + h$, $a^2 + k$,) $a^2\cos^2\theta + a^2\sin^2\theta + h\sin^2\theta\cos^2\psi + k\sin^2\theta\sin^2\psi$,
hence by concinnating these expressions, and substituting for $\alpha, \beta, \gamma$, their respective values $a_1\cos p$, $b_1\sin p\cos q_1$, $c_1\sin p\sin q$, we obtain the value of $R^2$ given in the text

(*p*) If $u$ be zero, the first member of equation (4) is reduced to $\alpha^2$, and the second to zero, and if $u$ be $\infty$ the first member becomes $\alpha^2 + \beta^2 + \gamma^2$, while the second member is $\infty$, consequently in the passage from zero to infinity there must be a value of $u$, which renders the first member equal to the second

(*q*) This follows at once from the equation $A_1bc = Ab_1c_1$

# BOOK II

## CHAPTER I

(a) We have

$$\frac{k}{h} = \frac{v}{u} \quad \text{as } v = iu,\ k = ih$$

## CHAPTER II

(a) $\phi = \frac{dv}{dt} = g\frac{(k^2 - v^2)}{k^2}$ $\quad \frac{k^2 dv}{k^2 - v^2} = g\,dt = \frac{k}{2}\left(\frac{dv}{k+v} + \frac{dv}{k-v}\right),$

$$gt = \frac{k}{2}\left[\log(k+v) - \log(k-v)\right] = \frac{k}{2}\log\left(\frac{k+v}{k-v}\right)$$

(b) If both sides of the equation

$$\frac{k-v}{k+v} = e^{\frac{-2gt}{k}}$$

be multiplied by $e^{\frac{gt}{k}}$, there results

$$\left(\frac{k-v}{k+v}\right)e^{\frac{gt}{k}} = e^{\frac{-gt}{k}},\ \text{and} \quad (k-v)\,e^{\frac{gt}{k}} = (k+v)\,e^{\frac{-gt}{k}},\ \text{and}$$

$$k\left(e^{\frac{gt}{k}} - e^{\frac{-gt}{k}}\right) = v\left(e^{\frac{gt}{k}} + e^{\frac{-gt}{k}}\right),$$

hence we obtain the value of $v$ given in the text

(c) If both members of the equation $v = \dfrac{e^{\frac{gt}{k}} - e^{\frac{-gt}{k}}}{e^{\frac{gt}{k}} + e^{\frac{-gt}{k}}}$ be multiplied by $dt$, and then integrated, we shall obtain

$$x = \int v\,dt = \frac{k^2}{g}\log\left(e^{\frac{gt}{k}} + e^{\frac{-gt}{k}}\right) + \text{const},$$

and as by hypothesis $x = 0$, when $t = 0$,

$$\text{const} = -\frac{k^2}{g}\log(e^0 + e^{-0}) = -\frac{k^2}{g}\log 2 = \frac{k^2}{g}\log\tfrac{1}{2},$$

hence it is evident, that the value of $x$ is that given in the text

(d) $g\,dx = vg\,dt = \frac{k^2 v\,dv}{k^2 - v^2}$ $\quad x = -\frac{k^2}{2g}\log\frac{k^2 - v^2}{k^2} = \frac{k^2}{2g}\log\frac{k^2}{k^2 - v^2}$

(*e*) From note (*d*) it appears that

$$x = \frac{k^2}{g} \log \left( e^{\frac{gt}{k}} + e^{\frac{-gt}{k}} \right) - \frac{k^2}{g} \log 2,$$

which when $e^{\frac{-gt}{k}}$ is neglected, becomes, as $\log e^{\frac{gt}{k}} = \frac{gt}{k}$, the value of $x$ given in the text

(*f*) It appears from No 124, that the accelerating force varies (every thing else being the same) as $\frac{1}{k^2}$, but as in this case it varies as $\frac{1}{\text{mass}}$, it follows that $\sqrt{\text{mass}}$ varies as $k$, hence as $k$ is the uniform velocity to which the body continually approximates, the greater the mass or the density, the greater will be this final velocity

(*g*) Generally $\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \&c$ hence as

$$\tfrac{1}{2}\left( e^{\frac{gt}{k}} + e^{\frac{-gt}{k}} \right) = 1 + \frac{g^2t^2}{2k^2} + \frac{g^4t^4}{1\ 2\ 3\ 4k^4},\ \log \tfrac{1}{2}\left( e^{\frac{gt}{k}} + e^{\frac{-gt}{k}} \right) =$$

$$\frac{g^2t^2}{2k^2} + \frac{g^4t^4}{1\ 2\ 3.4k^4} - \frac{g^4t^4}{8k^4} + \&c = \frac{g^2t^2}{2k^2} - \frac{g^4t^4}{12k^4} + \&c$$

(*h*) By integrating $\frac{kdv}{k^2+v^2} = -\frac{gdt}{k}$, we obtain, as $v = a$ when $t = 0$, i e, at the commencement of the motion,

$$\text{arc}\left(\text{tang} = \frac{v}{k}\right) - \text{arc}\left(\text{tang} = \frac{a}{k}\right) = -\frac{gt}{k},$$

hence from the known expression for the tangent of the difference of two arcs we have

$$\frac{ak - vk}{k^2 + av} = \text{tang}\ \frac{gt}{k} = \frac{\sin\frac{gt}{k}}{\cos\frac{gt}{k}},$$

$$k\left(a \cos\frac{gt}{k} - k \sin\frac{gt}{k}\right) = v\left(a \sin\frac{gt}{k} + k \cos\frac{gt}{k}\right),$$

from which it is easy to obtain the value of $v$ given in the text

(*i*) The integral of the differential equation

$$dx = vdt = \frac{k\left(a \cos\frac{gt}{k} - k \sin\frac{gt}{k}\right) dt}{a \sin\frac{t}{k} + k \cos\frac{t}{k}}$$

is

$$x=\frac{k^2}{g}\left(\log\ \left(a\sin\frac{gt}{k}+k.\cos\frac{gt}{k}\right)\right)+\text{const},$$

when $t=0$, and $x=0$ this expression then becomes

$$\frac{k^2}{g}\ \log\ k+\text{const}\ =0,$$

consequently,

$$\text{const}\ =-\frac{k^2}{g}\ \log\ k,$$

hence as $-\frac{k^2}{g}\ \log k=\frac{k^2}{g}.\log\ \frac{1}{k}$, it is easy to infer the value of $x$ given in the text

($k$) By integrating we obtain

$$x=\quad\frac{k^2}{2g}\log\ \ (k^2+v^2)+\text{const},$$

when $x=0$, then $v=a$, hence

$$\text{const.}=\frac{k^2}{2g}\ \log(k^2+a^2),$$

and, consequently, by substituting this expression for const, we obtain the value of $x$ given in the text

($l$) By expanding into a series the value of $v$ given above, we obtain

$$v=\frac{k\left[a\left(1-\frac{g^2t^2}{2k^2}+\ \&c\right)-k\left(\frac{gt}{k}-\ \frac{g^3t^3}{1\ 2\ 3\ k^3}+\&c.\right)\right]}{a\left(\frac{gt}{k}-\frac{g^3t^3}{1\ 2\ 3\ k^3}\ +\ \&c\right)+k\ \left(1-\frac{g^2t^2}{2k^2}+\&c\right)};$$

this expression becomes, when $\frac{1}{k}=0$,

$$k\left(\frac{a-gt}{k}\right)=a-gt$$

And by a like expansion of the expression for $x$, we obtain

$$x=\frac{k^2}{g}\ \log\left[\frac{a}{k}\left(\frac{gt}{k}-\frac{g^3t^3}{1.2\ 3\ k^3}+\&c.\right)+1-\frac{g^2t^2}{1\ 2k^2}+\frac{g^4t^4}{1.2\ 3\ 4k^4}-\&c.\right],$$

and as generally

$$\log(1+u)=\frac{u}{1}\ -\frac{u^3}{3}+\frac{u^5}{5}-\&c.,$$

this value of $x$ becomes, when $\frac{1}{k}=0$,

$$\frac{k^2}{g}\left(\frac{a}{k}\frac{gt}{k} - \frac{g^2t^2}{1.2\,k^2}\right) = at - \frac{gt^2}{2},$$

all the higher powers being neglected as being equal to cipher

(*m*) In this case as $v = 0$, we have evidently

$$\text{arc}\left(\text{tang} = \frac{a}{k}\right) = \frac{g\theta_1}{k}, \text{ and } \quad \theta_1 = \frac{k}{g} \text{ arc } \left(\text{tang} = \frac{a}{k}\right)$$

(*n*) When the numerator and denominator are multiplied by this quantity, we obtain

$$\theta' = \frac{k}{2g}\log \frac{k^2}{(\sqrt{a + k^2} - a)^2},$$

which is evidently equal to $\frac{k}{g}$ log $\frac{k}{\sqrt{a^2 + k^2} - a}$

(*o*) In fact, as $k^2 = \frac{D\imath}{\gamma\rho}$, and $k'^2 = \frac{D\imath'}{\gamma\rho}$, we have

$$k^2 \quad k'^2 \quad \imath \quad \imath'$$

(*p*) $\imath\, dt = dx = \frac{1}{k}\left(\sqrt{ak} - gt\right)^2 dt \qquad \imath = \frac{-1}{3gk}\left(\sqrt{ak} - gt\right)^3 +$ const, and since $x = 0$, when $t = 0$,

$$\frac{1}{3gk}\left(\sqrt{ak}\right)^3 = \text{const},$$

consequently,

$$\imath = \frac{a\sqrt{ak}}{3g} + \frac{1}{3gk}\left(gt - \sqrt{ak}\right)^3,$$

and as $v = \frac{1}{k}\left(\sqrt{ak} - gt\right)^2$ the sign of the second member is always the same, direction of motion is the same as before

(*q*) $x = c\cos t\sqrt{\frac{g}{\alpha}}$, when $x = 0$, $t\sqrt{\frac{g}{\alpha}} = \frac{\pi}{2}$, and $t = \frac{\pi}{2}\sqrt{\frac{\alpha}{g}}$

(*r*) The integral of the value of $\frac{d^2x}{dt^2}$ is in this case

$$\frac{1}{2}\frac{dx^2}{dt^2} = g\imath^2.\left(\frac{1}{\imath + h - x}\right) + c$$

Now as $\frac{dx}{dt} = 0$ when $x = 0$, we must have $c = -\frac{2g\imath^2}{\imath + h}$, and when $x = h$, the expression for $\frac{dx^2}{dt^2}$ becomes

$$2gr^2\left(\frac{1}{r}-\frac{1}{r+h}\right)=2gr^2\cdot\frac{h}{r\,(r+h)}=2gh\;\frac{r}{r+h}$$

(s) By reducing to a common denominator we obtain

$$\frac{dx^2}{dt^2}=2gr^2\frac{r}{(r+h)\,(r+h-x)}\qquad dx^2\frac{(r+h-x)}{x}=\frac{2gr^2\,dt^2}{r+h},$$

and if numerator and denominator of the first member of this equation be multiplied by $r+h-x$, and then the square root be taken, we shall obtain the expression in the text

(t) Since

$$\frac{r+h-2x}{2\sqrt{(r+h)\,x-x^2}}\;dx+\frac{r+h}{2\sqrt{(r+h)\,x-x^2}}\;dx=\frac{r+h-x}{\sqrt{(r+h)x-x^2}}\cdot dx$$
$$=\sqrt{\frac{2gr^2}{r+h}}\;dt,$$

we shall have by integrating

$$\sqrt{(r+h)\,x-x^2}+\tfrac{1}{2}\frac{(r+h)}{r^2}\,\text{arc}\left(\cos=\frac{r+h-2x}{r+h}\right)=\sqrt{\frac{2gr^2}{r+h}}\;t,$$

and because

$$\sqrt{1-\left(\frac{r+h-2x}{r+h}\right)^2}=2\sqrt{\frac{(r+h)x-x^2}{r+h}},$$

it is evident that

$$\text{arc}\left(\cos=\frac{r+h-2x}{r+h}\right)=\text{arc}\sin\frac{2\sqrt{(r+h-x^2}}{r+h},$$

therefore we shall have

$$t\sqrt{\frac{2gr^2}{r+h}}=\sqrt{(r+h)x-x^2}+\tfrac{1}{2}(r+h)\text{arc}\left(\sin=\frac{2\sqrt{(r+h)x-x^2}}{r+h}\right),$$

now as the sine is very small, it may be assumed equal to the arc, consequently

$$t\sqrt{\frac{2gr^2}{r+h}}=\sqrt{(r+h)\,x-x^2}+\tfrac{1}{2}\,(r+h)\cdot\frac{2\sqrt{(r+h)\,x-x^2}}{r+h}=$$

$2\sqrt{(r+h)\,x-x^2}=2\sqrt{(r+h)-x)\,x}=$ (when $r+h-x$ is reduced to $r$) to $2\sqrt{rx}$

(u) Since $\frac{dz}{dt}=0$ and $z=h$, we have by formula (c)

$$f^2=k^2=\frac{2b^2}{c-\alpha}-\frac{2b^2}{c-h}+\frac{2a^2}{\alpha}-\frac{2a^2}{h}.$$

Now as

$$c-h=\frac{cb}{a+b},\quad \frac{2b^2}{c-h}=\frac{2b\,(a+b)}{c},\ \text{and}\ \frac{2a^2}{h}=2a\frac{(a+b)}{c},$$

and, consequently,

$$\frac{2b^2}{c-h}+\frac{2a^2}{h}=\frac{2a\,(a+b)+2b\,(a+b)}{c}=\frac{2\,(a+b)^2}{c};$$

hence we obtain

$$f^2=\frac{2b^2}{c-a}+\frac{2a^2}{a}-\frac{2\ (a+b)^2}{c}.$$

(*v*) $$h=\frac{ac}{a+b}=\frac{ac}{a\,(1+\sqrt{75})}=\frac{c}{1+\sqrt{75}}.$$

(*w*) This is the case, because the mean motion of the moon in its orbit is equal to its motion about its axis. See Vol II No 433, note

(*x*) The projectile describes a *curve*, because it is acted on by dissimilar forces See No 144, note

(*y*) From the equation

$$(2a^2-2b^2+c\gamma)^2=8a^2c\gamma,$$

we obtain by solving it for $\gamma$,

$$\gamma-2a\sqrt{\frac{2\gamma}{c}}=2\frac{(b^2-a^2)}{c},\ \text{and}\qquad \sqrt{\gamma}-\sqrt{\frac{2}{c}}\ a=\pm\sqrt{\frac{2}{c}}b,$$

consequently,

$$\gamma=\frac{2}{c}(a\pm b)^2;$$

and if we substitute $8a^2c\gamma$ for $(2a^2-b^2+c\gamma)^2$ in the expression under the radical, it becomes

$$\sqrt{2a^2c-2a\sqrt{2c\gamma}\,z+\gamma z^2}=\sqrt{(a\sqrt{2c}-z\sqrt{\gamma})^2}=a\sqrt{2c}-z\sqrt{\gamma},$$

now, if for $\sqrt{\gamma}$, its value $\sqrt{\frac{2}{c}}\cdot(a\pm b)$, be put, the expression becomes $a\sqrt{2c}-(a\pm b)z\sqrt{\frac{2}{c}}$, from which it is easy to perceive how the denominator in the expression for $dt$, may be rendered rational.

(*z*) See equation (c) of No 139

(*a'*) If $2a^2=k^2a$, the value of $z$, when velocity is cipher, is $\infty$, and, consequently, in this case, a body projected at the distance $a$ from the centre with a velocity $a^2=\frac{k^2a}{2}$ will ascend to $\infty$,

where its velocity becomes cipher; if $2a^2 \angle k^2\alpha$ then the body, when it has attained an infinite distance, has still a finite velocity equal to $k^2\alpha - 2a^2$, it is evident also, that when $2a^2 > k^2\alpha$, that $z$ can never become infinite.

If a body falls freely without any initial velocity from a distance $\alpha$, towards a centre of force varying inversely as the square of the distance, $\frac{dz^2}{dt^2}$ the square of the velocity, is $= 2a^2\left(\frac{1}{z} - \frac{1}{\alpha}\right)$, if $\alpha$ is $\infty$, then $\frac{dz^2}{dt^2} = \frac{2a^2}{z}$, now it is evident from what will be established in No. 239, that $a^2 =$ the square of the velocity in circle at unit of distance from the centre of force, and $\frac{a^2}{z} =$ the square of the velocity of a circle whose radius $= z$, consequently, we have, when a body falls from an infinite distance to the centre of force, the velocity at any point of the descent, to the velocity in a circle at the same distance $\sqrt{2} : 1$. Hence, if a body be projected from the centre of force with a velocity which is to that in a circle at the same distance $\sqrt{2} : 1$, the body will ascend to an infinite distance. This indeed is evident from the expression $2a^2 = k^2\alpha$, or $k^2 = \frac{2a^2}{\alpha}$, for $\frac{a^2}{\alpha}$ is the square of the velocity of a circle at distance $\alpha$, and the converse of this is true, namely, if a body falls freely from an infinite distance, when it arrives at the distance $\alpha$ from centre, it will have acquired the velocity $k$. If $2a^2$ is $>$ than $k^2\alpha$, then $z$ the distance at which the body is arrested $= \frac{2a^2\alpha}{2a^2 - \alpha k^2}$.

## CHAPTER III.

(a) If the forms of the functions $ft, f't, f''t$ are all the same, so that

$$x = aft, \quad y = bft, \quad z = cft,$$

it is evident that the point will move in a right line, and

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad \frac{c}{\sqrt{a^2+b^2+c^2}},$$

express the cosines of the angles which the direction of this line

makes with $x$, $y$, $z$. The constant quantities $a$, $b$, $c$ depend on the nature of the function $ft$, if, for example, $ft = t$, then $a$, $b$, $c$ represent the uniform velocities parallel to $x$, $y$, $z$, and the uniform velocity of the point $= \sqrt{a^2 + b^2 + c^2}$, if $ft = t^2$, then $a$, $b$, $c$ are proportional to the accelerating forces parallel to $x$, $y$, $z$, and the point will be moved with a motion uniformly accelerated, and represented by $\sqrt{a^2 + b^2 + c^2}$.

If $x = aft + bf't$, $y = cft + df't$, $z = eft + gf't$, the point will move in a curved line, which however is of *single* curvature, for by eliminating $t$, we obtain an equation of the form $a'x + b'y + c'z$, which is the equation of a plane. The simplest case of this form is $x = at + bt^2$, $y = ct + dt^2$, $z = et + gt^2$, and eliminating $t$ between the two first eqnations, we obtain an equation of the second order between $x$ and $y$, which from the relation that exists between the three first terms, is evidently that of a parabola.

If, as in text, $x = ft$, $y = f't$, $z = f''t$, the curve described will be of double curvature.

$$(b)\ v = \frac{ds}{dt},\ \cos a = \frac{dx}{ds}, \qquad v \cos a = \frac{ds}{dt}\,\frac{dx}{ds} = \frac{dx}{dt} = p$$

$$(c)\ \omega = v\,dt = \frac{ds}{dt}dt,\ \cos\alpha = \frac{dx}{ds},\ \therefore\ \omega\cos\alpha = \frac{dx}{ds}\,\frac{ds}{dt}\,dt =$$
$\frac{dx}{dt}dt = dx = p\,dt$, by equations (1), in like manner

$$\varepsilon\cos a = \tfrac{1}{2}u\cos a\,dt = \left(\text{as } u = \frac{p'}{\cos a}\right)\tfrac{1}{2}p'dt,$$

consequently

$$\omega\cos\alpha + \varepsilon\cos a = p\,dt + \tfrac{1}{2}p'dt = x' - x$$

($d$) If each of the three preceding equations be squared, and then added together, there results

$$(x' - x)^2 + (y' - y)^2 + (z' - z)^2 = \omega^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) +$$
$$\varepsilon^2(\cos^2 a + \cos^2 b + \cos^2 c) + 2\,\omega\varepsilon(\cos\alpha\cos a + \cos\beta\cos b + \cos\gamma + \cos c),$$

and it is evident that the factor of $\omega\varepsilon$ is equal to the cosine of the angle contained between MH and MK.

($e$) Let $\delta$ denote the infinitely small angle TMM$'$, then the component of $v$ in the direction MM$' = v\cos\delta$, $= v\left(1 - \frac{\delta^2}{1.2} + \&c\right)$, and as by

hypothesis $\delta$ is infinitely small, its square and higher powers may be neglected, and this renders the component of $v$ in the direction $MM' = v$

($f$) See *Harte's Translation of the System of the World*, Book 2, chapter III, and *Mechanique Celeste*, Book 10, No 15 When the body is let fall freely at the equator, the deviation is — $l$ to $h^{\frac{3}{2}}$, $h$ being the height of the tower, for if $r$ be the radius of equator, and $a$, $a'$, the arcs described by the bottom and top of the tower during the fall, we have

$$a \quad a' \quad r \quad r+h \quad a'-a=\frac{ah}{r},$$

now $a$ is — $l$ to the time of the fall, and it varies as $\sqrt{h}$, consequently $a'-a$, or the deviation is — $l$ to $h^{\frac{3}{2}}$, for $r$ is constant

($g$) In equation (d) by substituting $\frac{ds^2}{dt^2}$ for $v^2$, and multiplying both sides by $dt^2$, we obtain

$$\frac{ds^2}{dt^2}dt^2 (= ds^2) = k^2 + 2F(x, y, z) - 2F(a, b, c)\,dt^2.$$

$$dt = \frac{ds}{\sqrt{k^2 + 2F(x, y, z) - 2F(a, b, c)}},$$

in which by substituting for $x$, $y$, $z$, their values in functions of $s$, we obtain the expression given in the text

($h$) When the elevation of the point A above the horizontal plane passing through D and C is equal to $h$, the general expression for the velocity at this point, namely, $\sqrt{2g(h \pm z)}$ becomes then $2g\ (h-h)$, i e $\frac{ds}{dt} = \sqrt{2g\ h-h}$, and $dt = \frac{ds}{\sqrt{2g\ (h-h)}}$, which is infinite, but if the elevation of this point above the plane is less than $h$, then the velocity after passing this point becomes negative, which indicates that the moveable must descend towards the point of departure, where it is in precisely the same circumstances as at the commencement of the motion —See No 181

($i$) As $\cos\lambda = V\frac{dL}{dx}$, $\cos\mu = V\frac{dL}{dy}$, $\cos\nu = V\frac{dL}{dz}$, there results by multiplying equations (3) by $\delta x$, $\delta y$, $\delta z$ respectively, and then adding them together, the expression in the text, for

$$\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z = X\delta x + Y\delta y + Z\delta z + N(\cos\lambda\,\delta x + \cos\mu\,\delta y$$

$+ \cos\nu\,\delta z)$ i e (by substituting for $\cos\lambda$, $\cos\mu$, $\cos\nu$,)

$$\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z = X\delta x + Y\delta y + Z\delta z +$$

$$NV\left(\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z\right)$$

($l$) $\delta ds = \frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta dy + \frac{dz}{ds}\delta dz$. Multiplying both sides by $v = \frac{ds}{dt}$, and observing that $\delta dx = d\delta x$, &c , we obtain

$$v\,\delta ds = \frac{dx}{ds}\frac{ds}{dt}d\delta x + \frac{dy}{ds}\frac{ds}{dt}d\delta y + \frac{dz}{ds}\frac{ds}{dt}d\delta z,$$

which $\left(\text{as } \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{dt}, \text{ \&c}\right)$, becomes the expression in the text

($n$) If $r$ be infinitely small, then $e^{-\frac{r}{a}} = 1$, consequently the expression of the force is then equal to A, on the other hand, when $r$ is a considerable multiple of $\alpha$, $e^{-\frac{r}{a}}$ is cipher

($k$) When $z$ is considerable, the expression for $\frac{dz}{dt}$ the velocity of a body that has been projected from the surface of a sphere is, by No 143,

$$\frac{dz}{dt} = \sqrt{k^2 - 2g\alpha},$$

∴ in this case

$$30950^m = \sqrt{k^2 - 2g\alpha},$$

and as $g = 27\tfrac{1}{2}$ the terrestrial gravity G, and $\alpha = 110\,r$, $r$ denoting the radius of the earth, we obtain

$$k^2 = (30950^m)^2 + \underline{55\,G\;110\,r},$$

by means of which it is easy to compute the diminution in the text

## CHAPTER IV

($a$) The velocity along the side MM$'$ $= v\cos\delta$, therefore the velocity lost is equal to $v\,(1 - \cos\delta) = v\left(\frac{\delta^2}{2} - \frac{\delta^4}{1\,2\,3\,4} + \text{\&c}\right)$, which is evidently an infinitely small quantity of the second order

(*b*) By substituting for $\frac{dx^2}{dt^2}$ its value $\frac{dx^2}{ds^2}\frac{ds^2}{dt^2}$, and for $\frac{d\frac{dy}{dx}}{dt}$ its value $\frac{d\frac{dy}{dx}}{ds}\frac{ds}{dt}$, we obtain $\left(\text{as } v = \frac{ds}{dt} \text{ and } d\frac{dy}{dx} = \frac{d^2y\,dx - d^2x\,dy}{dx^2}\right)$

$$\frac{d^2y\,dx - d^2x\,dy}{dt^2} = v^2\frac{dx^2}{ds^2}\frac{d\frac{dy}{dx}}{ds}ds, \therefore \frac{d^2y\,dx - d^2x\,dy}{ds.dt^2} = v^2\left(\frac{d^2y\,dx - d^2x\,dy}{ds^3}\right)$$

(*c*) The value of $x - x'$ given in No. 20, is

$$x - x' = \frac{\rho^2}{ds^4}\left(dy(dx\,d^2y - dy\,d^2x) - dz(dz\,d^2x - dx\,d^2z)\right),$$

now if for $dx\,d^2y - dy\,d^2x$, $dz\,d^2x - dx\,d^2z$, we substitute their values, which may be deduced from equations (2), there results

$$x - x' = \rho\cos\gamma = \frac{\rho^2}{ds^4}\frac{ds^3}{v^2}\Big[Q\Big\{dy\left(\frac{dx}{ds}\cos q' - \frac{dy}{ds}\cos q\right) - dz\left(\frac{dz}{ds}\cos q - \frac{dx}{ds}\cos q''\right)\Big\}$$
$$- P\Big\{dy\left(\frac{dx}{ds}\cos\omega' - \frac{dy}{ds}\cos\omega\right) - dz\left(\frac{dz}{ds}\cos\omega - \frac{dx}{ds}\cos\omega''\right)\Big\}\Big],$$

hence, eliminating quantities which occur both in numerator and denominator, we obtain the value of $\frac{v^2}{\rho}\cos\gamma$ given in the text; and corresponding expressions may be obtained for $\frac{v^2}{\rho}\cos\gamma'$, $\frac{v^2}{\rho}\cos\gamma''$

(*d*) The coefficient of Q in the value of $\frac{v^2}{\rho}\cos\gamma$ is equal to

$$\frac{dy}{ds}\frac{dx}{ds}\cos q' - \frac{dy^2}{ds^2}\cos q - \frac{dz^2}{ds^2}\cos q + \frac{dx}{ds}\frac{dz}{ds}\cos q'',$$

and as

$$\frac{dy}{ds}\cos q' + \frac{dz}{ds}\cos q'' = -\frac{dx}{ds}\cos q,$$

we have

$$\frac{dx}{ds}\frac{dy}{ds}\cos q' + \frac{dx}{ds}\frac{dz}{ds}\cos q'' = -\frac{dx^2}{ds^2}\cos q,$$

consequently the coefficient of Q

$$= -\left(\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right)\cos q = -\cos q;$$

in the same manner it may be shown that the coefficient of $-$ P is

$-\cos\omega$, &c. Now it is easy to verify that P is the resultant of the forces $\frac{v^2}{\rho}$ and Q, for by squaring these equations, and then adding them together, we obtain

$$\left(\frac{v^2}{\rho}\right)^2(\cos^2\gamma+\cos^2\gamma'+\cos^2\gamma'')+\frac{2v^2}{\rho}\text{Q}\,(\cos\gamma\cos q+\cos\gamma'\cos q'+\cos\gamma''\cos q'')+\text{Q}^2(\cos^2 q+\cos^2 q'+\cos^2 q'')=\text{P}^2(\cos^2\omega+\cos^2\omega'+\cos^2\omega''),$$

now as the factor of $\left(\frac{v^2}{\rho}\right)$, and also the factor of $\text{Q}^2$ and $\text{P}^2$, are respectively equal to unity, and as the factor of $\frac{2v^2}{\rho}\text{Q}$ is cipher, it is evident that the force P is the resultant of $\frac{v^2}{\rho}$ and Q.

(*f*) As the areas are proportional to the times, the velocity $v$ varies as $\frac{1}{p}$, $p$ being the perpendicular let fall from the centre of force on the tangent, it is easy to show that this expression coincides with the general expression for central force given in terms of the perpendicular and chord of curvature, for if this chord be equal to $c$, and if the radius vector be denoted by $r$, we have by hypothesis,

$$v^2=\frac{h^2}{p^2},\quad c=\rho\cos\theta\quad\cdot\text{R}=\frac{h^2}{p^2c},=\left(\text{as } c=\frac{2p\,dr}{dp}\right)\frac{2h^2\,dp}{p^3\,dr}.$$

See Newton's Principia, Book I., Sections 2, 3.

(*g*) As the cosine of the angle which the radius makes with the vertical is equal to $\frac{z}{r}$, the part of the weight $mg$ which acts in the direction of the string is equal to $mg\frac{z}{r}$, and this must be added to the centrifugal force in order to obtain the entire tension, as long as the moveable is below the horizontal plane passing through C, $mg\frac{z}{r}$ is positive, and directed from the centre, but in the upper part, $mg\frac{z}{r}$ is negative, which shows that it is there directed to the centre; on the other hand, $\frac{2mg(h+z)}{r}$ which expresses the centrifugal force, is always directed from the centre; hence in the lower half of the circle, the tension is the sum of $\frac{2mg(h+z)}{r}$ and $\frac{mgz}{r}$, and in the upper half it is the difference of these two expressions. If the pen-

dulum falls *freely* from the horizontal plane, $h$ vanishes, and the entire tension is $\frac{3mgz}{r}$, the entire tension is thrice that produced by the weight, when sustained by a force equal and contrary to the tangential force, and the tension arising from the centrifugal force is to that produced by the weight 2 1 If $h = r$, which is the same as if the moveable descended freely from the highest point of the circle, $\theta = mg\left(\frac{2r + 3z}{r}\right)$, is at the lowest point where $z = r$, equal $5mg$, i e the entire tension is five times the weight, when $\theta = mg$, i e when the entire tension is equal to the weight, $mg\left(\frac{2h + 3z}{r}\right) = mg$, $z = \frac{r - 2h}{3}$, hence if there is no initial velocity, i e when the body falls freely, the strain is equal to the weight when the moveable has fallen freely through one-third of the height, $\theta$ will evidently vanish in the upper half of the circle when $z = \frac{2h}{3}$, but as $z$ can never exceed $r$, it is evident that the value of $h$ may be such that $\theta$ will never vanish, e g if $r \angle \frac{2h}{3}$ the tension will never vanish, the moveable will *revolve* about the centre, and the tension will be the sum of $2mg\left(\frac{h+z}{r}\right)$ and $\frac{mgz}{r}$ in the lower half, and the difference of these quantities in the upper half of the circle

($h$) If $z = \frac{2h}{3}$ is less than $r$, the weight at this point resolved in the direction of the radius is equal and opposite to the centrifugal force, beyond this point the tension due to the weight is greater than the centrifugal force, and therefore if the rod was not inflexible, the moveable would part from the circle when $z = \frac{2h}{3}$, and describe a parabola (No 209) having a common tangent with the circle at the point of departure If the moveable was made to describe an entire cycloid in its oscillations instead of a circular arc, it would be easy to show that if it moved from its horizontal base without any initial velocity, the tension at any point arising from its centrifugal force is equal to the component of its weight acting in the direction of the string, and at the lowest point the entire tension is equal to twice the weight of the moveable

($i$) In proceeding from the poles to the equator, as the attraction varies in the inverse ratio of the square of the distance, it must diminish, and this diminution is nearly proportional to the increment of distance For if the attraction at the pole be denoted by A, and the radius by $m$, the attraction at any other point by A$'$, and its distance from the centre of the earth by $m + dm$, we have

$$A' : A :: m^2 : (m + dm)^2$$

and

$$A' = A\frac{m^2}{(m + dm)^2} = A\left(1 - \frac{2dm}{m}\right) q\ p$$

A $-$ A$'$ or the decrement of gravity is proportional to $dm$, but $dm$ varies as the square of the cosine of $\mu$ $q$ $p$. See Nos 254 and 193, note

## CHAPTER V

($a$) If both sides of the equation $\frac{ds}{dt} = \sqrt{k^2 + 2g(z - c)}$ be multiplied by $dt$, we obtain

$$\frac{ds}{dt}dt = ds = \sqrt{k^2 + 2g(z - c)}\ dt$$

($b$) For ED is equal $a(1 + \cos\alpha)$, if $k^2 = 2ga(1 + \cos\alpha)$, the initial velocity is equal to that acquired in falling through the height ED, and if we substitute for $k^2$ this value, the expression for $dt$ given in the text, will be $dt = \frac{-a d\theta}{\sqrt{2ga(1 + \cos\theta)}}$

($c$) By hypothesis, $1 + \cos\theta = 1 + \cos 2\psi = 2\cos^2\psi$, and $d\theta = 2d\psi$,

$$2ga(1 + \cos\theta) = 4ga\cos^2\psi, \quad dt = \frac{-2a\,d\psi}{2\sqrt{ga}\ \cos\psi} = -\sqrt{\frac{a}{g}}\ \frac{d\psi}{\cos\psi},$$

now let $\sin\psi = x$, then $\frac{d\psi}{\cos\psi} = \frac{dx}{1 - x^2}$, the integral of this last quantity is $\frac{1}{2}\log\frac{1 - x}{1 + x} + C = \frac{1}{2}\log\frac{1 - \sin\frac{1}{2}\theta}{1 + \sin\frac{1}{2}\theta} + C$, when $t = 0$, we have $\frac{1}{2}\log\frac{1 - \sin\frac{1}{2}\alpha}{1 + \sin\frac{1}{2}\alpha} + C = 0$, substituting the value that may be obtained in this way for C, we obtain the expression in text.

($d$) It is evidently the same thing whether the moveable sets out from the point M, which is very near to B, with a small initial velocity denoted by $h$, or whether it falls without any initial velocity

from a point so much higher than M, as that when it has reached M it would have acquired the velocity $h$

($e$) When $t\sqrt{\frac{g}{a}}$ is a multiple of $2\pi$, $\theta=\alpha$, and when $t\sqrt{\frac{g}{a}}$ is an odd multiple of $\pi$, $\theta=-\alpha$, and in each case $\frac{d\theta}{dt}=0$

($f$) By making this substitution, we have $\theta'=\alpha\cos(t+\mathrm{T})\sqrt{\frac{g}{a}}$ $=\alpha\cos t\sqrt{\frac{g}{a}}\cos\mathrm{T}\sqrt{\frac{g}{a}}-\alpha\sin t\sqrt{\frac{g}{a}}\sin\mathrm{T}\sqrt{\frac{g}{a}}$, which expression, when $\mathrm{T}=\pi\sqrt{\frac{a}{g}}$, becomes $-\alpha\cos t\sqrt{\frac{g}{a}}$

($g$) As $ad\theta$ is the differential of the arc of the circle, $\frac{ad\theta}{dt}$ expresses the velocity, and therefore as $\frac{d\theta}{dt}=\pm\alpha\sqrt{\frac{g}{a}}$ at the lowest point, the velocity at this point $=\pm\alpha\sqrt{ga}$, and as $b=\frac{1}{2}a\alpha^2$, we have $a=\frac{2b}{\alpha^2}$, and $v=\sqrt{2gb}$.

($h$) Retaining the fourth powers of $\alpha$ and $\theta$, the expression for

$$dt=-\sqrt{\frac{a}{g}}\frac{d\theta}{\sqrt{\alpha^2-\theta^2-\frac{\alpha^4}{12}+\frac{\theta^4}{12}}}$$

which is evidently reducible to the expression in the text, and as

$$\left[1-\tfrac{1}{12}(\alpha^2+\theta^2)\right]^{-\frac{1}{2}}=1+\tfrac{1}{24}(\alpha^2+\theta^2),$$

the expression for $dt$ is evidently that given in text

($i$) The integral of $-\frac{d\theta}{\sqrt{\alpha^2-\theta^2}}$ is arc $\cos\frac{\theta}{\alpha}$, that of $-\frac{\alpha^2}{24}$ $d$ $\frac{d\theta}{\sqrt{\alpha^2-\theta^2}}=\frac{\alpha^2}{24}\text{arc}\cos\frac{\theta}{\alpha}$, that of $-\frac{\theta^2 d\theta}{24\sqrt{\alpha^2-\theta^2}}=\frac{\theta}{48}\sqrt{\alpha^2-\theta^2}+\frac{\alpha^2}{48}\text{arc}\cos\frac{\theta}{\alpha}$, the integral of

$$dt=\sqrt{\frac{a}{g}}\left[\text{arc}\cos\frac{\theta}{\alpha}+\frac{3\alpha^2}{48}\text{arc}\cos\frac{\theta}{\alpha}+\frac{\theta}{24}\sqrt{\alpha^2-\theta^2}\right],$$

the last term vanishes at the limits $\alpha,-\alpha$, and the other terms become at these limits

$$\sqrt{\frac{a}{g}}\left(2\,\text{arc}\cos = 1 + \frac{\alpha^2}{16}\,2\,\text{arc}\cos = 1,\right) = \sqrt{\frac{a}{g}}\left(\pi + \frac{\alpha^2\pi}{16}\right)$$

$$= \pi\sqrt{\frac{a}{g}}\left(1 + \frac{\alpha^2}{16}\right)$$

(*k*) Let $\tau$ be the given time, $\text{T}'$ the duration of each oscillation when the amplitude is infinitely small, and $\text{T}$ the duration of the oscillations where the amplitudes are only very small, we have

$$n\text{T}' = n'\text{T} = \tau \text{ i e}, n\pi\sqrt{\frac{g}{a}} = n'\pi\sqrt{\frac{g}{a}}\left(1 + \frac{\alpha^2}{16}\right) \quad n \quad n' \quad 1 + \frac{\alpha^2}{16} \quad 1$$

(*l*) $\cos^2\theta = 1 - 2x + x^2 \quad \sin^2\theta = 2x - x^2$, and $d\theta = \dfrac{dx}{\sqrt{2x - x^2}}$,

$\therefore\ dt = -\sqrt{\dfrac{a}{g}}\dfrac{dx}{\sqrt{2\beta - 2x}\ \sqrt{2x - x^2}}$ which is evidently equal to the expression in the text, now when $x = \beta$, $1 - \cos\theta = 1 - \cos\alpha$, $\theta = \alpha$, and when $x = 0$, $1 - \cos\theta = 0 \quad \theta = 0$

(*m*) If in the expression $x^{n-2}\sqrt{\beta x - x^2}\,dx$, we multiply and divide by $\sqrt{\beta x - x^2}$, we obtain $\int x^{n-2}\,dx = \beta\displaystyle\int\frac{x^{n-1}dx}{\sqrt{\beta x - x^2}} - \int\frac{x^n dx}{\sqrt{\beta x - x^2}}$

(*n*) For the manner in which this integral is obtained, see Lacroix's *Traite du Calcul Differential et Integral*, tom 2, page 314

(*o*) By differentiating with respect to *t*, we obtain

$$\frac{d\theta}{dt} = \left[-\alpha\gamma\sqrt{\frac{g}{a}}\sin t\gamma\sqrt{\frac{g}{a}} + \frac{\alpha g}{2k}\cos t\gamma\sqrt{\frac{g}{a}} - \frac{\alpha g}{2k}\cos t\gamma\sqrt{\frac{g}{a}} - \right.$$

$$\left. - \frac{\alpha g\sqrt{ga}}{4\gamma k^2}\sin t\gamma\sqrt{\frac{g}{a}}\right]e^{\frac{-gt}{2k}}$$

Now in this expression the factors of the cos mutually destroy each other, and those of the sine are

$$-\alpha\gamma\sqrt{\frac{g}{a}} - \frac{\alpha g\sqrt{ga}}{4\gamma k^2}, = \left(\text{as } 4k^2 = \frac{ga}{1-\gamma^2}\right)$$

$$-\alpha\gamma\sqrt{\frac{g}{a}} - \frac{\alpha(1-\gamma^2)}{\gamma}\sqrt{\frac{g}{a}} = \frac{\alpha}{\gamma}\sqrt{\frac{g}{a}}$$

(*p*) When the first oscillation is completed,

$$T\gamma\sqrt{\frac{g}{a}}=\pi, \quad T=\frac{\pi}{\gamma}\sqrt{\frac{a}{g}}$$

Now in a vacuum the value of $T$ is $\pi\sqrt{\frac{a}{g}}$, consequently, it is increased in a resisting medium in the ratio of 1 $\gamma$

(*q*) In the expressions for the amplitude, the quantity by which $e^{-\frac{gt}{2k}}$ is multiplied becomes the same at the end of each oscillation, but $e^{-\frac{gt}{2k}}$ evidently diminishes as $t$ increases, therefore the amplitudes continually diminish, now as

$$T=\frac{\pi}{\gamma}\sqrt{\frac{a}{g}}, \quad nT\gamma\sqrt{\frac{g}{a}}=n\pi,$$

therefore, the value of $\theta$ after the $n^{th}$ oscillation is equal to

$$\alpha.\cos n\pi\ e^{-\frac{n\pi\sqrt{ga}}{2\gamma k}}=(\text{as}\cos n\pi=(-1)^n),\ \alpha\ (-1)^n e^{-\frac{n\pi\sqrt{ga}}{2\gamma k}};$$

but the value of $\theta$ in this case is by hypothesis $(-1)^n\alpha_n$, hence then we obtain the value $\alpha_n$ given in the text, from which it evidently appears that these values of $\alpha$, constitute a geometrical progression of which the ratio is $e^{-\frac{\pi\sqrt{ga}}{2\gamma k}}$

(*r*) In this case when the cos and sin are expressed in exponentials, the value $\theta$ is equal to

$$\theta=\alpha.\left[\frac{e^{-t\beta\sqrt{\frac{g}{a}}}+e^{t\beta\sqrt{\frac{g}{a}}}}{2}-\frac{\sqrt{ga}}{2\beta k}\left(\frac{e^{-t\beta\sqrt{\frac{g}{a}}}-e^{t\beta\sqrt{\frac{g}{a}}}}{2}\right)\right]e^{-\frac{gt}{2k}},$$

from which it is evident, that if $2k \angle \sqrt{ga}$, $\theta$ will be cipher when $t$ is infinite

(*s*) The integral of the equation

$$2\frac{d^2\theta}{dt^2}\ d\theta+\frac{2g}{a}\sin\theta d\theta=\mu\frac{d\theta^2}{dt^2}d\theta,$$

is

$$\frac{d\theta^2}{dt^2}+\frac{2g}{a}\cos\theta=\mu\int\frac{d\theta^2}{dt^2}d\theta,$$

and as by hypothesis, $\int\frac{d\theta^2}{dt^2}d\theta=y$, and $\frac{d\theta^2}{dt^2}=\frac{dy}{d\theta}$, this equation becomes, by substituting these values,

$$\frac{dy}{d\theta}-\frac{2g}{a}\cos\theta-\mu y=0$$

Let $y=\Theta z$, $\Theta$ being a function of $\theta$, then substituting for $y$ and $\frac{dy}{d\theta}$ their respective values $\Theta z$ and $\frac{\Theta dz}{d\theta}+\frac{zd\Theta}{d\theta}$, the preceding equation becomes, by multiplying by $d\theta$,

$$\Theta dz+zd\Theta-\frac{2g}{a}\cos\theta d\theta-\mu\Theta zd\theta=0$$

Now if we assume (Lacroix's *Traite Elementaire*, No 257),

$$\Theta dz-\mu\Theta zd\theta=0,\ zd\Theta-\frac{2g}{a}\cos\theta d\theta=0,$$

from the first equation we obtain $z=e^{m\theta}$, and substituting this value of $z$ in the second equation, there results

$$d\Theta=\frac{2g}{a}\cos\theta d\theta\, e^{-\mu\theta},\ .\ \Theta=\frac{2g}{a}\sin\theta\, e^{-m\theta}+\frac{2mg}{a}\int\sin\theta d\theta\, e^{-m\theta}=$$

$$\left(-\frac{2\mu g}{a}\cos\theta\, e^{-m\theta}-\frac{2g\mu^2}{a}\int\cos\theta d\theta\, e^{-\mu\theta}+c,\right.$$

and, by continuing the partial integration in this manner, the value of $\Theta$ will be found equal to

$$\frac{2g}{a}e^{-\mu\theta}\sin\theta[(1-\mu^2+\mu^4-\mu^6+\&c.)-\mu\cos\theta(1-\mu^2+\mu^4-\mu^6+\&c.)],$$

but we know from the expansion of a binomial, that

$$\frac{1}{1+\mu^2}=1-\mu^2+\mu^4-\mu^6+\&c\ ,$$

therefore,

$$\Theta=2ge^{-\mu\theta}\left(\frac{\sin\theta-\mu\cos\theta}{1+\mu^2}\right)+c,$$

consequently, $y=\Theta z=\Theta\ e^{\mu\theta}$

$$=\frac{2g}{a}\ \left(\frac{\sin\theta-\mu\cos\theta}{1+\mu^2}\right)+ce^{\mu\theta}$$

(*t*) If in equation (5), $-\alpha_1$ be substituted for $\theta$, then as the velocity is by supposition cipher, the second member of this equation becomes cipher, by means of which, the expression in the text can be deduced, now, when in this equation, the square and higher powers of $\mu$ are neglected, $e^{\mu\alpha}=1+\mu\alpha$, consequently, the members at each side of this equation become

$$\cos\alpha_1(1+\mu\alpha_1)-\mu\sin\alpha_1=\cos\alpha(1-\mu\alpha)+\mu\sin\alpha,$$

which is evidently the expression of the text

($u$) As $\delta$ is very small, $\cos\delta = 1$, and $\cos\alpha_1 = \cos\alpha + \delta\sin\alpha$, and $\sin\alpha_1 = \sin\alpha + \cos\alpha\,\delta$, $\alpha_1\cos\alpha_1 (= (\alpha-\delta)\cos(\alpha-\delta)) = \alpha\cos\alpha - \delta\cos\alpha + \alpha\delta\sin\alpha$, $\cos\alpha_1 - \mu(\sin\alpha_1 - \alpha_1\cos\alpha_1) = \cos\alpha + \delta\sin\alpha - \mu\sin\alpha + \mu\alpha\cos\alpha = \cos\alpha + \mu\sin\alpha - \mu\alpha\cos\alpha$, $\delta\sin\alpha = 2\mu(\sin\alpha - \alpha\cos\alpha)$

($v$) $\sin\alpha = \frac{\alpha}{1} - \frac{\alpha^3}{1\,2\,3}$, and $\alpha\cos\alpha = \frac{\alpha}{1} - \frac{\alpha^3}{1\,2}$ (neglecting the fourth powers), $\sin\alpha - \alpha\cos\alpha = \frac{\alpha^3}{3}$, $\frac{1}{\sin\alpha} = \frac{1}{\alpha - \frac{\alpha^3}{1\,2\,3}} = \frac{\alpha^{-1}}{1} + \frac{\alpha^{-2}\alpha^3}{1\,2\,3}$,

$$\delta = 2\mu\left(\frac{\sin\alpha - \alpha\cos\alpha}{\sin\alpha}\right) = \frac{2\mu\alpha^3}{3}\left(\frac{\alpha^{-1}}{1} + \frac{\alpha^{-2}\alpha^3}{1\,2\,3}\right) =$$

(as the fourth power is neglected) $\frac{2\mu\alpha^2}{3}$, consequently,

$$\alpha_1 = \alpha - \delta = \alpha - \frac{2\mu\alpha^2}{3}$$

($t$) For the integral of this equation, see examples on the differential and integral calculus, page 387

($u$) By substituting for $\sin 2t\sqrt{\frac{g}{a}}$, and dividing all the terms by $a$ and $\sqrt{ga}$, which occur as factors in all the terms, we obtain the expression in text when $v = 0$, and as the factor $\sin t\sqrt{\frac{g}{a}} = \sin\pi$, when $t = \pi\sqrt{\frac{a}{g}}$, it is evident that for this value of $t$, $v = 0$

($v$) If $t'\sqrt{\frac{g}{a}} = \frac{\pi}{2} + \delta$, then $\cos t'\sqrt{\frac{g}{a}} = \cos\left(\frac{\pi}{2} + \delta\right) = -\delta$, $\cos 2t'\sqrt{\frac{g}{a}} = \cos(\pi + 2\delta) = -1$, the given equation becomes

$$-\left(1 - \frac{\alpha\mu}{3}\right)\delta + \frac{\alpha\mu}{4} - \frac{\alpha\mu}{12} = 0\,,$$

which, as the product $\alpha\delta$ is neglected by hypothesis, gives

$$\delta = \frac{\alpha\mu}{6}$$

($x$) By substituting for $t'\sqrt{\frac{g}{a}}$ its value $\frac{\pi}{2} + \frac{\alpha\mu}{6}$, and neglecting the cube and higher powers of $\alpha_1$, we obtain

$$\sin t'\sqrt{\frac{g}{a}}=\sin\left(\frac{\pi}{2}+\frac{\alpha\mu}{6}\right)=1-\frac{\alpha^2\mu^2}{1\,2\,6}, \sin 2t'\sqrt{\frac{g}{a}}=\sin\left(\pi+\frac{\alpha\mu}{3}\right)=\frac{\alpha\mu}{3},$$

and if these values be substituted in the general expression for $v$, we obtain, inasmuch as the cube of $\alpha$ is neglected,

$$v=\left(\alpha-\frac{\alpha^2\mu}{3}\right)\sqrt{ga}$$

($y$) In this case the general value of $\theta$ becomes, by substituting $\pi$ for $t\sqrt{\frac{g}{a}}$, equal to

$$\left(\alpha-\frac{\alpha^2\mu}{3}\right)\cos\pi+\frac{\alpha^2\mu}{4}+\frac{\alpha^2\mu}{12}\cos 2\pi, \qquad \alpha_1=\alpha-\frac{2\mu\alpha^2}{3}$$

($z$) See *Mechanique Celeste*, Book III Nos 24, 25, 38

($a'$) When $\gamma=1$ this equation gives $\operatorname{tang} t'\sqrt{\frac{g}{2a}}=\frac{-2k}{\sqrt{2ga}}$;

$$\text{but} -\frac{2k}{\sqrt{2ga}}=\operatorname{tang}\left(\frac{\pi}{2}+\frac{\sqrt{2ga}}{2k}\right), \text{ for}$$

$$\operatorname{tang}\left(\frac{\pi}{2}+\frac{\sqrt{2ga}}{2k}\right)=\frac{\operatorname{tang}\frac{\pi}{2}+\operatorname{tang}\frac{\sqrt{2ga}}{2k}}{1-\operatorname{tang}\frac{\pi}{2}\operatorname{tang}\frac{\sqrt{2ga}}{2k}}=$$

$\left(\text{as tang } \frac{\pi}{2} \text{ is infinite}\right) -\frac{1}{\operatorname{tang}\frac{\sqrt{2ga}}{2k}}$, but when $k$ is very great,

$$\operatorname{tang}\frac{\sqrt{2ga}}{2k}=\frac{\sqrt{2ga}}{2k}, \quad \frac{1}{-\operatorname{tang}\frac{\sqrt{2ga}}{2k}}=\frac{-2k}{\sqrt{2ga}} \quad t'\sqrt{\frac{g}{2a}}=\frac{\pi}{2}+\frac{\sqrt{2ga}}{2k}.$$

($b'$) If any of the exponents of this series were negative, then when $x=0$, $s$ would be infinite, and if any of them were cipher when $x=0$, $s$ would be infinite, also as $\frac{ds}{dx}=$ the secant of the angle which a tangent to the curve makes with the axis of $x$, at the lowest point when $\frac{dy}{dx}=\frac{ds}{dx}$, this tangent is perpendicular to the axis of $x$, and therefore $\frac{ds}{dx}$ is $\infty$, hence it is at once evident, that the least of the exponents $\alpha, \beta, \gamma$, &c, must be less than unity.

($c'$) As $x=hx'$, $dx=hdx'$, we have $\frac{x^{a-1}\,dx}{\sqrt{h-x}}=\frac{h^{a-1}\,x'^{a-1}\,hdx'}{\sqrt{h-hx'}}=$

$h^{a-1}\dfrac{x'^{a-1}dx'}{\sqrt{1-x'}}$, and the limits of the new integral are evidently zero and unity.

(d′) It has been proved in a former note, that one of the exponents, $\alpha, \beta, \gamma$, &c., is less than unity, it appears from what is established here, that the value of the least exponent is $\frac{1}{2}$, hence then it follows, that $\int_0^1 \frac{x'^{a-1}dx'}{\sqrt{1-x'}} = \int_0^1 \frac{dx'}{x'^{\frac{1}{2}}\sqrt{1-x'}} = \int_0^1 \frac{dx'}{\sqrt{x'-x'^2}}$,
and the integral of $\dfrac{dx'}{\sqrt{x'-x'^2}} = -2\,\text{arc}\left(\text{tang} = \dfrac{\sqrt{x'}}{\sqrt{1-x'}}\right) =$ (when taken between the limits of 1 and 0) $\pi$, as stated in the text.

(e′) As $u = \sqrt{1+\frac{dy^2}{dx^2}+\frac{dz^2}{dx^2}}$, $\delta u = \left[\frac{dy}{dx}\frac{\delta dy}{dx} + \frac{dz}{dx}\frac{\delta dz}{dx}\right]\frac{1}{u}$,
now the entire integral of $\frac{\lambda}{u}\frac{dy}{dx}\frac{d\delta y}{dx}dx = \frac{\lambda}{u}\frac{dy}{dx}\delta y - \int d\left(\frac{\lambda}{u}\frac{dy}{dx}\right)\delta y$,
and as at the limits $\beta$, $\alpha$, $\delta y$ is cipher, $\frac{\lambda}{u}\frac{dy}{dx}\delta y$ vanishes, we have

$$\int_\alpha^\beta \frac{\lambda}{u}\frac{dy}{dx}\frac{d\delta y}{dx}dx = -\int_\alpha^\beta d\left(\frac{\lambda}{u}\frac{dy}{dx}\right)\delta y$$

(f′) The first equation (c) gives $\lambda\frac{dy}{dx} = u\,a$, and as $u = \sqrt{1+\frac{dy^2}{dx^2}}$, by multiplying both sides by $dx$, we obtain

$$\lambda\frac{dy}{dx}dx = \sqrt{dy^2+dx^2}\;a$$

(g′) It is evident that this equation may be made to coincide with that given in No 72, by changing the origin of the coordinates on the axis of $x$.

(h′) Let $x-\alpha = az^2$, then $dx = 2azdz$, and $\dfrac{(x-\alpha)dx}{\sqrt{a((x-\alpha)-(x-\alpha)^2)}}$
$= \dfrac{2az^2dz}{\sqrt{1-z^2}}$, the integral of this last is $-az\sqrt{1-z^2} + a$ arc sin $=$ $z + c$, $=$ (by substituting for $z$ its value $\dfrac{\sqrt{x-\alpha}}{\sqrt{a}}$) $-a\sqrt{\dfrac{x-\alpha}{a}}$
$\left(\sqrt{1-\dfrac{x-\alpha}{a}}\right) + a$ arc sin $= \sqrt{\dfrac{x-\alpha}{a}} + c = -\sqrt{a(x-\alpha)-(x-\alpha)^2}$

$+ a$ arc cos $= \sqrt{\frac{a - x + \alpha}{a}} + c$, but arc cos $= \frac{\sqrt{a - x + \alpha}}{\sqrt{a}} = \frac{\text{arc}}{2}$

cos $= \left(\frac{2a - 2x + 2\alpha}{a} - 1\right) = \frac{\text{arc}}{2}$ cos $= \frac{a - - 2x + 2\alpha}{a}$, consequently, the integral is evidently that given in the text

($i'$) As $dy = \frac{(x-\alpha)dx}{\sqrt{a(x-\alpha) - (x-\alpha)^2}}$ $\frac{dy^2}{dx^2} + 1 = u^2 =$

$$\frac{a(x-\alpha) - (x-\alpha)^2 + (x-\alpha)^2}{a(x-\alpha) - (x-\alpha)^2} = \frac{a}{a - (x-\alpha)}$$

($k'$) Let $x - \alpha = y^2\sqrt{a}$ and $dx = 2\sqrt{a}\,ydy$ hence $\frac{\sqrt{a}\,dx}{\sqrt{a(x-\alpha) - (x-\alpha)^2}}$ $= \frac{2aydy}{\sqrt{ay^2 - y^4a}} = \frac{2\sqrt{a}dy}{\sqrt{1-y^2}} = 2\sqrt{a}$ arc cos $= \sqrt{1-y^2} = 2\sqrt{a}$. arc cos $= \sqrt{1 - \frac{x-\alpha}{a}}$, $2\sqrt{a}$ arc cos $= \sqrt{\frac{a - x + \alpha}{a}} = \sqrt{a}$ arc cos $=$ $\left(\frac{2a - 2x + 2\alpha}{a} - 1\right) = \sqrt{a}$ arc cos $= \frac{a - 2x + 2\alpha}{a}$, equal, when $x = \beta$, the expression given in text

($l'$) Now arc cos $= \frac{a - 2\beta + 2\alpha}{a} =$ arc sin $= \sqrt{\frac{a^2 - (a - 2\beta + 2\alpha)^2}{a^2}}$ $= \sqrt{\frac{a^2 - a^2 + 4a(\beta-\alpha) - 4(\beta-\alpha)^2}{a^2}} = 2\sqrt{\frac{a(\beta-\alpha) - (\beta-\alpha)^2}{a^2}}$, when the radius is infinite the sine equal the arc, and if in the expression for $\beta' - \alpha'$ given above, there be substituted for $\frac{a}{2}$ arc cos $= \frac{a - 2\beta + 2\alpha}{a}$, $\frac{2\sqrt{a(\beta-\alpha) - (\beta-\alpha)^2}}{a}$, the terms which express the value of $\beta' - \alpha'$ will be equal, and have contrary signs, therefore, $\beta' - \alpha'$ in this case is cipher, and the value of $y$ becomes evidently equal to $a'$, for the second member of the equation vanishes, likewise in this same case when $a$ is $\infty$, the part of the expression for $t'$, in which $a$ does not occur as a factor, vanishes, relatively to the other part, so that it becomes equal to $\frac{\sqrt{a}}{\sqrt{2g}}$.

$$\frac{2\sqrt{a(\beta-\alpha)}}{a} = \frac{\sqrt{2(\beta-\alpha)}}{\sqrt{g}}$$

($m'$) By substituting for $\frac{x d^2 x + y d^2 y + z d^2 z}{dt^2}$ its value

$$-\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = -v^2,$$

we obtain the value of N given in the text

($n$) $r\,d\psi$ is the arc of a circle described with a radius $= r$ in $dt$, and $\frac{r\,d\psi}{dt} = u$, and $x dy - y dx = r^2 d\psi = c dt$, $\frac{r\,d\psi}{dt} = u = \frac{c}{r} =$ (as $r = \sqrt{a^2 - z^2}$) $\frac{c}{\sqrt{a^2 - z^2}}$

($o'$) $z = a\cos\theta$, $\cos\theta = 1 - \frac{\theta^2}{2}$, $z = a - \frac{a}{2}\theta^2$, $k^2 = b + 2g\gamma$, by substituting for $k^2$ and $\gamma$, their values, we obtain $\beta^2 ga = b + 2ga - ga\alpha^2$, $b = -2ga + ga(\alpha^2 + \beta^2)$, $c^2 = k^2(a^2 - \gamma^2) = \beta^2 ga(a^2 - (a - \frac{1}{2}a\alpha^2)^2) = \beta^2 ga(a^2 - a^2 + a^2\alpha^2) = \beta^2 ga^3\alpha^2$, for by supposition the fourth power of $a$ is neglected. Now $dz = -a d\theta$, and $a^2 - z^2 = a^2 - (a - \frac{1}{2}a\theta^2)^2 = a^2\theta^2$, $2gz = 2ga - ga\theta^2$, $(a^2 - z^2)(2gz + b) - c^2 = a^2\theta^2(2ga - ga\theta^2 - 2ga + ga(\alpha^2 + \beta^2)) - ga^3\alpha^2\beta^2, = ga(a^2\theta^2(\alpha^2 + \beta^2 - \theta^2) - a^2\alpha^2\beta^2)$, $\sqrt{(a^2 - z^2)(2gz + b) - c^2} = a\sqrt{g(\alpha^2 - \theta^2)(\theta^2 - \beta^2)a}$, as $dz = -a^2\theta d\theta$, by substituting for $\sqrt{(a^2 - z^2)(2gz + b) - c^2}$ and also for $dz$ their values, we obtain the expression for $dt$ given in the text, and as

$$d\psi = \frac{c dt}{a^2 - z^2} = \frac{\sqrt{ga^3}\,\alpha\beta dt}{a^2\theta^2}$$

by substituting for $dt$ its value, we obtain for $d\psi$ the expression in text.

($p'$) It appears from equation (b) that the increment of the angle $\psi$ will be always positive, it will consequently increase indefinitely, with respect to $\theta$, as $\frac{d\theta}{dt^2} = \frac{g}{a}\frac{(\alpha^2 - \theta^2)}{\theta}\frac{(\theta^2 - \beta^2)}{\theta}$, $\theta$ always lies between $\alpha$ and $\beta$, therefore when these two quantities are equal, $\theta$ must also be equal to each of them.

($q'$) The base of the cone will be circular because $\theta$ is constant, and the motion will be uniform, because $\psi$ is proportional to the time $t$, when a revolution is completed we have

$$2\pi = \mathrm{T}\sqrt{\frac{g}{a}} \text{ and } \mathrm{T} = 2\pi\sqrt{\frac{a}{g}}$$

($r'$) When $\theta = \alpha$, $\cos 2t\sqrt{\frac{g}{a}} = 1$, $t = 0$, and when $\theta = \beta$, $\cos 2t\sqrt{\frac{g}{a}} = -1$, $2t\sqrt{\frac{g}{a}} = (\text{when } \theta = \beta)\,\pi$, $t = \frac{\pi}{2}\sqrt{\frac{a}{g}}$

By substituting for $\cos 2t\sqrt{\frac{g}{a}}$, and observing that $\cos^2 t\sqrt{\frac{g}{a}} + \sin^2 t\sqrt{\frac{g}{a}} = 1$, we obtain $\theta^2 = \frac{1}{2}(\alpha^2 + \beta^2)\left(\sin^2 t\sqrt{\frac{g}{a}} + \cos^2 t\sqrt{\frac{g}{a}}\right) + \frac{1}{2}(\alpha^2 - \beta^2)\left(\cos^2 t\sqrt{\frac{g}{a}} - \sin^2 t\sqrt{\frac{g}{a}}\right) = \alpha^2\cos^2 t\sqrt{\frac{g}{a}} + \beta^2\sin^2 t\sqrt{\frac{g}{a}}$

($t'$) Dividing this value of $d\psi$ by $\alpha^2\cos^2 t\sqrt{\frac{g}{a}}$, there will result

$$d\psi = \sqrt{\frac{g}{a}}\frac{\beta}{\alpha}\;\frac{dt}{\cos^2 t\sqrt{\frac{g}{a}}} - \left(1 + \frac{\beta^2}{\alpha^2}\,\tan^2 t\sqrt{\frac{g}{a}}\right),$$

now $\dfrac{dt\sqrt{\frac{g}{a}}}{\cos^2 t\sqrt{\frac{g}{a}}} = d\,\text{tang}\;t\sqrt{\frac{g}{a}}$, . if we call $\text{tang}\;t\sqrt{\frac{g}{a}} = x$, we shall have

$$d\psi = \frac{\frac{\beta}{\alpha}\,dx}{1 + \frac{\beta^2}{\alpha^2}\,x^2}, \text{ and } \psi = \text{arc tang} = \frac{\beta}{\alpha}\,x,$$

consequently,

$$\text{tang}\,\psi = \frac{\beta}{\alpha}x \text{ and } \alpha\,\text{tang}\,\psi = \beta\,\text{tang}\,t\sqrt{\frac{g}{a}},$$

when

$$\psi = \frac{\pi}{2},\; t\sqrt{\frac{g}{a}} = \frac{\pi}{2}, \text{ and } t = \frac{\pi}{2}\sqrt{\frac{a}{g}}.$$

($u'$) From the equation

$$\alpha\,\text{tang}\,\psi = \beta\,\text{tang}\,t\sqrt{\frac{g}{a}},$$

we obtain

$$\alpha^2\,\frac{1 - \cos^2\psi}{\cos^2\psi} = \beta^2\;\frac{\sin^2 t\sqrt{\frac{g}{a}}}{\cos^2 t\sqrt{\frac{g}{a}}},$$

consequently,

$$\alpha^2 \cos^2 t \sqrt{\frac{g}{a}} = \left(\alpha^2 \cos^2 t \sqrt{\frac{g}{a}} + \beta^2 \sin^2 t \sqrt{\frac{g}{a}}\right) \cos^2 \psi,$$

but the quantity within the brackets $= \theta^2$, consequently,

$$\therefore \alpha^2 \cos^2 t \sqrt{\frac{g}{a}} = \theta^2 \cos^2 \psi, \text{ and } x^2 = a^2\theta^2 \cos^2 \psi = a^2\alpha^2 \cos^2 t \sqrt{\frac{g}{a}},$$

in like manner it may be shown, that

$$y^2 = a^2\theta^2 \sin^2 \psi = a^2\beta^2 \sin^2 t \sqrt{\frac{g}{a}}, \qquad \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = a^2$$

## CHAPTER VI

(*a*) As the medium is uniform, and every thing consequently the same on both sides of the vertical plane, there is no reason why the projectile should deviate to one side rather than to the other of this plane

(*b*) $t = \dfrac{x}{a\cos\alpha}$, and if this value of $t$ be substituted in the expression for $y$, we obtain $y = x \operatorname{tang}\alpha - \frac{1}{2} g \dfrac{x^2}{a^2 \cos^2\alpha}$ as $\dfrac{g}{a^2} = \dfrac{1}{2h}$ there results for $y$ the expression given in the text, and as $\dfrac{dy}{dx} = \operatorname{tang}\alpha - \dfrac{x}{2h\cos^2\alpha}$, at the vertex where $\dfrac{dy}{dx} = 0$, $x = 2h \sin\alpha \cos\alpha$, $y = h\sin^2\alpha$, it appears from these expressions for the coordinates of the vertex of the parabola described by the projectile, that when the velocity of projection or $h$ is given, the locus of the vertices of all parabolas which can be described, is an ellipse whose minor axis is vertical and equal to $h$, and whose major axis is horizontal, and equal to $2h$, for we have $x^2 = 4h^2 \sin^2\alpha(1 - \sin\alpha) = 4hy - 4y^2$ $4y^2 + x^2 - 4hy = 0$, which evidently is the equation of such an ellipse as has been described, in which the origin of the coordinates is at the lower extremity of the minor axis From the equation $x = ta\cos\alpha$, $y = ta\sin\alpha - \frac{1}{2}gt^2$, it appears that when $t$ the time is given, the locus of the points arrived at in a given time is a right line parallel to the direction of projection, and intersecting the axis of $y$ at a distance from the origin equal to $\dfrac{gt^2}{2}$ As the direction of projection bisects the angle between vertical and line drawn from point of projection

to the focus, and as $h$ is always equal to the distance of the origin from focus, if $x''$ and $y''$ denote the coordinates of the focus, and $\alpha$, as before, the angle between the horizontal axis and direction of projection, $x'' = h \cos(2\alpha - 90°)$, $y'' = \sin(2\alpha - 90°)$, i e, $x'' = -h \sin 2\alpha$, $y'' = h \cos 2\alpha$, $x''^2 + y''^2 = h^2$, $\frac{y''}{x''} = -\cot 2\alpha$, the first equation shows, that when the velocity of projection is given, the locus of the foci is a circle radius equal to $h$, the second proves that when the angle of the projection is given, this locus is a right line, likewise in the same circumstances, that is, when direction of projection is given, it appears from the equations $x = 2h \sin\alpha \cos\alpha$, $y = h \sin^2 \alpha$, that the locus of the vertices of the described parabolas is a right line of which the equation is $y = \frac{x}{2} \operatorname{tang} \alpha$

(c) $\frac{dx}{dt} = a \cos\alpha$, $\frac{dy}{dt} = a \sin\alpha - gt$, $v^2 = \frac{dx^2 + dy^2}{dt^2} = a^2 (\sin^2\alpha + \cos^2\alpha) - 2a(gt \sin\alpha) + g^2t^2 = a^2 - 2agt \sin\alpha + g^2t^2$, and as $a^2 = 2gh$, $2agt \sin\alpha - g^2t^2 = 2gy$, $v^2 = 2g\ (h - y)$, as we know from other considerations If the equation $y = x \operatorname{tang}\alpha - \frac{x^2}{4h \cos^2\alpha}$ be solved for $h$, we obtain $h = \frac{x^2}{4 \cos^2\alpha\,(x \operatorname{tang}\alpha - y)}$, hence we can determine the velocity, with which a body should be projected in a given direction, in order to reach a given point

(d) It appears from inspection of the value of $z$, that the part under the radical is the equation of a parabola, and it follows from what is established in the text, that all parabolas described, with a velocity of projection equal to $\sqrt{2gh}$, will touch the concavity of this parabola.

It is evident, that the *maximum* range on a *given* plane, passing through the point of projection, is when the direction of projection bisects the angle between this plane and the vertical, consequently, the given plane passes through the focus of the parabola, which is described, and if $\phi$ be the inclination of the given plane to the horizon, we have $\phi = 2\alpha - 90°$, and $\operatorname{tang} \alpha = \operatorname{tang} \frac{1}{2}(\phi + 90°)$, $\beta = \frac{2h}{\operatorname{tang}\alpha} = \frac{2h}{\operatorname{tang}\frac{1}{2}(\phi + 90°)}$ and the greatest range which is equal $\frac{\beta}{\cos\phi} = \frac{2h}{\cos\phi \operatorname{tang}\frac{1}{2}(\phi + 90)} = \frac{2h}{1 + \sin\phi}$, for $\operatorname{tang} \frac{1}{2}(\phi + 90) =$

$$\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}}=\frac{\sin\frac{\phi}{2}+\cos\frac{\phi}{2}}{\cos\frac{\phi}{2}-\sin\frac{\phi}{2}},\qquad \text{as } \cos\phi=\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2},\ \cos\phi$$

$$\tan\tfrac{1}{2}(\phi+90)=1+2\sin\frac{\phi}{2}\cos\frac{\phi}{2}=1+\sin\phi$$

(e) $m=\dfrac{4\pi D r^3}{3}$, and the surface of the sphere is equal to $4r^2\pi$, if these values be substituted in the value of $\dfrac{R}{m}$, the result will be of the form in the text.

(f) By substituting for $\dfrac{R}{m}$ in the equation $\dfrac{d^2x}{dt^2}=-\dfrac{R}{m}\dfrac{dx}{ds}$, we obtain

$$\frac{d^2x}{dt^2}=-c\frac{ds}{dt}\frac{dx}{ds}\frac{ds}{dt}=-c\frac{ds}{dt}\frac{dx}{dt},$$

multiplying both sides by $dt$, and integrating, we obtain

$$\log\frac{dx}{dt}=-cs+\log.A\qquad \frac{dx}{dt}=A\ e^{-cs}$$

Now when $s=0$, $\dfrac{dx}{dt}=a\cos\alpha$, $\qquad A=a\cos\alpha$, consequently, the value of $\dfrac{dx}{dt}$ is that given in the text

(g) In fact, by substituting for $\dfrac{R}{m}$ in the value of $\dfrac{d^2y}{dt^2}$, there results

$$\frac{d^2y}{dt^2}=c\frac{ds}{dt}\frac{dy}{ds}\frac{ds}{dt}-g=c\frac{ds}{dt}\frac{dy}{dt}-g,$$

and it is evident, from inspection of these equations, that we may assume $\dfrac{dy}{dx}=p\ \dfrac{dx}{dt}$, $p$ being a new unknown

(h) Differentiating the equation $\dfrac{dy}{dt}=p\dfrac{dx}{dt}$, and dividing by $dt$, there results

$$\frac{d^2y}{dt^2}=\frac{dp}{dt}\frac{dx}{dt}+p\frac{d^2x}{dt^2}=-c\frac{ds}{dt}\frac{dy}{dt}-g=-cp\frac{ds}{dt}\frac{dx}{dt}-g=p\frac{d^2x}{dt^2}-g,$$

$$\frac{dx}{dt}\frac{dp}{dt}=-g$$

(i) Multiplying both sides of equation (2) by $dx$, we obtain

$$dp=-\frac{dx}{2h\cos^2\alpha}e^{2cs},\qquad dx=-2h\cos^2\alpha e^{-2cs}dp,\ \text{also } \frac{dy}{dt}.dt=$$

$$p\frac{dx}{dt}\ dt,\ \text{i e},\ dy=pdx,\ \text{and}\ \frac{dx}{dt}.dt\ \frac{dp}{dt}\ dt\,(=dx\,dp)=-gdt^2.$$

(*h*) $v^2 = \frac{dx^2 + dy^2}{dt^2} = (1+p^2)\frac{dx^2}{dt^2} = \left(\text{as } \frac{dx^2}{dt^2} = g \cdot \frac{dt'}{dp'}\right), (1+p^2)$
$g \cdot \frac{dt'}{dp^2}$

(*l*) $(\sqrt{1+p'^2} - p')\ (\sqrt{1+p'^2} + p') = 1$, therefore, the sum of the logarithms of the two factors is cipher, consequently, one of them is equal to the other affected with a contrary sign.

$cdx' =$ by sustituting for $p$ and $dp$

$$= \frac{-dp'}{-p'\sqrt{1+p'^2} + \log(\sqrt{1+p'^2} - p') - \gamma}$$

$$= \frac{-dp}{-p'\sqrt{1+p'^2} - \log(p' + \sqrt{1+p'^2}) - \gamma} = \frac{dp'}{P'}.$$

(*m*) For very great values of $p'$, $p'$ may be assumed equal to $\sqrt{1+p'^2}$, for in this case when $\sqrt{p'^2+1}$ is expanded, the first term is $p'$, and the powers of $p'$ which occur in the second and following terms are very small with respect to it, therefore,

$$\gamma + p'^2 + \log 2p', \text{ or } \gamma + p'^2 + \tfrac{1}{2}\log p'^2 + \log 2 = P';$$

and from the equation $e^x = 1 + x + x^2 + \&c$, it appears, as is stated in the text, that the logarithm of a very great number is always very small relatively to this number.

(*n*) In this case the expression becomes, by substituting $-p'$ for $p$,

$$cv^2 = \frac{g(1+p'^2)}{\gamma + p'\sqrt{1+p'^2} + \log(p' + \sqrt{1+p'^2})},$$

which becomes, by neglecting $\gamma + \log 2$ and $\frac{1}{2} \log p^2$,

$$cv^2 = g, \text{ and } \quad v^2 = \frac{g}{c}$$

(*o*) By taking the differential of $du$, we have evidently

$$du = dy \cos\beta - dx \sin\beta = (dy - dx \operatorname{tang}\beta)\ \cos\beta,$$

and if the values of $dy$ and $dx$ be substituted in this equation, and the sign of the common denominator be changed, we have the expression for $du$ given in the text

(*p*) By integrating this equation, we obtain $\frac{dy}{dx} = \frac{-1}{4ch\cos^2\alpha} e^{2x}$ $+ c$, and because when $x, y = 0$, $\operatorname{tang}\alpha = \frac{-1}{4ch\cos^2\alpha} + c$, $c = \operatorname{tang}\alpha$

$+\frac{1}{4ch\cos^2\alpha}$, by integrating a second time, we obtain $y=-\frac{1}{8c^2h\cos^2\alpha}e^{c2x}+\tang\alpha\ x+\frac{x}{4ch\cos^2\alpha}+c'$, and when $x$ and $y=0$ in this equation, we have $c'=-\frac{1}{8c^2h}$, hence by concinnating, we obtain for $y$ the the expression given in the text As $e^{2cx}=1+2cx+4c^2x^2+8c^3x^3$, &c $\left(\frac{e^{2cx}-2cx-1}{8c^2h\cos^2\alpha}\right)=\frac{x^2}{2h\cos^2\alpha}$, when $c=0$, hence $y=\tang\alpha x-\frac{x^2}{2h\cos^2\alpha}$ which is, as we know, the equation of a projectile in a vacuum See on this subject the twenty-first Number *Du Journal de l'Ecole Polytechnique,* page 191, and Vol II. Nos 358, 359

($q$) From this equation we obtain $-\frac{gdt^2}{dx^2}=\frac{dp}{dx}=\frac{e^{2cx}}{2h\cos^2\alpha}\cdot dt=\frac{e^{cx}dx}{\sqrt{2gh}\cos\alpha}$ and $t=\frac{e^{cx}}{\sqrt{gh}\ c\cos\alpha}+A$, when $t=0$ $x=0$ $A=-\frac{1}{\sqrt{2gh}\ c\cos\alpha}$, $\cdot$ the value of $t$ is that given in the text

($a$) The planet was in this case supposed to move in the circumference of a circle of which the sun occupied a position at *some distance from the centre,* it was on this account that the motion was said to be performed in an *eccentric* circle

($b$) It is easy to determine when $\theta-nt$, i e the equation of the centre is a *maximum,* for if about the focus of the ellipse, which the sun is supposed to occupy, as centre, a circle be described whose radius is a mean proportional between the semiaxes of the ellipse, the area of this circle is equal to that of the ellipse, and if we conceive a body to move in this circle with the mean angular velocity $n$ from the radius which coincides with the perihelion distance, at the instant the earth passes through the perihelion, and in the same direction, it is evident that as the earth's velocity in this point is greater than the mean angular velocity, the earth will *precede* the body which moves in the circle, and it will continue to precede it, until the angular motion of the body becomes equal to that of the earth, i e. until the earth's angular motion becomes equal to the mean angular motion, after this, the angular motion of the earth being less than the

mean angular motion, the body will begin to gain on the earth, and will evidently overtake it when the earth arrives at its aphelion, hence it is evident that the earth precedes the body by the greatest quantity, and the equation of the centre is a maximum when the angular motion of the earth is equal to that of the body, this is evidently, when $r$ the radius of the circle, is a mean proportion between the semiaxes, or $r = a(1-e^2)^{\frac{1}{4}}$ For if $n$ be, as in the text, the mean angular velocity, we have $\frac{r^2 n}{2}$ = the area described in an indefinitely small portion of time in the circle, and if $r'$ and $n'$ be the corresponding quantities in the ellipse, we have $\frac{r'^2 n'}{2}$ = the indefinitely small area described in the ellipse in the same time, consequently we have $n$ $n'$ as the synchronous areas in circle and ellipse divided by the squares of their corresponding radii $r$ and $r'$, but in the circle and ellipse the synchronous areas are equal, being as the whole areas divided by the periodic times, i e in a ratio of equality, hence the angular motions $n$, $n'$ are equal, and $\theta - nt$ the equation of the centre a maximum, when $r = r'$, i e at the points where the circle, which has been described with a radius equal to $a(1-e^2)^{\frac{1}{4}}$, intersects the ellipse It appears from this value of $r$, that the greatest value of the equation of the centre depends on the eccentricity, as is stated in the text

(c) From equation (2) there results, $\cos\theta = \frac{a(1-e^2)}{er} - \frac{1}{e}$, $d\cos\theta = -\sin\theta d\theta = -\frac{a(1-e^2)dr}{er^2}$, therefore as $\sin^2\theta = 1 - \cos^2\theta = \frac{e^2r^2 - a^2(1-e^2)^2 - r^2 + 2ra(1-e^2)}{e^2r^2}$, $\sin\theta = \frac{\sqrt{a^2e^2-(r-a)^2}\,(\sqrt{1-e^2})}{er}$

and, consequently, $d\theta = -\frac{d\cos\theta}{\sin\theta} = \frac{a(1-e^2)dr}{r\sqrt{a^2e^2-(r-a)^2}.\sqrt{1-e^2}}$, which is evidently equal to the expression in the text, and as $nadt = \frac{r^2 d\theta}{a.\sqrt{1-e^2}} = \frac{r\,dr}{\sqrt{a^2e^2-(r-a)^2}}$, by substituting for $r$ and $dr$ their values, there results $nadt = \frac{a(1-e\cos u)\,ae\sin u du}{\sqrt{a^2e^2 - a^2e^2\cos^2 u}} = \frac{a^2e(1-e)\cos u\sin u du}{ae\sin u}$, $ndt = (1-e)\cos u du$, likewise, by sub-

stituting for $r$ and $dr$ their values in the expression for $d\theta$, it becomes

$$d\theta=\frac{a^2\sqrt{1-e^2}\;e\sin u\,du}{a\,(1-e\cos u)\;ae\sin u}=\frac{\sqrt{1-e^2}\;du}{1-e\;\cos u}=\frac{\sqrt{1-e^2}\;du}{1-e\;\cos^2\frac{1}{2}u+e\sin^2\frac{1}{2}u},$$

now if both the numerator and denominator of this expression be divided by $\cos^2\frac{1}{2}u$, and if we observe that $\cos^2\frac{1}{2}u+\sin^2\frac{1}{2}u=1$, we shall obtain $d\theta=\sqrt{1-e^2}\cdot\frac{du}{\cos^2\frac{1}{2}u}$ — by $\frac{\sin^2\frac{1}{2}u}{\cos^2\frac{1}{2}u}+\frac{\cos^2\frac{1}{2}u}{\cos^2\frac{1}{2}u}-\frac{e\cos^2\frac{1}{2}u}{\cos^2\frac{1}{2}u}+\frac{e\sin^2\frac{1}{2}u}{\cos^2\frac{1}{2}u}=$ (by substituting for $\tan\frac{1}{2}u$ and for $\frac{du}{\cos^2\frac{1}{2}u}$ their respective values), the expression in the text. Now in this value of $d\theta$, let $1-e=m$, $1+e=n$, then $\frac{2\sqrt{1-e^2}dz}{1-e+(1+e)z^2}=\frac{2\sqrt{mn}\;dz}{m+nz^2}=$ $\left(\text{by making }\frac{\sqrt{n}}{\sqrt{m}}z=y\right)\frac{2dy}{1+y^2}$, consequently we have $\frac{\theta}{2}=\text{arc tang}$ $=y=\text{arc tang}=z\frac{\sqrt{1+e}}{\sqrt{1-e}}$, and $\text{tang}\,\frac{1}{2}\theta=\frac{\sqrt{1+e}}{\sqrt{1-e}}\text{tang}\,\frac{1}{2}u$

(*d*) $\text{Cos}\,int\;\cos i'nt=\frac{\cos(i+i')nt+\cos(i-i')nt}{2}$, $\sin int\;\sin i'nt$ $=\frac{\cos(i+i')nt-\cos(i-i')nt}{2}$, and it is evident that if these values be substituted for $\cos i'nt\;\cos int$, and $\sin i'nt\;\sin int$, and if the integrals be taken between the limits 0 and $\pi$, the results will be those indicated in the text, and as $\cos^2 int=\frac{\cos 2int+1}{2}$, $\sin i^2nt=$ $\frac{1-\cos 2int}{2}$, it is evident when these values are substituted in the expressions $\int_0^\pi\cos i^2nt\;d\,nt$, $\int_0^\pi\sin^2 int\;d\,nt$, that the results will be in each case $\frac{\pi}{2}$ When $i=0$, $\int_0^\pi\cos int\;d\,nt=\int_0^\pi d\,nt$, and therefore $A_0\int_0^\pi d\,nt=A_0\pi=\int_0^\pi r\;d\,nt$

(*e*) The integral of $(\theta-nt)\sin int\;d\,nt$ is $=\frac{(\theta-nt)\;\cos int}{i}+$ $\frac{1}{i}\int\cos int\;d(\theta-nt)$, and when taken between the limits $nt=0$, $nt=\pi$, it is evident, from the value of $\theta-nt$ given above, that it is

reduced to $\frac{1}{i}\int \cos int\, d(\theta - nt)$, hence it appears that the value of $B_i$ is that given in the text.

(*f*) As $int = iu - ie\sin u$, we have $\cos int = \cos(iu - ie\sin u)$, and if we remark that $d(\theta - nt) = d\theta - d\,nt = \frac{\sqrt{1-e^2}}{1-e\cos u}du - (1-e\cos u)du$, it is easy to deduce the value of $B_i$ given in the text.

(*g*) As $\cos(iu - ie\sin u) = \cos iu\cos(ie\sin u) + \sin iu\sin(ie\sin u)$ by expanding $\cos(ie\sin u)$, $\sin(ie\sin u)$ into a series according to the formula expressing the sine and cosine of an arc in terms of the arc, we will obtain the expression for $\cos(iu - ie\sin u)$ given in the text.

(*h*) By substituting for $r$ and $d\,nt$ in the value of $A_0$ we obtain $A_0 = \frac{a}{\pi}\int_0^\pi (1-e\cos u)^2du = \frac{a}{\pi}\int_0^\pi \left(1 - 2e\cos u + \frac{e^2}{2} + \frac{e^2}{2}\cos 2u\right)du$, equal at the limits $u = 0$, $u = \pi$, $\frac{a}{\pi}(\pi + \frac{1}{2}e^2\pi)$, i.e. $a(1 + \frac{1}{2}e^2)$.

(*i*) By substituting these values, we obtain, $v^2 = c^2\left(\frac{d\frac{1}{r}}{d\theta}\right)^2 + \frac{c^2}{r^2} = \frac{c^2e^2\sin^2\theta}{a^2(1-e^2)^2} + \frac{c^2(1 + 2e\cos\theta + e^2\cos^2\theta)}{a^2(1-e^2)^2}$, $\therefore v^2a^2(1-e^2)^2 = (1 + 2e\cos\theta + e^2)c^2$, but $\frac{2a(1-e^2)}{r} = 2 + 2e\cos\theta$, and $\frac{2a(1-e^2)}{r} - 1 + e^2 = 1 + 2e\cos\theta + e^2$ $\therefore \left(\frac{2a}{r} - 1\right)(1-e^2) = 1 + 2e\cos\theta + e^2$, and $\therefore v^2a^2(1-e^2)^2 = c^2\left(\frac{2a}{r} - 1\right)(1-e^2)$, and $v = \frac{c}{a\sqrt{1-e^2}}\sqrt{\frac{2a}{r} - 1}$, $\sin\delta = \frac{c}{vr} = \frac{a\sqrt{1-e^2}}{r\sqrt{\frac{2a}{r}-1}}$, it appears from this expression, that at the mean distance the velocity is equal to velocity in circle at same distance, and also that at this same distance the angle $\delta$ is a minimum.—See page 374.

(*k*) Since $\sin\phi = \sin\gamma\sin(\theta + \omega)$, as $\sin\gamma$ is always + when $\sin\phi$ is +, then $\theta + \omega$ is $\angle 180$.

(*l*) From the equation $\tan\psi = \cos\gamma\tan\lambda$ we obtain $\tan^2\psi\cos^2\lambda = \cos^2\gamma\sin^2\lambda$, i.e. $\tan^2\psi(1 - \sin^2\lambda) = \cos^2\gamma\sin^2\lambda$, $\sin^2\lambda = \frac{\tan^2\psi}{\cos^2\gamma + \tan^2\psi}$, and $\sin\phi = \sin\gamma\sin\lambda = \frac{\sin\gamma\tan\psi}{\sqrt{\cos^2\gamma + \tan^2\psi}}$

(*m*) The comets do not appear to have exerted any sensible attraction on the planetary masses, or in any way to have deranged their motions, when they are observed through very powerful telescopes, and under circumstances in which we ought only to perceive a part of the illuminated atmosphere, we are not able to discover any phases, stars are said to have been seen through the densest part of the nucleus

(*n*) See Dorpat Catalogue of Double Stars, by Struve; Philosophical Transactions, 1802 and 1803, *Connaissance des temps*, 1830

(*o*) The force $\rho\frac{ds^2}{dt^2}$ acts in the direction of the tangent, and this force resolved in the directions of the axes of $x$ and $y$, is equal to $\rho\frac{ds^2}{dt^2}\frac{dx}{ds}$, $\rho\frac{ds^2}{dt^2}\frac{dy}{ds}$, which since $\frac{dx}{ds}\frac{ds}{dt}$, $\frac{dy}{ds}\frac{ds}{dt}$, are respectively equal to $\frac{dx}{dt}$ and $\frac{dy}{dt}$, become the second members of equations (2) given in the text

(*p*) If equations (2) be respectively multiplied by $dx$ and $dy$, and then added together, there results, as

$$dx = \frac{dx}{dt}.dt,\ ds = \frac{ds}{dt}\,dt, \&c.$$

$$\frac{dx\,d^2x + dy\,d^2y}{dt^2} + \mu\left(\frac{x\,dx + y\,dy}{r^3}\right) = -\rho\left(\frac{dx^2 + dy^2}{dt^2}\right)ds,$$

which, since $\frac{dx^2 + dy^2}{dt^2} = \frac{dr^2 + r^2d\theta^2}{dt^2}$ and $r = \sqrt{x^2 + y^2}$, is equal to

$$\tfrac{1}{2}\,d\left(\frac{dr^2 + r^2d\theta^2}{dt^2}\right) - \mu\,d\frac{1}{r} = -\rho\left(\frac{dr^2 + r^2d\theta^2}{dt^2}\right)ds,$$

and in like manner, if the first equation (2) be multiplied by $y$, and then taken from the second multiplied by $x$, we obtain

$$\frac{xd^2y}{dt^2} - \frac{yd^2x}{dt^2} = d.\left(\frac{xdy - ydx}{dt^2}\right) = -\rho\frac{ds}{dt}\left(\frac{xdy}{dt} - \frac{ydx}{dt}\right);$$

and as $\frac{xdy}{dt} - \frac{ydx}{dt} = r^2\frac{d\theta}{dt}$, we obtain

$$d\ r^2\frac{d\theta}{dt} = -\rho\frac{ds}{dt}r^2d\theta$$

(*q*) See No. 232

(q) $\theta = nt + \varepsilon + \theta_1$, now $\theta_1$ is a periodic function arranged according to the sines of the increasing multiples of $nt + \varepsilon - \omega$, when $\theta$ is increased by 360, it is evident that we arrive at the same point as before If there was no perturbation the value of the time lapsed between two consecutive returns to the perihelion would be equal to that of a return to the same fixed point.

(s) If the first of these equations be multiplied by $1 - e\cos u$, and the second by $ae\sin u$, we obtain by adding them together,

$$(1-e\cos u)^2 da - a\cos u(1-e\cos u)de + ae\sin u(1-e\cos u)du + ae\sin u(d\varepsilon - d\omega) + ae\sin^2 u de - ae\sin u(1-e\cos u)du,$$

which as $ae(\sin^2 u + \cos^2 u) = ae$, is evidently the expression in the text

(t) As by the third equation (a)

$$\text{tang}^2 \tfrac{1}{2} u = \frac{1-e}{1+e} \text{tang}^2 \tfrac{1}{2}(\theta - \omega),$$

if this value be substituted in the expression for $\cos u$, there results

$$\cos u = \frac{1 - \dfrac{(1-e)\sin^2\frac{1}{2}(\theta-\omega)}{(1+e)\cos^2\frac{1}{2}(\theta-\omega)}}{1 + \dfrac{(1-e)\sin^2\frac{1}{2}(\theta-\omega)}{(1+e)\cos^2\frac{1}{2}(\theta-\omega)}},$$

$$= \frac{(1+e)\cos^2\frac{1}{2}(\theta-\omega) - (1-e)\sin^2(\theta-\omega)}{(1+e)\cos^2\frac{1}{2}(\theta-\omega) + (1-e)\sin^2(\theta-\omega)},$$

which as $\cos^2\frac{1}{2}(\theta-\omega) + \sin^2\frac{1}{2}(\theta-\omega) = 1$, $\cos^2\frac{1}{2}(\theta-\omega) - \sin^2\frac{1}{2}(\theta-\omega) = \cos(\theta-\omega)$ is evidently equal to the expression given in the text, hence we have

$$(1-e\cos u)^2 = \frac{(1-e^2)^2}{(1+e\cos(\theta-\omega)^2)},$$

$$a(e-\cos u) = \frac{(e^2-1)\,a\cos(\theta-\omega)}{1+e\cos(\theta-\omega)},$$

$$ae\sin u = \frac{ae\sqrt{1-e^2}\sin(\theta-\omega)}{1+e\cos(\theta-\omega)};$$

now if these values of $(1-e\cos u)^2$, $a(e-\cos u)$, $ae\sin u$ be substituted in the equation

$$(1-e\cos u)^2 da + a(e-\cos u)de + ae\sin u(d\varepsilon - du) = 0,$$

we will obtain, by multiplying by $1 + e\cos\theta - \omega$, and dividing by $1-e^2$, the equation (f)

(v) It appears from the equation

$$\{\, r^2 d\theta = \sqrt{\mu a(1-e^2)}dt \,\}$$

that the synchronous areas described about different centres of forces, vary as the square roots of the parameters of the conic sections described, multiplied into the square roots of the sum of the masses of the attracting and attracted body, see Princip, Book I. sec 3, prop. 16

$$(u)\ v^2 = \frac{dr^2 + r^2 d\theta^2}{dt^2} = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad d\left(\frac{dr^2 + r^2 d\theta^2}{dt^2}\right) - 2\mu d\,\frac{1}{r} =$$

$$\left(-2\rho\left(\frac{dr^2 + r^2 d\theta^2}{dt^2}\right)ds\right) = -\mu\ d\,\frac{1}{a} \quad d\,\frac{1}{a} = 2\rho\left(\frac{2}{r} - \frac{1}{a}\right)ds$$

$$d\,r^2 d\theta = -\rho r^2 d\theta ds = \sqrt{\mu}\ d\sqrt{a(1-e^2)}dt, \quad -\rho\sqrt{\mu a(1-e^2)}\ dt\,ds$$

$$= \sqrt{\mu}\ d\sqrt{a(1-e^2)}\,dt$$

$$(x)\ d\,\frac{1}{a} = 2\rho\left(\frac{2}{r} - \frac{1}{a}\right)ds \quad -\frac{da}{a} = 2\rho\left(\frac{2a - r}{r}\right)ds,$$ and if in this equation there be substituted the value of $r$, we shall obtain the expression given in the text

$$d\sqrt{a(1-e^2)} = -\rho\sqrt{a(1-e^2)}ds, \text{ i e } \frac{da(1-e^2) - 2aede}{2\sqrt{a(1-e^2)}} = -$$

$$\rho\sqrt{a(1-e^2)}ds \qquad 2ede = \frac{2\rho a(1-e^2)ds + da(1-e^2)}{a} =$$

(by substituting for $da$ its value)

$$2\rho(1-e^2)ds - 2\rho(1 + 2e\cos(\theta-\omega) + e^2)ds,$$

which, by obliterating the quantities that destroy each other, becomes, by dividing by $e$, equal to the value of $de$ given in the text

Equation (e) becomes, by performing the differentiations,

$$r\cos\theta\cos\omega de - re\cos\theta\sin\omega d\omega + r\sin\theta\ \sin\omega de + re\sin\theta\cos\omega d\omega =$$

$$da(1-e^2) - 2\,aede,$$

that is, by concinnating and substituting for $da$ and $de$ in the second member,

$$r\cos(\theta-\omega)de + re.\sin(\theta-\omega)d\omega = -2\rho a(1 + 2e\cos(\theta-\omega) + e^2)ds$$

$$+ 4\rho ae(e + \cos(\theta-\omega))ds\ (= -2\rho a(1-e^2)ds),$$

now if in the first member of this equation we substitute for $r$ and $de$ their values, the preceding equation becomes by dividing both sides by the common factor, $a(1-e^2)$

$$-\frac{2\rho(e\cos(\theta-\omega) + \cos(\theta-\omega)^2)ds + \sin(\theta-\omega)ed\omega}{1 + e\cos(\theta-\omega)} = -2\rho ds,$$

and consequently

$$ed\omega\sin(\theta-\omega)=-2\rho(1+e\cos(\theta-\omega)ds+2\rho(e\cos(\theta-\omega)+\cos(\theta-\omega)^2)ds$$

$$=-2\rho\,(1-\cos(\theta-\omega)^2)ds=-2\rho\sin(\theta-\omega)^2 ds$$

$$ed\omega=-2\rho\sin(\theta-\omega)ds$$

If in equation (f) there be substituted for $da$, $de$, and $ed\omega$ their values already obtained, there results

$$\left[-2\rho a\left(\frac{1+2e\cos(\theta-\omega)+e^2}{1+e\cos(\theta-\omega)}\right)+2\rho a(e\cos(\theta-\omega)+\cos(\theta-\omega)^2)+\right.$$

$$\left.\frac{2\rho a\sin(\theta-\omega)^2}{\sqrt{1-e^2}}\Big)\right]ds+\frac{ae\sin(\theta-\omega)}{\sqrt{1-e^2}}d\varepsilon,$$

reducing the two first terms to the same denominator they become

$$\frac{-2\rho a\,(1+2e\cos(\theta-\omega)+e^2)+2\rho a(e\cos(\theta-\omega)+\cos(\theta-\omega)^2)+2\rho a(e^2\cos(\theta-\omega)^2+e\cos(\theta-\omega)}{1+e\cos(\theta-\omega)}\Big)\cdot$$

$$=(\text{as } e\cos(\theta-\omega)^3=e\cos(\theta-\omega)\,(1-\sin(\theta-\omega)^2)$$

$$-\frac{2\rho a\,(1+e^2)\,\sin^2(\theta-\omega)+e\cos(\theta-\omega)\sin(\theta-\omega)^2}{1+e.\cos(\theta-\omega)},$$

and thus equation (f) becomes

$$-2\rho a\left[\frac{[(1+e^2)+e\cos(\theta-\omega)]\sin(\theta-\omega)^2}{1+e\cos(\theta-\omega)}+\frac{2\rho a\ \sin(\theta-\omega)^2}{\sqrt{1-e^2}}\right]ds$$

$$+ae.\frac{\sin(\theta-\omega)\,d\varepsilon}{\sqrt{1-e^2}}=0,$$

dividing by $\sin(\theta-\omega)$, and then reducing to a common denominator, we obtain

$$-2\rho a[\sin(\theta-\omega)\,(1+e\cos(\theta-\omega)+e^2)\sqrt{1-e^2}-(1+e\cos(\theta-\omega)\sin$$

$$(\theta-\omega)]ds+ae\,(1+e.\cos(\theta-\omega))d\varepsilon=0,$$

multiplying both sides by $1+\sqrt{1-e^2}$, there results

$$-2\rho a\sin(\theta-\omega)\,[e^2\sqrt{1-e^2}-e^3\cos(\theta-\omega)-e^4]ds+ae\,(1+e(\theta-\omega)$$

$$(1+\sqrt{1-e^2}))d\varepsilon=0,$$

hence we obtain the value of $d\varepsilon$ given in the text, and in order to obtain the expression for $ds$ we have

$$\frac{dr}{d\theta}=\frac{a(1-e^2)e\sin\theta-\omega)}{(1+e.\cos(\theta-\omega))^2}\qquad r^2+\frac{dr^2}{d\theta^2}=$$

$$(a\ \ (1-e^2))^2\frac{1+2e\ \cos(\theta-\omega)+e^2\,(\sin^2(\theta-\omega)+\cos^2(\theta-\omega)}{(1+e\cos\theta-\omega)^4},$$

$$\cdot\ ds=\sqrt{r^2+\frac{dr^2}{d\theta^2}}\,d\theta=a(1-e^2)\frac{\sqrt{1+2e\cos(\theta-\omega)+e^2}}{(1+e\cos(\theta-\omega))^2}\,d\theta$$

It appears from the value of $\delta a$ that the radius vector diminishes while $\delta n$ the angular velocity increases, and the variations of the quantities $e$, $\omega$, $\varepsilon$ are only periodic, the variation of the absolute velocity becomes by substituting $3\rho an\theta$ for $\delta n$, and $-2\rho a^2\theta$ for $\delta a$, the expression in the text Likewise, as $\delta r=-2\rho a^2\theta$, the sum of these continual diminutions from $\theta=0$ to $\theta=2\pi$, is $\pi\rho a^2$.

($y$) From the equation $r^2d\theta=cdt$, it appears that when the areas are proportional to the times of their description, the angular velocity $d\theta=\frac{cdt}{r^2}$, varies inversely as the square of the distance, and conversely, when the angular velocity varies inversely as the square of the distance, the areas are proportional to the times, for then $cdt=r^2d\theta=$ a constant quantity, and generally, in different orbits the angular velocity $d\theta=\frac{cdt}{r^2}$, varies as the synchronous area divided by the square of the distance

$r^2d\theta=cdt\quad \frac{rd\theta}{dt}=(v\sin\delta)=\frac{c}{r}$, and at the initial distance where $r=\gamma$, $\delta=\alpha$, $v=\sqrt{2gh}$ we have $\sqrt{2gh}\sin\alpha=\frac{c}{\gamma}$.

($z$) Multiplying both sides by $dx$ and integrating, we obtain for the first equation $\frac{dx^2}{dt^2}=-\frac{kx^2}{\gamma}+c$, $\frac{dy^2}{dt^2}=-\frac{ky^2}{\gamma}+c'$, hence it is easy to obtain by the known rules the values of $x$ and $y$ given in the text

($a'$) When $t$ is increased by $2\pi\sqrt{\frac{\gamma}{k}}$, the values of $x$ and $y$ remain the same as before, for in this case, the value of $x$ is $\gamma.\cos$
$\left((t+2\pi)\sqrt{\frac{\gamma}{k}}\right)\sqrt{\frac{k}{\gamma}}-\sqrt{\frac{2gh}{k\gamma}}.\cos\alpha.\sin\left(t+2\pi\sqrt{\frac{\gamma}{k}}\right)\sqrt{\frac{k}{\gamma}}$

$=\gamma\left(\cos t\sqrt{\frac{k}{\gamma}}.-\sqrt{\frac{2gh}{k\gamma}}\cos\alpha\sin t\sqrt{\frac{k}{\gamma}}\right)$, the same is true for the value of $y$.

It appears from this, that when the force is proportional to the distance the periodic time varies as $\sqrt{\frac{\gamma}{k}}$.

($b'$) By squaring both sides of these equations, and then adding them together, this result is obtained, if $\alpha=90$ and $k\gamma \angle 2gh$, the trajectory will be an ellipse, of which $\gamma$ will be the major semiaxis, but if $k\gamma > 2gh$, $\gamma$ will be the minor semiaxis

When the force becomes repulsive, i e, when R is changed into $-$R, the values of $x$ and $y$ will not be expressed in periodic functions of $t$, but will increase indefinitely.

($c'$) Equation (3) $c^2\left(\frac{d\frac{1}{r}}{d\theta}\right)^2+\frac{c^2}{r^2}+2\int \mathrm{R}dr=b$, becomes, by substituting for $c^2$, $\int\mathrm{R}dr$, and $b$, their respective values, $\gamma^2\,2gh\,.\sin^2\alpha.$
$\left(d\left(\frac{\frac{1}{r}}{d\theta}\right)^2+\frac{1}{r^2}\right)+k\gamma\left(1-\frac{\gamma^2}{r^2}\right)=2gh,=\left(\text{as }\gamma\frac{d\frac{1}{r}}{d\theta}=\frac{dz}{d\theta}\right),2gh\sin^2\alpha.$
$\left(\frac{dz^2}{d\theta^2}+z^2\right)+k\gamma\,(1-z^2)=2gh$, hence dividing by $2gh\sin^2\alpha$, and concinnating, we obtain the expression $\frac{dz}{d\theta}+\left(1-\frac{k\gamma}{2gh\sin^2\alpha}\right)z^2=\frac{1}{\sin^2\alpha}-\frac{k\gamma}{2gh\sin^2\alpha}$, which, by substituting $\pm n^2$ for $1-\frac{k\gamma}{2gh\sin^2\alpha}$, and observing that $\frac{1}{\sin^2\alpha}=\cot^2\alpha+1$, becomes the expression given in the text

($d'$) In order to integrate this differential in the case where the superior signs are employed, let $\frac{\cot^2\alpha+n^2}{n^2}=l^2$, and then $nd\theta=\frac{dz}{\sqrt{l^2-z^2}}$, of which the integral is arc $\sin=\frac{nz}{\sqrt{\cot^2\alpha+n^2}}+\mathrm{c}$, and when $\theta=0$, $z=1$ $\mathrm{c}=-$ arc $\sin=\frac{n}{\sqrt{\cot^2\alpha+n^2}}$, hence the value of $n\theta$ is that given in the text

In the second case, the integral is $\pm\,n\theta=\log\,(\pm\,nz+\sqrt{\cot^2\alpha-n^2+n^2z^2})+\mathrm{c}$, and when $\theta=0$, and $z=1$, we have $\mathrm{c}=-\log\,(\pm\,n+\cot\alpha)$, and therefore,

$$\pm n\theta = \log\left(\frac{\pm nz \pm \sqrt{\cot^2\alpha - n^2 + n^2 z^2}}{\pm(n+\cot\alpha)}\right)$$

From the first value of $n\theta$ we obtain

$$\sin n\theta = \frac{nz\ \cot\alpha - n\sqrt{\cot^2\alpha + (1-z^2)n^2}}{\cot^2\alpha + n^2},$$

$$\cos n\theta = \frac{n^2 z + \cot\alpha\sqrt{\cot^2\alpha + (1-z^2)n^2}}{\cot^2\alpha + n^2},$$

hence, multiplying the first of these equations by $\cot\alpha$, and the second by $n$, we shall have

$$\cot\alpha\ \sin\ n\theta = \frac{nz\cot^2\alpha - n\ \cot\alpha\sqrt{\cot^2\alpha + (1-z^2)\ n^2}}{\cot^2\alpha + n^2},$$

$$n\ \cos\ n\theta = \frac{n^3 z + n\ \cot\alpha\sqrt{\cot^2\alpha + (1-z^2)\ n^2}}{\cot^2\alpha + n^2},$$

and by adding them we obtain

$$\cot\alpha\sin n\theta + n\cos n\theta = nz\left(\frac{\cot^2\alpha + n^2}{\cot^2\alpha + n^2}\right) = nz.$$

($e'$) $\dfrac{ndz}{d\theta} = n\ \cot\alpha\ \cos n\theta - n^2 \sin n\theta$, when $z$ is a maximum we have $\dfrac{ndz}{d\theta} = 0$, and $\cot\alpha\ \cos n\theta - n\ \sin n\theta = 0$, consequently, $\cot\alpha = n\,\text{tang}\,n\theta$ It is easy to perceive, that this value of tang $n\theta$ gives $z$ a *maximum*, and, consequently, $r$ a *minimum*, substituting this value of $\cot\alpha$ in the expression for $nz$, we obtain

$$nz = n\ \frac{\sin^2 n\theta}{\cos n\theta} + n\ \cos n\theta = n\,.\left(\frac{\sin^2 n\theta + \cos^2 n\theta}{\cos . n\theta}\right) =$$

$$n\ \sqrt{\text{tang}^2 n\theta + 1},\ \text{and}\ z = \sqrt{\frac{1}{n^2}\cot^2\alpha + 1}$$

At the point where $r$ is a minimum, we have $\dfrac{dz}{d\theta} = -\dfrac{dr}{d\theta}\ \dfrac{1}{r^2} = 0$, therefore, at this point $\dfrac{dr}{d\theta} = 0$, consequently, the curve has an apse at this point

($f'$) When $z = 0$, $r$ is $\infty$, the value of tang $n\theta$ for $z = 0$, is $\dfrac{-n}{\cot\alpha} = -n\,\text{tang}\,\alpha$, consequently, if from the centre we draw a line making with the initial distance $\gamma$ an angle $\theta$, such that tang $n\theta =$

$-n$ tang $\alpha$, the line drawn making this angle will meet the curve at an infinite distance, and the curve has an asymptote parallel to this line, for if $p$ be a perpendicular let fall from centre on tangent, we have in general $\frac{1}{p^2} = \frac{dz^2}{d\theta} + z^2$, in the case of an asymptote $z$ vanishes, and the value of $\frac{dz}{d\theta} = \frac{1}{p} = -\sqrt{\cot\alpha^2 + n^2}$, consequently, if a line be drawn from the centre at right angles to the direction of the radius which makes with $\gamma$ an angle $\theta$ such, that tang $n\theta = -n$ tang $\alpha$, and equal to $p$, and if through the extremity of $p$ a line be drawn parallel to the infinite radius, this line will be an asymptote

($g'$) In the case of the logarithmic value of $\pm n\theta$, if we suppose $n$ to be affected with a negative sign, we will have

$$-n\theta = \log.\left(\frac{-nz + \sqrt{\cot^2\alpha - n^2 + n^2z^2}}{-n + \cot\alpha}\right),$$

and therefore

$$(-n + \cot\alpha)\,e^{-n\theta} = -nz + \sqrt{\cot^2\alpha - n^2 + n^2z^2},$$

but we have also

$$(n + \cot\alpha)\,e^{n\theta} = nz + \sqrt{\cot^2\alpha - n^2 + n^2z^2},$$

hence we obtain, by taking the first equation from the second,

$$(n + \cot\alpha)\,e^{n\theta} + (n - \cot\alpha)\,e^{-n\theta} = 2nz = \frac{2n\gamma}{r},$$

consequently, by substituting for $e^{n\theta}$, $e^{-n\theta}$, their respective values, we shall have

$$2nz = \begin{bmatrix} (n + \cot\alpha)\left(1 + \frac{n\theta}{1} + \frac{n^2\theta^2}{1\ 2} + \frac{n^3\theta^3}{1\ 2\ 3} + \&c,\right) \\ (n - \cot\alpha)\left(1 - \frac{n\theta}{1} + \frac{n^2\theta^2}{1\ 2} - \frac{n^3\theta^3}{1\ 2\ 3} + \&c,\right) \end{bmatrix}$$

and from the inspection of these it is evident that $z$ continually increases

The value of $\frac{dz}{d\theta} = \left(\frac{n + \cot\alpha}{2}\right)e^{n\theta} - \left(\frac{n - \cot\alpha}{2}\right)e^{-n\theta}$, if in order to determine the greatest value of $z$, we put this expression equal to cipher, we will obtain $e^{n\theta} = \frac{\sqrt{n - \cot\alpha}}{\sqrt{n + \cot\alpha}}$ and $z = \frac{1}{n}\sqrt{n^2 - \cot^2\alpha}$, if $n > \cot\alpha$ this value is real, and $z$ is a minimum for this value of $e^{n\theta}$,

and beyond this value of $\theta$, $z$ increases, and the curve which is traced has two values that are symmetrical on each side of this value of $e^{n\theta}$ The curve has an apse at this point, for $-\frac{r^2 d\theta}{dr}=\frac{d\theta}{dz}$ $=\infty$ in this case, but $\frac{r\,d\theta}{dr}$ is the tangent of the angle under the tangent and radius vector, and therefore at this point this angle is right If the motion commenced from this point, we would have $\alpha=90$, and the equation of the spiral would be $r=\frac{2\gamma}{e^{n\theta}+e^{-n\theta}}$, from inspection of this equation it is evident, that the branches at each side of the apse are symmetrical, for the curve remains the same when $n\theta$ is changed into $-n\theta$, the branches intersect in an indefinite number of points, all of which exist on the line drawn from the centre to the apse, the distance of any one of these intersections such as the $m^{th}$ from the centre is determined by the equation

$$r=\frac{2\gamma}{e^{m\pi}+e^{-m\pi}}$$

From the general expression for the time it follows, that the time to the centre, when the number of revolutions is infinite,

$$=\frac{\gamma}{n\sqrt{2gh}}=\frac{\gamma}{\sqrt{2gh-k\gamma}}.$$

If in the preceding value of $z=\frac{1}{n}\sqrt{n^2-\cot^2\alpha}$, we suppose $\cot^2\alpha>n^2$, then the value of $z$ is impossible, and the curve does not admit of either a maximum or minimum for $z$, however, the curve has an asymptote, for we shall find that the value of $\frac{1}{p^2}\left(=z^2+\frac{dz^2}{d\theta^2}\right)$ when $z=0$ or $r=\infty$, is equal to $(\cot^2\alpha-n^2)$

From the preceding discussion it appears, that in the equation $z=\frac{1}{2n}(n+\cot\alpha)\,e^{n\theta}+\frac{1}{2n}(n-\cot\alpha)\,e^{-n\theta}$, the curve which it denotes is different according as $n$ is $>$ or $\angle$ than $\cot\alpha$, when $n>\cot\alpha$ the curve has an apse, in the second case it has an infinite radius and asymptote In both cases we have $2gh\sin^2\alpha\angle k\gamma$, but as $2gh-k\gamma=k\gamma\cot^2\alpha-2n^2gh$; in the first case also we have $2gh\angle k\gamma$, and in the second $2gh>k\gamma$

If in the equation

$$\left[2nz = \begin{Bmatrix} n + \cot\alpha \\ n - \cot\alpha \end{Bmatrix} \begin{pmatrix} 1 + \frac{n\theta}{1} + \frac{n^2\theta^2}{1\,2} + \frac{n^3\theta^3}{1\,2\,3} + \&c \\ 1 - \frac{n\theta}{1} + \frac{n^2\theta^2}{1\,2} - \frac{n^3\theta^3}{1\,2\,3} - \&c. \end{pmatrix}\right]$$

we suppose $n=0$, i. e., $2gh \sin^2\alpha = k\gamma$, we shall have $z = \frac{\gamma}{r} = \cot\alpha\ \theta$, and $\gamma = \cot\ \alpha\ r\ \theta$, which is the equation of the reciprocal or hyperbolic spiral. In order to determine the time, if in the equation $r^2 d\theta = cdt$, we substitute $\frac{\gamma^2}{\cot^2\alpha\ \theta^2}$ for $r^2$, we obtain $\frac{\gamma^2\ d\theta}{\cot^2\alpha\theta.^2} = cdt$ $ct = \frac{-\gamma^2}{\cot^2\alpha}\frac{1}{\theta} + c$, and between the limits $\theta$, $\theta'$, we have

$$ct = \frac{\gamma^2}{\cot^2\alpha}\left(\frac{1}{\theta} - \frac{1}{\theta'}\right).$$

If T denote the entire time to the centre, we shall have, commencing with $\theta = 2\pi$,

$$cT = \frac{\gamma^2}{\cot^2\alpha}\ \frac{1}{2\pi}\left[\frac{1}{1\,2} + \frac{1}{2\,3} + \frac{1}{3\,4} + \&c\right] = \frac{\gamma^2\ \text{tang}^2\alpha}{2\pi}$$

The asymptote to this curve is determined by the equation $\frac{1}{p^2} = z^2 + \frac{dz^2}{d\theta^2} =$ in this case $z^2 + \cot^2\alpha$, when $z=0$, or $r=\infty$, we have $p$ the perpendicular from the centre on the asymptote $=$ tang $\alpha$

If in the expression for $z$, we suppose $n = \cot\alpha$, i. e., $2gh = k\gamma$, the value of $z$ becomes $= e^{\cot\alpha\ \theta}$, from which it is evident, that the curve described is the logarithmic spiral, and if the base $a = e^{\mp\cot\alpha}$, the equation of the curve will be $r = a^\theta$, in order to determine the time, if in the equation $cdt = r^2 d\theta$, we substitute for $r^2$, and integrate between the limits $\theta=0$, $\theta = -2\pi$, we obtain $ct = \left(1 - \frac{1}{a^{2\pi}}\right)$, and in like manner between the limits $\theta = -2\pi$ $\theta = -4\pi$, we find $ct' = \frac{1}{2.a^{2\pi}}\left(1 - \frac{1}{a^{2\pi}}\right)$, and so on for successive revolutions, therefore, if T denote the time to the centre, it will be expressed by the series

$$\frac{1}{2c}\left(1 - \frac{1}{a^{2\pi}}\right)\left[1 + \frac{1}{a^{2\pi}} + \frac{1}{a^{4\pi}} + \&c\right],$$

$=$ by substituting for $c$, and taking the sum of this series

$$\frac{\cos ec^t\ \alpha}{\gamma\, 2.\sqrt{2gh}}$$

($h'$) This value is evidently equal to

$$d\theta = \frac{c'd\rho}{\sqrt{k^2\gamma^4 - \beta c^2 - k^2\gamma^4 + 2k\gamma^2 c^2\rho - c^4\rho^2}},$$

which by assuming $y = c^2\rho - k\gamma^2$, and by eliminating becomes

$$\frac{dy}{\sqrt{k^2\gamma^4 - \beta c^2 - y^2}},$$

and the integral of this is equal to

$$\omega + \text{arc}\cos = \frac{-y}{\sqrt{k^2\gamma^4 - \beta c^2}} = 0,$$

equal by substituting for $y$,

$$\omega + \text{arc}\cos = \frac{k\gamma^2 - c^2\varrho}{\sqrt{k^2\gamma^4 - \beta c^2}} = 0.$$

The value of $r$ deduced from this equation is

$$r = \frac{c^2}{k\gamma^2 - \sqrt{k^2\gamma^4 - c^2\beta}\ \cos(\theta - \omega)};$$

consequently, when $\theta = \omega$, $r$ is a *maximum*, when $\theta = \omega + \pi$, $r$ is a *minimum*. $\omega + \pi$ is substituted for $\omega$ to embrace the case of the comets

($i'$) By substituting for $r$ and $r\ \cos(\theta - \omega)$, their respective values in equation (a), and squaring both sides of the resulting expression, we obtain

$$k^2\gamma^4\,(x'^2 + y'^2) = c^4 + x'^2.(k^2\gamma^4 - c^2\beta) - 2c^2x'\ \sqrt{k^2\gamma^4 - c^2\beta}$$

($k'$) If in general $4p$ denotes the principal parameter of an ellipse, we will have $ae = \sqrt{a^2 - 2ap} = a - p + (\quad)\frac{1}{a}$ &c

$a(1 - e)$ the shortest distance $= p + (\quad)\frac{1}{a} -$ &c, consequently, when $a$ is $\infty$, we shall have $a(1 - e) = p$

($l'$) Substituting for $p$ and dividing by $\gamma$, we obtain

$1 + \cos\omega = 2\sin^2\alpha$, i e, $2\cos^2\frac{1}{2}\omega = 2\sin^2\alpha$, for $1 + \cos\omega = 2\cos^2\frac{1}{2}\omega$

($m'$) Since $\int\frac{d\psi}{\cos^2\psi} = \frac{\sin\psi}{\cos\psi}$, by partially integrating, we obtain

$$\int \frac{d\psi}{\cos^4\psi} = \frac{\sin\psi}{\cos^3\psi} - \int \frac{\sin\psi}{\cos\psi}\,\frac{2\sin\psi\ \cos\psi d\psi}{\cos^4\psi} = -\left(2\int \frac{\sin^2\psi d\psi}{\cos^4\psi}\right) =$$

$$-2\int \frac{d\psi}{\cos^4\psi} + \int \frac{2d\psi}{\cos^2\psi},$$ and as $\int \frac{d\psi}{\cos^2\psi} = \text{tang}\,\psi$, we have $\int \frac{d\psi}{\cos^4\psi}$

$= \text{tang}\,\psi\ \ (1+\text{tang}^2\psi) + 2\,\text{tang}\,\psi - 2\int \frac{d\psi}{\cos^4\psi}$, by integrating this last quantity in the same way, we shall obtain a similar result multiplied by 2, and by continuing this partial integration, we will arrive at the following series,

$$\text{tang}\,\psi(1+\text{tang}^2\psi)(1-2+4-8+\&c.)+2\text{tang}\,\psi(1-2+4-8+\&c.) +e,$$ but $(1-2+4-8+\&c.) = (1+2)^{-1} = \frac{1}{3}$,

hence then we shall have this integral,

$$= \frac{\text{tang}\ \psi(1+\text{tang}^2\psi) + 2\,\text{tang}\,\psi}{3} + e = \frac{nt}{\sqrt{2}}$$

($n'$) It appears from equation (e), compared with $n = \frac{\sqrt{\mu}}{p\sqrt{p}}$, that the times, in which comets moving in different parabolic orbits, describe equal anomalies, vary in the sesquiplicate ratio of the perihelion distance.

Also by means of equation $n = \frac{\sqrt{\mu}}{p\sqrt{p}}$, $i = \frac{\sqrt{\mu}}{l\sqrt{l}}$, the synchronous areas described by the earth and comet may be compared, for the sector described by the comet at the perihelion distance $p$, is to the synchronous sector, described by a body moving in a circle of which the radius $= p$ with an angular velocity equal to $n$, as $\sqrt{2}$ 1 See *Mechanique Celeste*, Book II. Chap IV.

## CHAPTER VII

($\alpha$) When $r^2$ is neglected in consequence of its minuteness, we have $\rho^{-3} = (a^2 - 2ar\cos\lambda)^{-\frac{3}{2}} = a^{-3} + 3a^{-4}r\cos\lambda$, consequently,

$$\frac{fm'r\sin\lambda}{\rho^3} = \frac{fm'r\sin\lambda}{a^3},\ \frac{fm'a}{\rho^3} = \frac{fm'}{a^2} + \frac{3fm'r\cos\lambda}{a^3}$$

therefore,

$$\frac{fm'a - fm'r\cos\lambda}{\rho^3} = \frac{fm'}{a^2} + \frac{2fm'r\cos\lambda}{a^3}$$

(*b*) See *Mechanique Celeste*, Tom IV, and also the translation of the System of the World, Vol I Book I Chap XV, and Vol II Chap XI

(*c*) The force $\phi'$ resolved in the direction CM is equal to $\phi' \cos \lambda$, for $\lambda$ is the angle between CM and AC, and as the direction of the force $\phi$ is perpendicular to AC, the angle which it makes with CM must be the complement of $\lambda$, consequently this force resolved in the direction of CM $= \phi \sin \lambda$, and if we substitute for $\phi'$ and $\phi$ their respective values $\frac{2fmr \cos\lambda}{a^3}$, $\frac{fm'r \sin \lambda}{a^3}$, and if we observe, that these components of $\phi'$ and $\phi$ act in opposite directions along the radius CM, and consequently must be affected with opposite signs, the result of the two must be equal to $(2 \cos^2 \lambda - \sin^2 \lambda) \frac{fm'r}{a^3}$, which, by substituting $\frac{m}{75}$, for $m'$ is the expression in the text. These results may be applied to the disturbance produced in the lunar motions by the sun, and if we suppose that $r$ is equal to the radius of the moon's orbit, and $a$ the mean distance of the sun from earth, it appears from this expression, that when $\lambda = 0$, i e. in the syzygies, the force $\phi$ vanishes, and the force $\phi'$ is a maximum and equal to $\frac{2fmr}{75a^3}$, on the contrary, when $\lambda = 90$, i. e in the quadratures, the force $\phi'$ vanishes, and $\phi = \frac{fmr}{75\, a^3}$, hence the greatest value of $\phi'$ is to the greatest value of $\phi$ as 2 1; but we should always keep in mind, that the force $\phi$ increases the gravity, while it is diminished by the action of the force $\phi'$, as they act in opposite directions, and as they respectively vanish, $\phi'$ in the quadratures and $\phi$ in the syzygies, there must be some intermediate position where they are equal; this is easily determined, for since $2 \cos^2 \lambda - \sin^2 \lambda = 3 \cos^2 \lambda - 1$, the value of the force $\phi' \cos \lambda - \phi \sin\lambda$ is $(3 \cos^2 \lambda - 1) \frac{fmr}{75\, a^3}$, consequently, when $3 \cos^2 \lambda - 1 = 0$, i e, when $\cos \lambda = \frac{1}{\sqrt{3}}$, $\phi' \cos \lambda = \phi \sin \lambda$, and from syzygies to an elongation $= \frac{1}{\sqrt{3}}$, $\phi \sin \lambda$, or the additious force predominates over $\phi' \cos \lambda$, or the ablatious force, and from this point to the quadratures, the latter exceeds the former, and therefore, during an entire revolution, the ablatious force predominates, it

would be easy to compute the actual diminution, for by multiplying $(3\cos^2\lambda - 1)\dfrac{fm'r}{a^3}$ by $d\lambda$, and then integrating, the result is $(\frac{3}{2}\lambda - \lambda + \frac{3}{2}\sin 2\lambda)\cdot\dfrac{fm'r}{a^3}$, which is the effect of the disturbing forces, while the angle $\lambda$ is described, and for an entire circumference this becomes (because the periodical part vanishes), $\dfrac{fmr}{a^3}\pi$; and therefore, the mean disturbing force acting on the moon in the direction of the radius vector is $-\dfrac{m'r}{a^3}$, it is easy to show that this quantity is equal to $\frac{1}{358}$th part of the moon's gravity to the earth, for if T$'$, T, denote the periodic times of the moon and earth, we have

$$\frac{m}{a^2} \text{ F the central force of the moon} \quad \frac{a}{T^2} \quad \frac{r}{T'^2} \qquad \frac{mr}{a^3} = 1 \quad \frac{T'^2}{T^2}.$$

Now $\dfrac{T'^2}{T^2} = \dfrac{1}{179}$ $\quad \dfrac{mr}{a^3} = \dfrac{F}{179}$, and consequently, the mean disturbing force $= \dfrac{F}{358}$. The effect of this diminution of the moon's gravity by a $\frac{1}{358}$th part is to increase her distance from the earth, and as the force by hypothesis acts in the direction of the radius vector, the mean area is not altered, but the angular velocity which varies inversely as the square of the distance is diminished by a $\frac{1}{179}$th part, and the periodic time is increased by the same quantity.

(*d*) Let $C = 2\pi r' =$ circumference of the moon's orbit, A the arc described by the moon in a minute in her orbit, V its versed sine, $r'$ the radius of her orbit, then as she is supposed to revolve uniformly, we have

$$A \quad C \quad 1'' : n, \qquad A = \frac{C}{n}, \; V = \frac{A^2}{2r'} = \frac{2\pi^2 r'}{n^2};$$

and by substituting for $r'$, we obtain the expression in text, now if the same space V be described at the surface, and at the distance of the moon, and if $f$ and $t$ denote the force and time of descent at the surface of the earth, and $f', t'$, the corresponding quantities at the moon, we have

$$V = ft^2 = f't'^2, \qquad t^2 \quad t'^2 \cdot f' \quad f,$$
$$\text{i e, when } f \quad f' \quad (60)^2 \quad 1, \; t \quad t' \quad 1 \quad 60,$$

i. e., a body at the earth's surface would in a second fall through a space equal to what a body at the distance of the moon would fall through in a minute. The corrections of various circumstances, the consideration of which are omitted in order to simplify the demonstration, are—1st The disturbing action of the sun, which was adverted to in the preceding note 2ndly The circumstance of the revolution of the moon not being performed about the centre of the earth, but about the common centre of gravity of the earth and moon. See No 243, and Harte's Translation *Du Systeme du Monde*, Vol II. page 381

(*e*) If the figure of the earth be supposed to be that of an ellipsoid *l* generated by a revolution about its lesser axis, and if the equicapacious sphere be supposed to have the same centre, the greatest elevation of the sphere above the spheroid is half the depression beneath it, when the spheroid differs little from a sphere, for if $r$ be the radius of the sphere, $a$ and $b$ the major and minor semiaxes of the spheroid, then if $a - r = \rho$, $r - b = s$, we have $s = 2\rho$ q p, for

$$\frac{4\pi r^3}{3} = \frac{4\pi}{3}a^2 b = \frac{4\pi}{3}(r+\rho)^2(r-s), = \frac{4\pi}{3}(r^3 + 2r^2\rho - r^2 s),$$

when the squares and products of $\rho$ and $s$ are neglected, consequently, we must have $2\rho = s$, we can by means of this relation obtain the value of $r$ in terms of $a$ and $b$, for we have evidently $2a - 2r = r - b$, $r = \dfrac{2a+b}{3}$, the elevation of this equicapacious sphere above the sphere inscribed in the ellipsoid, and whose radius is therefore equal to $b$, is $= \dfrac{2a+b}{3} - b = \frac{2}{3}(a-b)$, now in figure 60 let AB be the minor axis, and CD the major axis of the spheroid, ABB a section of the inscribed sphere, let $d$ be any point of the spheroid, $do$, $dn$ lines drawn to the centre and parallel to OC, meeting the inscribed sphere in the points $m$ and $n$, we have by the nature of the ellipsoid,

CR : $dn$ :: OC : $dv$ :: rad : cos $doc$, q p

and by similar triangles $ndm$, $dvo$, $nd$ : $dm$ :: $od$ : $dv$ :: radius to cos $doc$,

RC . $dm$ :: CO$^2$ : $dv^2$, i e as square of radius as to the square of the cosine of $d$OC, or of the latitude $\lambda$ q p, as RC $= a - b$, we have

$dm = (a - b)\cos^2\lambda$, now the depression of a point of the equicapacious sphere below the spheroid, and existing in the radius $od$, is evidently equal to the difference between $dm$ and $r - b$, or to the difference between $dm$ and $\frac{2}{3}(a - b)$ which by substituting for $dm$ its value $(a - b)\cos^2\lambda$, is equal to $(a - b)(\frac{2}{3} - \cos^2\lambda)$, consequently, when $\frac{2}{3} = \cos^2\lambda$, this depression vanishes, or the equicapacious sphere intersects the spheroid, from the value of $dm$ it is evident the increase of the force of gravity in going from the equator to the poles, varies as the square of the cosine of latitude, as was stated in No 193. The value of the latitude resulting from the above equation, namely, $\frac{1}{\sqrt{3}}$, is remarkable also as being the distance from the quadratures at which the additious is equal to the ablatious force

($f$) As lines drawn from places on the opposite sides of the mountain to the same fixed star, are supposed to be parallel, and as the deviation on each side is supposed to be equal, the angular distance of the two stars must be double the deviation

($g$) By substituting for $p$ and $q$ their respective values, we obtain $P\ a\sin x = \frac{Qac\sin(\gamma - x)}{y}$, and then as $P = \frac{fmm_1}{r^2}$, $Q = \frac{fm'm_1}{y^2}$, we have

$$\frac{fmm_1}{r^2}a\ \sin x = \frac{fm'm_1\,ac\sin(\gamma - x)}{y^3},$$

which by substituting for $m'$ its value, and dividing by $m_1$ is the expression in the text, now in the equation $\frac{\sin x}{\sin(\gamma - x)} = \frac{\rho' r'^3}{\rho r c^2}$, it is evident that

$$\sin x = (\sin\gamma\cos x - \cos\gamma\sin x)\frac{\rho' r'^3}{\rho\ r c^2},$$

and

$$1 = (\sin\gamma\ \cot x - \cos\gamma)\frac{\rho' r'^3}{\rho r c^2}, \qquad \cot x = \frac{\rho r c^2}{\rho' r'^3\sin\gamma} + \cot\gamma,$$

when $c$ is least and $\gamma = 90$, cot $x$ is least, and $x$ the greatest

($h$) It is necessary that the level should be accurately *horizontal*, otherwise the effect of gravity would be complicated with that of the force of torsion

($i$) By substituting for $\cos(\gamma - \theta)$ its value, we obtain

$$z^2 = a^2 + c^2 - 2ac\cos\gamma\cos\theta - 2ac\ \sin\gamma\sin\theta,$$

$=$ (if the square of $\theta$ be neglected in the series expressing $\sin\theta$, $\cos\theta$,)

$a^2 + c^2 - 2ac\cos\gamma - 2ac\sin\gamma\,\theta, = b^2 - 2ac\sin\gamma\ \theta,$
and therefore,

$z^{-3} = (b^2 - 2ac\sin\gamma\ \theta)^{-\frac{3}{2}} = b^{-3} + 3ac\sin\gamma.\theta b^{-5}, \quad \frac{\sin(\gamma-\theta)}{z^3} =$

$(\sin\gamma - \cos\gamma\,\theta)(b^{-3} + 3ac\sin\gamma\,\theta b^{-5}) = \frac{\sin\gamma}{b^3} - \frac{\cos\gamma\,\theta}{b^3} + \frac{3ac\sin^2\gamma\,\theta}{b^5},$

now if the numerator of $\frac{\cos\gamma\ \theta}{b^3}$ be multiplied by the value of $b^2$, and if the denominator be multiplied by $b^2$, we obtain

$$\frac{\cos\gamma\ \theta}{b^3} = \frac{(a^2+c^2)\cos\gamma\ \theta - 2ac\cos^2\gamma\ \theta}{b^5},$$

hence then we have

$$\frac{\sin(\gamma-\theta)}{z^3} = \frac{\sin\gamma}{b^3} - [(a^2+c^2)\cos\gamma - 2ac - ac\sin^2\gamma]\ \frac{\theta}{b^5},$$

for $-2ac\,(\sin^2\gamma + \cos^2\gamma) = -2ac$

($k$) See note ($e$) of this Chapter, and No. 193

($l$) See the *Mechanique Celeste*, Livre 7, also Harte's translation of the System of the World, Vol II Chap V, and notes

($m$) When $2f\pi\rho'\left(\frac{yz\,dy\,dz}{(y^2+z^2)^{\frac{3}{2}}}\right)$ is integrated with respect to $z$, it becomes $-2\pi f\rho'\left(\frac{ydy}{\sqrt{z^2+y^2}} + Y\right)$, the value of $Y$ is determined by making $z=0$, which gives $Y = c$, and $-2\pi f\rho'\frac{ydy}{\sqrt{z^2+y^2}}$ integrated with respect to $y$, gives $y = -\sqrt{z^2+y^2} + z$, in which $z$ is evidently equal to $h$, hence then the value of $-2\pi f\rho'\int\frac{yz\,dy\,dz}{(y^2+z^2)^{\frac{3}{2}}}$ between the limits $o$ and $h$ for $z$, $o$ and $c$ for $y$, is the value of $k'$ given in the text

($n$) $g - g'$ (= by neglecting $\gamma'-\gamma$ and $h^2$) $k - \frac{k\,r^2}{(r+h)^2} - k' =$
$\frac{2kh}{r} - k'$

Now as $k$ differs from $g'$ by a small fraction, if in the fraction $\frac{2kh}{r}$, $g'$ was substituted for $k$, the result would differ from $\frac{2g'h}{r}$ by the product of $\frac{h}{r}$ into the difference between $g'$ and $k$, which as it is

a very small quantity, the product is a very small quantity, and consequently, may be neglected

(*o*) As $k'=2\pi f\rho' h$, if in this expression we substitute for $f$ its value derived from assuming $\frac{4\pi\rho f r}{3}=g'$ namely, $\frac{3g'}{4\pi\rho r}$ the result, namely, $k'=\frac{3\rho' h g'}{2\rho r}$ will differ from the actual value by a very small quantity of the second order, hence then if in the expression $g-g'=\frac{2kh}{r}-k'$ we substitute $g'$ for $k$, and $\frac{3\rho' h g'}{2\rho r}$ for $k'$, we shall obtain $g-g'=g'\left(\frac{2h}{r}-\frac{3\rho' h}{2\rho r}\right)$

(*p*) If $\rho'=\frac{\rho}{2}$ then the value $g-g'$ is $g'\left(\frac{2h}{r}-\frac{3h}{4r}\right)=g'\frac{5h}{4r}$

# BOOK III

## CHAPTER I

(*a*) It appears from what has been just established, that R is the resultant of the given forces P, P′, P″, &c, and of Q, consequently these forces may be replaced by R, and a force equal and directly contrary to Q

(*b*) As each body furnishes six equations, namely, X, Y, Z, L, M, N, respectively equal to cipher, the entire number of equations which thus result, will be six times the number of bodies, but for every point of contact, the elimination of R and of the corresponding quantities of the other bodies reduces this number by one, therefore, the entire number of distinct resulting equations will be six times the number of bodies minus the number of points of contact.

(*c*) When $L = 0$ the resultant of the forces parallel to $oy$ must coincide with it, and therefore be destroyed by the fixed point.

(*d*) $Q = P \sin \gamma$, $q = \pm (y \cos \lambda - x \cos \mu)$, therefore, $Qq = \pm P (y \sin \gamma \cos \lambda - x \sin \gamma \cos \mu)$ equal $\pm P (y \cos \alpha - x \cos \beta)$

(*e*) $P \cos \alpha = - P' \cos \alpha'$, therefore substituting $\frac{h}{l}$ for $\cos \alpha$, and $- \cos \delta$ for $\cos \alpha'$, we obtain the expression in the text.

(*f*) Since the friction is by experiment $= l$ to the pressure, it follows from what has been just established, that the coefficient $f$, and consequently $A$, are independent of the pressure, and therefore of the weight

## CHAPTER II

(*a*) The expression for L in No 261 is

$$P (x \cos \beta - y \cos \alpha) + P (x' \cos \alpha' - y' \cos \beta') + \&c = L,$$

now by substituting for Q, Q′, &c in the value of S, $P \sin \gamma$, $P' \sin \gamma'$, &c, we obtain the value for L, for $\cos \beta = \sin \gamma \cos \mu$, $\cos \alpha = \sin \gamma \cos \lambda$, &c

(*b*) If the squares of the three quantities $H \cos \delta$, $H \cos \delta'$, $H \cos \delta''$ be taken, and if they be then added, there results

$$H^2 (\cos^2 \delta + \cos^2 \delta_1 + \cos^2 \delta) = P^2 p^2,$$

$p^2$ being substituted for its value given in the text, therefore we have

$$\text{P}p \cos\delta = \text{P}(x\cos\beta - y\cos\alpha),$$

consequently,

$$\cos\delta = \frac{1}{p}(x\cos\beta - y\cos\alpha)$$

(*c*) If in the equation $\text{A}u + \text{B}v + \text{C}w$, we substitute for $v$, $w$, their values $\frac{u\cos\beta}{\cos\alpha}$, $\frac{u\cos\gamma}{\cos\alpha}$, and then multiply by $\cos\alpha$, we obtain the equation $\text{A}\cos\alpha + \text{B}\cos\beta + \text{C}\cos\gamma = 0$, now from the equation $\text{A}x + \text{B}y + \text{C}z = 0$, we obtain $\text{B} = -\left(\frac{\text{A}x + \text{C}z}{y}\right) = -\left(\frac{\text{A}\cos\alpha + \text{C}\cos\gamma}{\cos\beta}\right)$, consequently,

$$\text{A}x\cos\beta + \text{C}z\cos\beta = \text{A}y\cos\alpha + \text{C}y\cos\gamma,$$

and therefore,

$$\text{C} = \text{A}\left(\frac{x\cos\beta - y\cos\alpha}{y\cos\gamma - z\cos\beta}\right),$$

in a similar manner we may obtain the value of B.

(*d*) By expanding the binomials in the value of $p^2$ we obtain $x^2\cos^2\beta + x^2\cos^2\gamma + y^2\cos^2\alpha + y^2\cos^2\gamma + z^2\cos^2\alpha + z^2\cos^2\beta - 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma$, now $x^2(\cos^2\beta + \cos^2\gamma) = x^2 - x^2\cos^2\alpha$, $y^2(\cos^2\alpha + \cos^2\gamma) = y^2 - y^2\cos^2\beta$, $z^2(\cos^2\alpha + \cos^2\beta) = z^2 - z^2\cos^2\gamma$, consequently as $x^2 + y^2 + z^2 = r^2$, the value of $p^2$ will become

$$r^2 - r^2\cos^2\alpha\cos^2\lambda - r^2\cos^2\beta\cos^2\mu - r^2\cos^2\gamma\cos^2\nu - 2r^2\cos\alpha\cos\lambda\cos\beta\cos\mu - 2r^2\cos\alpha\cos\lambda\cos\gamma\cos\nu - 2r^2\cos\beta\cos\mu\cos\gamma\cos\nu$$

Now it is evident, that the parts of this expression which contain the cosines multiplied by $r^2$, are the square of $\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu$, and consequently it vanishes, therefore the value of $p^2 = r^2$, as was proposed to be verified.

(*e*) By making this substitution, the fourth equation of No 261 becomes

$$\text{P}(x\cos\beta - y\cos\alpha) + \text{P}'(x'\cos\beta' - y'\cos\alpha') + \&c - (\text{P}\cos\beta + \text{P}'\cos\beta' + \text{P}''\cos\beta'' + \&c)\, x_1 + (\text{P}\cos\alpha + \text{P}'\cos\alpha' + \text{P}''\cos\alpha'' + \&c)\, y_1,$$

now the two first terms of this expression are equal evidently to L, and the two last terms are equal to $\text{X}y_1 - \text{Y}x_1$, hence the value of $\text{L}_1$ is that given in the text.

(*f*) By taking the partial difference of the value of G with respect to $a_1$ we obtain, when $\text{G}_1$ is a *maximum*,

$$-2Y(L + Xy_1 - Yx_1) + 2Z(M + Zx_1 - Xz_1) = 0,$$

consequently,

$$x_1(Y^2 + Z^2) = X(Yy_1 + Zz_1) + YL - ZM,$$

and by adding $x_1X^2$ to each side of this equation, there results

$$x_1(X^2 + Y^2 + Z^2)(= x_1R^2) = X(Xx_1 + Yy_1 + Zz_1) + YL - ZM$$

The expressions for $R^2y_1$, $R^2z_1$, may be obtained in a similar manner

(*g*) From two of the preceding equations we obtain

$$\frac{R^2x_1 + ZM - YL}{X} = \frac{R^2y_1 + XL - ZN}{Y},$$

i e,

$$R^2(Xy_1 - Yx_1) + (X^2 + Y^2)L = Z(NX + MY),$$

and by adding $Z^2L$ to each member of this equation, there results

$$R^2(Xy_1 - Yx_1) + (X^2 + Y^2 + Z^2)L = Z(NX + MY + LZ),$$

which, by dividing by $R^2$, and observing that $X^2 + Y^2 + Z^2 = R^2$, becomes the expression in text

## CHAPTER III

(*a*) It is evident from the construction, that if $m$ and $n$ be the angles, which any of the forces such as P, makes with the adjacent sides MA, MM′, of the polygon, the tension along the side MA is equal to $\frac{P\sin n}{\sin(m+n)}$, and that along the side $MM' = \frac{P\sin m}{\sin(m+n)}$, now if $m'$ and $n'$ be the corresponding quantities at the summit M′ of the polygon, we have the tension along M′M evidently equal to $\frac{P'\sin n'}{\sin(m'+n')}$, consequently, $\frac{P\sin m}{\sin(m+n)} = \frac{P'\sin n'}{\sin(m'+n')}$, hence when the directions of the forces P, P′, P″, &c, applied to the summits of the polygon bisect the angles contained between the sides of the polygon, the tension in each side is the same throughout the entire polygon, and equal to any of the forces P, P′, P″, &c, divided by twice the cosine of the angle which it makes with the side of the cord to which it is applied, when P, P′, P″, &c, are all equal, the polygon is a regular figure, and conversely, when the polygon is a regular figure, the forces P, P′, P″, &c, must be equal, and each of them is to the tension as a side of the polygon to the radius of the circumscribed circle

(*b*) As $\alpha, \alpha', \alpha''$, &c., and also $e, g, \gamma, \gamma' \gamma''$, &c., are in this case $90^\circ$, and $\beta, \beta', \beta''$, &c., are cipher, it follows that the first equation (a) will be equal to

$$\text{H} \cos a + \text{K} \cos e = 0,$$

and the second will be

$$\text{H} \cos b + \text{and H} \cos f + \text{P} + \text{P}' + \text{P}'', \text{\&c} = 0,$$

and all the terms of the third equation will vanish, now if both equations (b) be squared and added together, we have

$$\text{H}^2 + \text{K}^2 + 2\text{HK} (\cos a \cos e + \cos b \cos f) = \Pi^2,$$

and the factor of $2\text{HK}$ is evidently equal to the cosine of the angle contained by the sides H and K, therefore, $\Pi$ is equal and contrary to the resultant of these forces, it follows from the equation $\text{H} \cos b + \text{K} \cos f + \Pi = 0$, that the directions of the extreme forces H and K meet in the vertical drawn through the centre of gravity of P, P′, P″, &c. In this case it is evident that $\sin n = \sin m'$, consequently if the equation which we obtained above in note (*a*), namely, $\frac{\text{P} \sin m}{\sin (m + n)} = \frac{\text{P}' \sin n'}{\sin (m' + n')}$, &c., be multiplied by $\sin n$ and $\sin m'$ respectively, it becomes

$$\frac{\text{P} \sin m \sin n}{\sin (m + n)} = \frac{\text{P}' \sin m' \sin n'}{\sin (m' + n')}$$

$$= \frac{\text{P}}{\cot m + \cot n} = \frac{\text{P}'}{\cot m' + \cot n'} = \frac{\text{P}''}{\cot m'' + \cot n''} + \text{\&c},$$

and it is remarkable, that in this case, namely, when all the forces are vertical, the horizontal tensions on each side of the polygon will be respectively equal to these fractions, consequently, the horizontal tension is constant throughout the entire polygon. If this equal horizontal tension be denoted by A, and if the tensions along the sides of the polygon be denoted by T, T′, T″, &c., we find $\text{T} \sin m = \text{A} = \text{T}' \sin m' = \text{T}'' \sin m''$, &c., so that the tension on each side varies inversely as the sine of the inclination of the side to the vertical.

(*d*) Since the sum of the tensions of all the strings resolved in the direction of the vertical is

$$\left[\frac{\delta}{\omega} \cos \gamma + \frac{\delta'}{\omega'} \cos \gamma' + \frac{\delta''}{\omega''} \cos \gamma'' + \text{\&c.}\right] \text{P},$$

and since this is by hypothesis equal to P, the quantity by which P is multiplied must be equal to 1.

(*e*) By substituting their values for $\cos\alpha$, $\cos\alpha'$, T, T$'$, $y'$, $x'$, the first equation (a) becomes

$$-T\frac{dx}{ds}+T\frac{dx}{ds}+Td\frac{dx}{ds}+\frac{dx}{ds}\,dT=d\left(T\frac{dx}{ds}\right)$$

The second equation (b) may, by similar substitutions, be derived from the equation (a), and with respect to the third there results by these substitutions,

$$T\left(y\frac{dx}{ds}-x\frac{dy}{ds}\right)+(T+dT)\left[(x+dx)\left(\frac{dy}{ds}+d\frac{dy}{ds}\right)-(y+dy)\left(\frac{dx}{ds}+d\frac{dx}{ds}\right)\right]=pxds,$$

which by obliterating equal quantities affected with opposite signs, is evidently equal to

$$d\,T\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)=pxds$$

(*f*) By adding together the values of $e^{\frac{x}{h}}$ and $e^{-\frac{x}{h}}$, and observing that $ds=\sqrt{1+\frac{dy^2}{dx^2}}\,dx$ we obtain the value of $ds$, and if the value of $e^{-\frac{x}{h}}$ be taken from $e^{\frac{x}{h}}$, and if we then multiply by $dx$, we obtain the value of $dy$

(*g*) By adding and subtracting the values of $l$ and $b$ we obtain

$$l+b=h\left(e^{\frac{k}{h}}-e^{-\frac{k'}{h}}\right),\quad l-b=h\left(e^{\frac{k'}{h}}-e^{-\frac{k}{h}}\right),$$

$$l^2-b^2=h^2\left(e^{\frac{(k+k')}{h}}+e^{-\frac{(k+k')}{h}}-2\right),$$

= by substituting $a$ for $k+k'$, the expression in the text, and since

$$(e^a-e^{-a})^2=e^{2a}+e^{-2a}-2,\text{ and }\frac{a}{h}=2\alpha,$$

there results

$$l^2-b^2=\frac{a^2}{4\alpha^2}(e^\alpha-e^{-\alpha})^2,\quad \sqrt{\frac{l^2-b^2}{a^2}}=\frac{1}{2\alpha}(e^\alpha-e^{-\alpha})=n$$

(*h*) $e^\alpha=1+\frac{\alpha}{1}+\frac{\alpha^2}{1.2}+\frac{\alpha^3}{1.2.3}+\&c$, $e^{-\alpha}=1-\frac{\alpha}{1}+\frac{\alpha^2}{1.2}-\frac{\alpha^3}{1.2.3.}-\&c$, $\frac{1}{2\alpha}(e^\alpha-e^{-\alpha})(=n)=1+\frac{\alpha^2}{6}$, $\alpha^2=6(n-1)$

(*i*) $k+k'=a$, $k-k'=2\beta h$, $k=\frac{a}{2}+\beta h$, $k'=\frac{a}{2}-\beta h$, $\frac{k}{h}=\frac{a}{2h}+\beta$, $\frac{k'}{h}=\frac{a}{2h}-\beta$, and if these values be substituted for $\frac{k}{h}$, $\frac{k'}{h}$ in the value of $b$, the result will be equation (e)

(*k*) The integral of the value of $dy'$ is $c\log(x'+c)+\sqrt{(x'+c)^2-c'^2}$ + c, when $x'=0$, $y'=0$, $c=-c\log(c+\sqrt{c^2-c'^2})$, $y'=$ the expression in the text

(*l*) $\gamma=c+\sqrt{c^2-c'^2}$, $\gamma'=c-\sqrt{c^2-c'^2}$, $\gamma\gamma'=c'^2=h^2$, for $e^{\frac{x}{c}}\times e^{-\frac{x}{c}}=1$

(*m*) By substituting for T its value $ph\frac{ds}{dx}$ there results $phd\frac{dy}{ds}\frac{ds}{dx}$ $=pdx$, i e, $hd\frac{dy}{dx}=dx$, $h\frac{dy}{dx}=x$, $2hy=x^2$, hence as $\mathrm{T}=ph\frac{ds}{dx}$ $=ph\sqrt{1+\frac{dy^2}{dx^2}}$, we obtain by substituting $\frac{x^2}{h^2}$ for $\frac{dy^2}{dx^2}$, the value of T given in the text

(*n*) $\sqrt{h^2+x^2}dx=\frac{h^2dx}{\sqrt{h^2+x^2}}+\frac{x^2dx}{\sqrt{h^2+x^2}}$, the integral of $\frac{h^2dx}{\sqrt{h^2+x^2}}$ taken between the limits $k$, $-k'$, is

$$h^2(\log\sqrt{h^2+k^2}+k)-\log(h^2+k'^2-k')=h^2\log\frac{\sqrt{h^2+k^2}+k}{\sqrt{h^2+k'^2}-k'},$$

in like manner as

$$\int_{-k'}^{k}\frac{x^2dx}{\sqrt{h^2+x^2}}=k\left(\frac{h^2+k^2}{2}\right)+k'\left(\frac{h^2+k'^2}{2}\right)-\tfrac{1}{2}\int\frac{h^2dx}{\sqrt{h^2+x^2}},$$

by substituting for $\int_{-k'}^{k}\frac{h^2dx}{\sqrt{h^2+x^2}}$ its value, and multiplying by 2 we obtain the expression for $2hl$ in the text, now if both numerator and denominator of the logarithmic function be multiplied by $\sqrt{h^2+k^2}+k$, there will result, when $k=k'$,

$$2hl=h^2\log\left(\frac{\sqrt{h^2+k^2}+k}{h}\right)^2+2k\sqrt{h^2+k^2},$$

from whence we can easily deduce the value of $hl$ given in the text

(o) If for $\sqrt{h^2+k^2}$ its value in a series be substituted, we shall obtain

$$\log\frac{k+\sqrt{h^2+k^2}}{h}=\log\frac{k+h+\frac{1}{2}\frac{k^2}{h}-\frac{1}{8}\frac{k^4}{h^3}}{h}=\log\left[1+\frac{k}{h}+\frac{1}{2}\frac{k^2}{h^2}-\&c\right],$$

now as in general $\log(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\&c$, we have

$$\log\left[1+\frac{k}{h}+\frac{1}{2}\frac{k^2}{h^2}\&c\right]=\left[\frac{k}{h}+\frac{1}{2}\frac{k^2}{h^2}-\&c-\frac{k^2}{2h^2}-\frac{2k^3}{2.2h^3}+\frac{k^3}{3h^3}\Big|\&c\right]$$

$$=\frac{k}{h}-\frac{k^3}{6h^3}+\&c\,,$$

hence, by substituting this value in the expression for $hl$, there results

$$hl=h^2\left[\frac{k}{h}-\frac{1}{6}\frac{k^3}{h^3}\right]+kh+\frac{k^2}{2h},$$

and if we multiply by $h$,

$$h^2l=2h^2k+\tfrac{1}{3}k^3,\qquad h^2=\frac{k^3}{3(l-2k)}=\frac{a^3}{8.3.(l-a)},$$

by substituting for $k$ its value $\frac{a}{2}$

($p$) By substituting this value of $\text{r}$ we obtain

$$dx=dx'+\omega\text{P}dx'+\omega p\left(\frac{l'-x'}{l'}\right)dx',\qquad x=x'+\omega\text{P}x'+\frac{\omega pl'x'}{l'}-\frac{\omega px'^2}{2l'}.$$

($q$) By performing this multiplication, there results by adding

$$d\text{T}\left(\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}+\frac{dz^2}{ds^2}\right)+\text{T}\left(\frac{dx}{ds}\,d\,\frac{dx}{ds}+\frac{dy}{ds}\,d.\frac{dy}{ds}+\frac{dz}{ds}\,d\,\frac{dz}{ds}\right)+$$

$$\varepsilon\left(\text{X}\frac{dx}{ds}ds+\text{Y}\frac{dy}{ds}ds+\text{Z}\frac{dz}{ds}ds\right)=0,$$

which from what is stated in text is evidently equal to equation (3)

($r$) If the first of equations (1) multiplied by $\frac{dy}{ds}$ be taken from the second multiplied by $\frac{dx}{ds}$ we shall have, as

$$d.\text{T}\frac{dx}{ds}=d\text{T}\frac{dx}{ds}+\text{T}\,d\,\frac{dx}{ds}\,\&c\,,$$

$$d\text{T}\left(\frac{dy}{ds}\frac{dx}{ds}-\frac{dx}{ds}\frac{dy}{ds}\right)+\text{T}\left(\frac{dx}{ds}d\,\frac{dy}{ds}-\frac{dy}{ds}d\,\frac{dx}{ds}\right)+\varepsilon\left(\text{Y}\frac{dx}{ds}ds-\text{X}\frac{dy}{ds}ds\right)=0,$$

consequently, as $ds$ is the independent variable, we can obtain from this expression the first equation (4)

(s) As $\varepsilon X = N \cos\lambda =$ (by substituting its value for $\cos\lambda$) $NV\frac{dL}{dx}$, by performing the same operation for $\varepsilon Y$, $\varepsilon Z$, then multiplying by $dx$, $dy$, $dz$ respectively, and finally adding them together, we obtain

$$\varepsilon(Xdx + Ydy + Zdz) = NV\left(\frac{dL}{dx}dx + \frac{dL}{dy}dy + \frac{dL}{dz}dz\right) = NVdL = 0$$

(t) By substituting their values for $X, Y, Z$, in the second members of equations (4), they will become

$$N(\cos\lambda\,dy - \cos\mu\,dx)\frac{ds^2}{T},\quad N(\cos\nu\,dx - \cos\lambda\,dz)\frac{ds^2}{T},$$
$$N(\cos\mu\,dz - \cos\nu\,dy)\frac{ds^2}{T},$$

and if these be multiplied by $\cos\nu$, $\cos\mu$, $\cos\lambda$ respectively, and then added together, the result is evidently cipher, hence the expression

$$\rho(\cos\nu\cos\nu' + \cos\mu\cos\mu' + \cos\lambda\cos\lambda'),$$

to which the preceding is equal must be cipher

(u) The coefficient of $N$ in the expression for $\frac{k}{\rho}$, becomes by expanding

$$\frac{dy^2}{ds^2}(\cos^2\lambda + \cos^2\nu) + \frac{dx^2}{ds^2}(\cos^2\mu + \cos^2\nu) + \frac{dz^2}{ds^2}(\cos^2\lambda + \cos^2\mu) -$$
$$2\frac{dy}{ds}\cdot\frac{dx}{ds}\cos\lambda\cos\mu - 2\frac{dx}{ds}\frac{dz}{ds}\cos\lambda\cos\nu - 2\frac{dy}{ds}\frac{dz}{ds}\cos\mu\cos\nu,$$

which, as $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$, becomes

$$\frac{dy^2}{ds^2} + \frac{dx^2}{ds^2} + \frac{dz^2}{ds^2} - \left(\frac{dy^2}{ds^2}\cos^2\mu + \frac{dx^2}{ds^2}\cos^2\lambda + \frac{dz^2}{ds^2}\cos^2\nu + 2\frac{dy}{ds}\frac{dx}{ds}\cdot\right.$$
$$\left.\cos\lambda\cos\mu + 2\frac{dx}{ds}\frac{dz}{ds}\cos\lambda\cos\nu + 2\frac{dy}{ds}\frac{dz}{ds}\cos\mu\cos\nu\right)$$

Now the expression between the brackets is equal to cipher, and the other expression is equal to unity, hence $N = \frac{k}{\rho}$.

(v) If these equations be multiplied by $x$ and $y$ respectively, there results by adding them together,

$$xd.T\frac{dx}{ds} + yd\,T\frac{dy}{ds} + N.\frac{(x^2+y^2)}{c}ds = 0,$$

and as $x^2 + y^2 = c^2$, this equation coincides with the first equation

(6), and if these equations be respectively multiplied by $\frac{dx}{ds}, \frac{dy}{ds}$, there results from their addition the second equation (6), for $xdx + ydy = 0$

($x$) The first equation (6) is evidently equal to

$$\left(x\frac{dx}{ds}+y\frac{dy}{ds}\right)dT + \left(x\ d\frac{dx}{ds}+y\ d\frac{dy}{ds}\right)T + Ncds = 0,$$

hence as $x\frac{dx}{ds}+y\frac{dy}{ds}=0$, $x.d\frac{dx}{ds}+y\ d\frac{dy}{ds} = -ds$, this equation is reduced to $T = cN$, in like manner the second equation (6) is equal to, by substituting $cds$ for $xdy - ydx$,

$$\left(\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}\right)dT + \left(\frac{dx}{ds}\ d\frac{dx}{ds}+dy\ d.\frac{dy}{ds}\right)T - \mu ds = 0,$$

consequently we have $dT = \mu ds$

*Notes to Paragraph III*

($a$) $\sigma' = \sigma + \frac{u\sigma}{\rho}$, by substituting for $\sigma'$, $\sigma$, their respective values, we obtain $\gamma(1+\delta') = \gamma(1+\delta) + \frac{u\,\gamma(1+\delta)}{\rho}$, dividing by $\gamma$, and neglecting $\frac{u\delta}{\rho}$ we obtain, by omitting a common quantity on both sides of the equation, $\delta' = \delta + \frac{u}{\rho}$, and, when the length of the mean filament is not changed, $\delta = 0$ and $\delta' = \frac{u}{\rho}$, i e the lengthening and contracting of the filaments, is proportional to $u$ their distance from the mean filament

($b$) Substituting for $\delta'$ in the value of T, it becomes

$$T = \alpha\lambda\int_{-\varepsilon}^{\varepsilon}\delta\, du + \alpha\lambda\int_{-\varepsilon}^{\varepsilon}\frac{udu}{\rho},$$

and performing the integration between the limits $\varepsilon$, $-\varepsilon$, the first term becomes $2\alpha\lambda\delta\varepsilon$, and the second becomes $\frac{\alpha\lambda\varepsilon^2}{2\rho} - \frac{\alpha\lambda\varepsilon^2}{2\rho} = 0$, and in the same manner, substituting for $\delta'$ in the value of $\mu$, it becomes

$$\mu = \alpha\lambda\int_{-\varepsilon}^{\varepsilon}\delta u du + \alpha\lambda\int_{-\varepsilon}^{\varepsilon}\frac{u^2du}{\rho},$$

integrating these quantities between $\varepsilon$, $-\varepsilon$, the first term vanishes, as in the preceding case, and the second becomes $\frac{2}{3}\frac{\alpha\lambda\varepsilon^3}{\rho}$

From the value of r it appears, that in the case of the same plate, it is proportional to $\delta$ the extension of the mean filament, and when $\delta = 0$, i.e. when the length of the filament is not changed, $r = 0$, and the forces which in this case act on K must be parallel forces reducible to two, but not directly opposed.

(c) The angle of contact is always equal to the element of the curve divided by the radius of curvature, and when the element is given, this angle or the curvature varies inversely as the radius of curvature, consequently $\mu$ varies in this case as the angle of contact.

(d) Making $\frac{dy}{dx} = z$, equation (2) will become

$$\frac{c^2dz}{(1+z^2)^{\frac{3}{2}}} = a\,dx - x\,dx,$$

and integrating so that we may have $z = 0$ when $x = 0$, we obtain

$$\frac{2c^2z}{\sqrt{1+z^2}} = 2ax - x^2,$$

consequently,

$$4c^4z^2 = (2ax - x^2)^2(1+z^2),$$

and,

$$(4c^4 - (2ax - x^2)^2)\,z^2 = (2ax - x^2)^2,$$

hence by substituting for $z$ its value, we obtain

$$\frac{dy}{dx} = \frac{(2ax - x^2)}{\sqrt{4c^4 - (2ax - x^2)^2}},$$

$ds^2 = \frac{(2ax - x^2)^2\,dx^2}{4c^4 - (2ax - x^2)^2} + dx^2 = \frac{4c^4dx^2}{4c^4 - (2ax - x^2)^2} =$ (when $c$ is very great, in which case $2ax - x^2$ may be neglected relatively to $c^2$,) $dx^2$, hence in this case we must have $2c^2dy = (2ax - x^2)\,dx$, and $6c^2y = 3ax^2 - x^3$

(e) $6c^2y = 3ax^2 - x^3$, which, when $x = a$ and $y = b$, becomes $3c^2b = a^3$, and from the equation $3Qc^2 = \alpha\omega\varepsilon^2$, we obtain $3c^2 = \frac{\alpha\omega\varepsilon^2}{Q}$, and $\alpha\omega\varepsilon^2b = a^3Q$, by substituting for $\alpha\omega$ its value $\frac{w}{\delta}$, we will have

$$\frac{w}{\delta}\varepsilon^2b = a^3Q \text{ and } b = \frac{a^3\delta Q}{\varepsilon^2w},$$

therefore, as $a\delta = h$ we shall obtain

$$b = \frac{a\,h\text{Q}}{\varepsilon' w},$$

$w$ being by hypothesis a weight equivalent to the force which draws the plate in the direction of its length, if we suppose it equal to Q, we will have $b = \frac{a'h}{\varepsilon}$, consequently $b \quad h \qquad a' \quad \varepsilon^2$

($f$) When $h = 0$, $y$ is *always* equal to cipher, therefore the spring cannot be bent, when $h$ does not vanish, $y$ and $\frac{dy}{dx}$ will have finite values, therefore the spring will bend, at the point B, $x = a$, $y = 0 = h \sin\frac{\pi a}{c}$, therefore $a = ic$, $i$ being a whole number

($g$) If the fourth power of $\frac{h}{c}$ be neglected, we shall have

$$\sqrt{1 + \frac{\pi^2 h^2}{c^2}\cos^2\frac{\pi x}{c}} = 1 + \tfrac{1}{2}\frac{\pi^2 h^2}{c^2}\cos^2\frac{\pi x}{c}$$

$$= \text{as } \cos^2\frac{\pi x}{c} = \tfrac{1}{2} + \tfrac{1}{2}\cos\frac{2\pi x}{c}$$

$$1 + \frac{\pi^2 h^2}{4c^2} + \tfrac{1}{4}\frac{\pi^2 h^2}{c^2}\cos\frac{2\pi x}{c},$$

if this value be substituted in the expression

$$l = \int_0^a \sqrt{1 + \frac{\pi^2 h^2}{c^2}\cos^2\frac{\pi x}{c}}\,dx,$$

and then integrated, there will result

$$l = a + \frac{\pi^2 h^2}{4c^2}a,$$

which, by substituting for $a$ its value $ic$, becomes the expression in the text

($h$) In the equation $y = fa \sin\frac{\pi x}{a}$, $y = 0$, when $x = 0$, and $x = a$, therefore, the curve will not cut the vertical between the two points A and B In the equation $y = f'a \sin\frac{2\pi x}{a}$, $y = 0$, when $x = 0$, $x = \frac{a}{2}$, and $x = a$, therefore the curve will cut the vertical in the middle point between A and B In the equation $y = \phi a \sin\frac{?\pi x}{a}$, $y = 0$ when

$i=0$, $i=a$, $i=\left(\frac{2}{a}-\frac{i}{}\right)a$, hence it appears, that the curve will cut the vertical AB, in the number $i+1$ points

When $k=0$, $y=0$, therefore the figure is rectilineal, but the least increase of $k$ above this gives a finite value for $y$, in which case the figure is not rectilineal, but when $l$ is $\angle\, c$, $k$ is impossible, therefore $y$ cannot have a possible value, consequently, the figure *must* be *rectilineal*

($i$) In general we have $P=\pi^2\frac{\alpha\omega\varepsilon^2}{3c^2}$, now it appears from No 312, that as long as $l$ is $\angle\, c$, the spring cannot be bent, but when the quantity $c$ is so diminished that $l$ may surpass $c$, then the spring may be bent by the weight, consequently, $l=c$ is the limit after which the spring is bent, hence, the value of P determined by the equation $P=\frac{\alpha\omega\varepsilon^2}{3l^2}$, gives the greatest weight which the spring can support without bending

($k$) $\sigma'=\sigma+\frac{u\sigma}{\rho}$, $=$ (by substituting for $\sigma$ its value $\gamma(1+\delta)$, and neglecting $\left.\frac{\delta u}{\rho}\right)$, $\gamma(1+\delta)+\frac{\gamma u}{\rho}$, in like manner $\sigma'=\gamma'(1+\delta')=$ $\left(\text{by substituting } \gamma' \text{ its value } \gamma+\frac{u\gamma}{r} \text{ and neglecting } \frac{\delta' u}{r}\right)$, $\gamma+\frac{u\gamma}{r}+\gamma\delta'$, hence we obtain, by dividing by $\gamma$ and obliterating common terms in the equation $\gamma(1+\delta)+\frac{\gamma u}{r}=\gamma+\frac{u\gamma}{r}+\gamma\delta'$, $\delta'=\delta+u\left(\frac{1}{\rho}-\frac{1}{r}\right)$

($l$) Substituting for $\delta'$ its value given above, we obtain

$$T=\alpha\int_{-k'}^{k}\delta v du+\alpha\int_{-k'}^{k} v u du\left(\frac{1}{\rho}-\frac{1}{r}\right),$$

the integral of the second term vanishes, and that of the first $=\alpha\omega\delta'$, in like manner by a similar substitution in the value of $\mu$, we obtain

$$\mu=a\int_{-k'}^{k}\delta v u du+a\int_{-k'}^{k} v u^2 du\left(\frac{1}{\rho}-\frac{1}{r}\right),$$

in this case, the integral of the first term vanishes, and that of the second is equal to $\frac{\alpha\omega q^2}{3}\left(\frac{1}{\rho}-\frac{1}{r}\right)$

($m$) In the first case, as the distances of the centre of gravity from the vertex and base of the triangle are respectively $\frac{2}{3}c$, $\frac{1}{3}c$, we

have in this case $k = \frac{1}{3}c$, $k' = \frac{2}{3}c$, now any line parallel to the base such as $v$, which is distant from the centre of gravity by a quantity $= u$, which may be either positive or negative, is to the base $a$ $\frac{2}{3}c + u$ $c$, $v = \frac{a}{c}(\frac{2}{3}c + u)$, and consequently

$$\int vu^2du = \frac{2}{3}\frac{au^3}{3} + \frac{a}{c}\frac{u^4}{4},$$

when $u = \frac{c}{3}$ this quantity becomes equal to $\frac{2}{9}\frac{ac^3}{3^3} + \frac{ac^3}{4\ 3^4}$, and when $u = \frac{2c}{3}$, it is equal to $\frac{2a}{9}\frac{2^3c^3}{3^3} + \frac{a}{4}\frac{2^4c^3}{3^4}$, hence when the quantity $\frac{\pi^2\alpha}{l^2}vu^2du$ is integrated between these limits $\frac{1}{3}c$, $-\frac{2}{3}c$, which are those of $k$, $-k'$ in the present case, it becomes equal to

$$\frac{\pi^2\alpha}{l^2}\left[2\left(\frac{1+2^3}{9\ 3^3}\right).ac^3 + \left(\frac{1-2^4}{4\ 3^4}\right)ac^3 = \frac{\pi^2\alpha ac}{36l^2}\right.$$

In the case when the form of the base is concave,

$$v = \frac{a}{c}\left[\frac{c}{3} + u\right], \qquad \int vu^2du = \frac{au^3}{9} + \frac{au^4}{c4},$$

which becomes, when $u = \frac{2}{3}c$, $\frac{1}{3}c$, which correspond to $k$ and $k'$ in this case,

$$\frac{2^3ac^3}{9\ 3^3} + \frac{2^4ac^3}{4.3^4}, \quad \frac{ac^3}{9\ 3^3} + \frac{ac^3}{4\ 3^4},$$

hence we obtain,

$$\frac{\pi^2\alpha}{l^2}\int_{-k'}^{k} vu^2du = \frac{(1+2^3)ac^3}{9\ 3^3} + \frac{(2^4-1)ac^3}{4\ 3^4} = \frac{\pi^2\alpha}{l^2}\frac{ac^3}{12l^2}$$

When the normal section is a square, as the centre of gravity is at the intersection of the diagonals, $k$, $k'$, the limits between which the integral should be taken are respectively equal to $\frac{f}{2}$, and, as all lines at any distance $u$ from the centre of gravity, drawn parallel to a side of the square are equal to the side, we have

$$u = f, \quad \int_{-k'}^{k} vu^2du = \frac{2f}{3}\frac{f^3}{2^3} = \frac{f^4}{12}, \qquad P = \frac{\pi^2\alpha f^4}{12l^2}$$

In the case of a circle, $v$ the line drawn at the distance $u$ from the centre of gravity which is that of the circle, is evidently equal to

$2\sqrt{k^2-u^2}$, and $\int_{-k'}^{k} vu^2du = 2\int_{-k'}^{k}\sqrt{k^2-u^2}\;u^2du$ = by multiplying and dividing by $\sqrt{k^2-u^2}$,

$$2\int_{-k'}^{k}\frac{k^2u^2du}{\sqrt{k^2-u^2}} - 2\int_{-k'}^{k}\frac{u^4du}{\sqrt{k^2-u^2}},$$

now if we make $u = kx$, these expressions become respectively

$$2k^4\int_{-k'}^{k}\frac{x^2dx}{\sqrt{1-x^2}}, \quad -2k^4\int_{-k'}^{k}\frac{x^4dx}{\sqrt{1-x^2}},$$

the integral of the former is

$$-k^4x\sqrt{1-x^2}+k^4\int\frac{dx}{\sqrt{1-x^2}},$$

and that of the latter

$$\tfrac{1}{2}k^4x^3\sqrt{1-x^2}+\tfrac{3}{4}k^4x\sqrt{1-x^2}-\tfrac{3}{4}k^4\int\frac{dx}{\sqrt{1-x^2}}.$$

Now when this integral is taken between the prescribed limits $k, -k'$, all the other terms will vanish but the term $\frac{k^4}{4}\int\frac{dx}{\sqrt{1-x^2}}$, which is evidently equal to $\frac{k^4\pi}{4}$, hence the value of P becomes that in the text

(n) If the area of the normal section in the case of the square and circle were equal, then $f^2=\pi k^2$, consequently, by substituting this value of $f^2$ in the expression for P in the case of the square, it becomes $=\frac{\pi^2ak^4}{12l^2}$, which is to the value of P in the case of the circle $=\frac{\pi^3ak^4}{4l^2} :: \pi : 3$

(o) By making this substitution we obtain

$$P=\frac{\pi^3a}{4l^2}(g'^4-g^4)=\frac{\pi^3}{4l^2}(g'^2+g^2)(g'^2-g^2),$$

now if $\pi(g'^2-g^2)$, the area of the normal section, be supposed equal to $\pi k^2$, then $g'^2=k^2+g^2$, consequently if $k^2$ be substituted in place of $g'^2-g^2$, and $k^2+g^2$ in place of $g'^2$, we shall obtain the expression in the text

(p) As $ds$ is supposed to be the independent variable, $dx=\frac{dx}{ds}ds$, $dy=\frac{dy}{ds}ds$, &c., consequently $Xdx+Ydy+Zdz=$

$$X\frac{dx}{ds}ds + Y\frac{dy}{ds}ds + Z\frac{dz}{ds}ds$$

($g$) Since in this case all the quantities in equation (b) vanish but $\cos\mu, \cos f, \cos g, \cos h$, P$'$, Q$'$, R$'$, and T, $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, and as these three last quantities are equal to $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, it is evident that the values of T $\cos\alpha'$, T $\cos\beta'$, T $\cos\gamma'$ are those given in the text.

($i$) From equation (1) we obtain

$$\beta\frac{dy}{dx} = \frac{4g\gamma\omega}{24}x^3 + 3cx^2 + 2c'x, \quad \beta\frac{d^2y}{dx^2} = \frac{3\,4\,g\gamma\omega}{24}x^2 + 2\,3\,cx + 2c',$$

$$\beta\frac{d^3y}{dx^3} = \frac{2\,3\,4\,g\gamma\omega}{24}x + 2\ 3c,$$

now since by hypothesis $\beta\frac{d^2y}{dx^2} = 0$, $\beta\frac{d^3y}{dx^3} = -Q$, when $x = a$, there results $\frac{1}{2}g\gamma\omega a^2 + 6ca + 2c' = 0$, $g\gamma\omega a + 2\,3\,c = -Q$

From the second of these equations we obtain, by substituting $q$ for $g\gamma\omega a$, $c = -\frac{1}{6}(Q + q)$, and by putting this value of c in the first equation, and substituting for $g\gamma\omega a$ its value $q$, there results

$$\tfrac{1}{2}qa - (Q + q)a + 2c' = 0, \qquad c' = \tfrac{1}{2}(Q + \tfrac{1}{2}q)a,$$

consequently if these values of c and c$'$ are substituted in equation (1), there results the value of $\beta y$ given in the text, for it was shown by the conditions of the question that $c'' = 0$. When $Q = 0$, this equation becomes

$$\beta y = \frac{qx^4}{24a} - \frac{q}{6}x^3 + \tfrac{1}{4}qax^2,$$

and when $x = a$, in which case $y = b$, there results

$$\beta b = \frac{q}{24}a^3 - \tfrac{1}{6}qa^3 + \tfrac{1}{4}qa^3 = \frac{qa^3}{8},$$

and when $q = 0$, this equation becomes

$$\beta y = -\tfrac{1}{6}Qx^3 + \tfrac{1}{2}Qax^2,$$

which, when $x = a$, and $y = b$, becomes

$$\beta b = \frac{Qa^3}{3},$$

and when $q = Q$, the preceding values of $b$ are evidently in the ratio of 3 : 8

(s) Since in this case $y = 0$ when $x = a$, equation (1) becomes, because $g\gamma\omega = \frac{q}{a}$, $0 = \frac{q}{24a}\, a^4 + ca^3 + c'a^2$, hence if it be multiplied by $x$, and then taken from equation (1), we obtain the expression in the text

$$(t)\ \beta\frac{dy}{dx} = \frac{q}{24a}(x^3 - a^3) + c(x^2 - a^2) + c'(x - a) + \frac{3qx^3}{24a} + 2cx^2 + c'x,$$

when $x = 0$, this expression becomes

$$\beta\frac{dy}{dx} = -\frac{q}{24}a^2 - ca^2 - c'a = 0, \qquad c' = -\left(\frac{q}{24} + c\right)a,$$

when $x = a$, this expression becomes

$$\beta\frac{dy}{dx} = \frac{qa^3}{8a} + 2ca^2 + c'a = 0, \qquad c' = -\left(\frac{q}{8} + 2c\right)a,$$

hence a comparison of these two values of $c'$ gives $c = -\frac{1}{12}q$, and consequently $c' = \frac{1}{24}aq$, by substituting these values of $c$ and $c'$, equation (2) becomes

$$\beta y = \frac{qx}{24a}(x^3 - a^3) - \frac{1}{12}qx\,(x^2 - a^2) + \frac{1}{24}aqx\,(x - a),$$

which may be made to assume the form

$$\beta y = \frac{qx}{24a}(x - a)[(x^2 + ax + a^2) - 2a(x + a) + a^2] =$$

$$\frac{qx}{24a}(x - a)\,(x^2 - ax) = \frac{qx^2}{24a}\,(x - a)^2$$

In the second case $\beta\,\frac{d^2y}{dx^2} = \frac{3qx^2}{24a} + 2cx + c' + 9\,\frac{qx^2}{24a} + 4cx + c' = 0$, when $x = 0$, hence it follows that $c' = 0$, and when $x = a$, $6c = -\frac{12q}{24}$, and $\quad c = -\frac{1}{12}q$, consequently, by substituting for $c$, equation (2) becomes

$$\beta y = \frac{qx}{24a}(x^3 - a^3) - \frac{1}{12}qx\,(x^2 - a^2) = \frac{qx}{24a}\,(x - a)\ [x^2 + ax + a^2 -$$

$$2a\,(x + a)] = \frac{qx}{24a}\,(x - a)[x^2 - ax - a^2],$$ which by changing the signs of the two factors of which this product is composed, becomes the value of $\beta y$ given in the text, and when $x = \frac{a}{2}$, the value of $y$, which in this case is $f$, becomes $= \frac{q}{24a}\,\frac{a}{2}\ \ \frac{a}{2}\,\frac{5a^2}{4} = \frac{5qa^3}{24\ 16}$ See note $a$, No 324

In this case also we have

$$\beta\frac{d^3y}{dx^3}=\frac{2\,3\,qx}{24\,a}+2c+\frac{2\,9\,qx}{24a}+4c,$$

which, when $x=0$, becomes $\beta\frac{d^3y}{dx^3}=6c=-\frac{1}{2}q$, and when $x=a$, $\beta\frac{d^3y}{dx^3}=\frac{q}{4}+2c+\frac{3}{4}q+4c=q-\frac{1}{2}q=\frac{q}{2}$

In the third case, as $\frac{dy}{dx}=0$ for $x=0$, we have

$$\beta\frac{dy}{dx}=-\frac{qa^3}{24}-ca^2-c'a=0,$$

and as in this same case $\beta\frac{d^2y}{dx^2}=0$ when $x=a$, we have

$$\frac{qa}{2}+6ca+2c'=0,$$

consequently, comparing the two values of $c'$ derived from these two equations, we obtain

$$\left[\frac{q}{24}+c\right]a=\left[\frac{q}{4}+3c\right]a,\qquad c=-\frac{5q}{48},\text{ and}\qquad c'=-\frac{qa}{4}+\frac{5q}{16}a$$

$=\frac{qa}{16}$; and, if these values be substituted in equation (2), we obtain

$$\beta y=\frac{qx}{24a}(x^3-a^3)-\frac{5qx}{48}(x^2-a^2)+\frac{qax}{16}(x-a)$$

$$=\frac{qx}{48a}(x-a)\left[2x^2+2ax+2a^2-5a(x+a)+3a^2\right]$$

$=\frac{qx}{48a}(x-a)\left[2x^2-3ax\right]=$ (by changing the signs of the factors)

$\frac{qx^2}{48a}(a-x)[3a-2x]$ The value of $\beta\frac{d^3y}{dx^3}$ ($=$ E when $x=0$), is in this case $\beta\frac{d^3y}{dx^3}=6c=-\frac{5q}{8}$, and $\beta\frac{d^3y}{dx^3}$ ($=-$Q when $x=a$) is in this case $\frac{q}{4}+\frac{3q}{4}+6c$, i e $q-\frac{5q}{8}=\frac{3q}{8}=-$Q

(u) By performing the integration with respect to $\sin\frac{n\pi x}{a}$, which is the variable of the second member of this equation, we find

$$\int\sin\frac{n\pi x}{a}dx=-\frac{a}{n\pi}\cos\frac{n\pi x}{a},\text{ and }\frac{-a}{n\pi}\int\cos\frac{n\pi x}{a}dx=\frac{-a^2}{n^2\pi}\sin\frac{n\pi x}{a},$$

hence if this operation be continued four times, the last integration gives

$$\frac{a^4}{n^4\pi^4} \sin\frac{n\pi x}{a},$$

from which, and by a suitable determination of the arbitrary constants introduced by the integration, from the consideration that $y = 0$ for $x = 0$, and for $x = a$, we obtain equation (b)

($v$) When $x' = \frac{a}{2}$, $\sin\frac{n\pi x'}{a} = \sin\frac{n\pi}{2}$, and as $\int_0^a \varphi x' dx' = a$ the expression $\int_0^a \sin\frac{n\pi x'}{a}\varphi x' dx'$ in this case $= a\sin\frac{n\pi}{2}$

($x$) In this case as $n = 2i - 1$, and $\sin(2i-1)\frac{\pi}{2} = -(-1)^i$, we have the first term of the second member of equation (b) equal to

$$-\frac{2qa^3}{\pi^4}\Sigma\frac{(-1)}{(2i-1)^4}\sin\frac{(2i-1)\pi x}{a}$$

($y$) If in this equation we substitute for $\omega$ its value $\frac{\pi x}{a}$, we shall obtain

$$\Sigma\frac{(-1)^i}{(2i-1)^4}\sin\frac{(2i-1)}{a}\pi x = \frac{\pi^4 x^3}{24a^3} \quad \frac{\pi^4 x}{32a},$$

which, by reducing to a common denominator, becomes the expression in the text

($z$) By differentiating we obtain

$$\beta\frac{dy}{dx} = \frac{q}{48}\ (3a^2 - 3\ 4\ x^2),$$

which, when $x = \frac{a}{2}$, i e at the middle point of the curve, is equal to cipher, therefore at this point the tangent is horizontal, and the sagitta then becomes equal to

$$\frac{q}{\beta\ 48}\left(\frac{3a^3}{2} - \frac{4a^3}{8}\right) = \frac{q\ a^3}{\beta\ 48}$$

($a'$) $\frac{dy}{dx} = \frac{q}{\beta\ 48}(3a^2 - 3\ 4\ x^2)$ ($=$ when $x = 0$, tang $\alpha$) $= \frac{qa}{16\ \beta}$,

and as $f'$ $\frac{a}{2}$ tang $\alpha$ rad $f' = \frac{a}{2}$ tang $\alpha$, but $f = \frac{qa^4}{48\ \beta} = \frac{a}{3}$ tang $\alpha$,

$f'$ $f$ 3 2, and in the second case of No 322, as

$$\beta\frac{dy}{dx}=\frac{q(a-x)(a^2+ax-x^2)-qx(a^2+ax-x^2)+qx(a-x)(a-2x)}{24a},$$

equal when $x=0$, $\frac{qa^3}{24a}$, in this case $f'=\frac{qa^3}{48}=\frac{q}{16}\,\frac{8a^3}{24}$, consequently it is to the value of $f$ in that No. as 8 : 5.

($b'$) By multiplying the infinite series given at the commencement of this number by the denominator, there results

$$1+2h\cos\theta+2h^2\cos 2\theta+2h^3\cos 3\theta+\&c.$$
$$1-2h\cos\theta+h^2$$

$$=1+2h\cos\theta+2h^2\cos 2\theta+2h^3\cos 3\theta+\&c.$$
$$-2h\cos\theta-4h^2\cos^2\theta-4h^3\cos 2\theta\cos\theta-4h^4\cos 3\theta\cos\theta$$
$$+h^2+2h^3\cos\theta+2h^4\cos 2\theta+2h^5\cos 3\theta+\&c.$$

Now since $2\cos^2\theta=\cos 2\theta+1$, $2\cos 2\theta\cos\theta=\cos 3\theta+\cos\theta$, $2\cos 3\theta\cos\theta=\cos 4\theta+\cos 2\theta$, by substituting these values in place of $\cos^2\theta$, $2\cos 2\theta\cos\theta$, $2\cos 3\theta\cos\theta$, and adding these lines together their sum will be equal to $1+h^2$.

This theorem is proved, *a priori*, in the *Theorie de la Chaleur, par Poisson*, No. 93.

When the values of $u$ are infinitely small, $4\sin^2\frac{1}{2}(\theta-\alpha)=u^2$, and if in the expression $\frac{g f\theta d\theta}{g^2+4\sin^2(\theta-u)}$, we substitute for $f\theta$, $d\theta$, and $4\sin^2\frac{1}{2}(\theta-\alpha)$ we will obtain the expression in the text, and as the integral of this function between the limits $\infty$ and 0, is $\frac{\pi}{2}$, and between the limits 0 and $-\infty$ is also $\frac{\pi}{2}$, between the limits $\infty$ and $-\infty$ the integral is $\pi$.

When $\alpha=\pi$, then $\cos n(\theta-\alpha)=\cos n\pi\cos n\theta$, ($=$ as $\cos n\pi=(-1)^n$) $(-)^n\cos n\theta$.

Since $\theta=\frac{\pi x'}{a}$, and as the limits relative to $\theta$ are $\pi$ and 0, the limits relative to $x'$ must be $a$ (when $\theta=\pi$) and when $\theta=\alpha$, it must be cipher.

Since $\cos n\pi\left(\frac{x\pm x'}{a}\right)=\cos\frac{n\pi x'}{a}\cos\frac{n\pi x}{a}\mp\sin\frac{n\pi x'}{a}\sin\frac{n\pi x}{a}$, when the first equation (5) is taken from the second, we obtain the expression in the text.

($c'$) When $f\theta=\theta$, this function becomes

$$\int_0^\pi \cos n\theta\, \theta d\theta = \sin\frac{n\theta}{n}\theta + \cos\frac{n\theta}{n},$$

which taken between the limits $\pi$ and 0, is

$$\cos\frac{n\pi - 1}{n^2},$$

consequently, since when $n$ is an even number of the form $2i$, $\cos n\pi = 1$, and when $n$ is an odd number, of the form $2i-1$, $\cos n\pi = -1$, in the former case $\cos\frac{n\pi-1}{n^2} = 0$, in the latter it is equal to $-\frac{2}{(2i-1)^2}$, hence the given equation becomes as $\int_0^\pi f\theta\, d\theta = \frac{\pi^2}{2}$, and $f\alpha = \alpha$,

$$\frac{\pi^2}{2} - 4\Sigma\frac{\cos(2i-1)\alpha}{(2i-1)^2}\alpha = \pi\alpha, \qquad \Sigma\frac{\cos(2i-1)\alpha}{(2i-1)^2} = \frac{\pi}{8}(\pi - 2\alpha);$$

now in the expression $\sin\frac{(2i-1\alpha}{(2i-1)^3} = \frac{\pi}{8}(\pi-\alpha)\alpha$, when $\alpha = \frac{\pi}{2} + \omega$, as $\cos(2i-1)\frac{\pi}{2} = 0$, $\sin(2i-1)\frac{\pi}{2} = (-1)^i$, $(\pi-\alpha)\alpha = \left(\frac{\pi^2}{4} - \omega^2\right)$, we have evidently, by changing all the signs, $\Sigma(-1)^i \sin\frac{(2i-1)\omega}{(2i-1)^3} = \frac{\pi}{8}(\omega^2 - \frac{1}{4}\pi^2)$, from $\omega = -\frac{1}{2}\pi$ to $\omega = \frac{1}{2}\pi$, for these limits $\omega = -\frac{1}{2}\pi$, $\omega = \frac{\pi}{2}$, correspond to $\alpha = 0$, $\alpha = \pi$

END OF VOL. I